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# PROCEEDINGS

OF

# THE LONDON MATHEMATICAL SOCIETY.

## VOL. XXIX.

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# THE LONDON MATHEMATICAL SOCIETY.

VOL. XXIX.

## THIRTY-FOURTH SESSION, 1897–98

(since the Formation of the Society, January 16th, 1865).

*November 11th*, 1897.

THE FOURTH ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Prof. E. B. ELLIOTT, M.A., F.R.S., President, in the Chair.

Present eighteen members.

Mr. J. B. Dale and Mr. G. B. Mathews were admitted into the Society.

The President briefly referred to the recent decease of Lt.-Col. J. R. Campbell, who had rendered signal service to the Society by his liberal gift of £500 to its funds. The deceased gentleman served on the Council in the Sessions 1893–94, 1894–95.

The Treasurer gave an abstract of his report. In the course of his remarks he pointed out that the publications of the Society had of recent years grown to such an extent as to tax its utmost resources; and that, were it not for the help of the funds munificently established many years ago by Lord Rayleigh, and more recently by the late Lt.-Col. Campbell, the Society would be compelled to materially restrict its sphere of activity. Its reception was moved by Mr. A. B. Kempe, and seconded by Mr. J. E. Campbell. With the motion was

coupled another nominating the Rev. T. R. Terry as Auditor, should he be able to serve. The reception of the report and the motion for nomination of the Auditor were carried unanimously.

The senior Secretary stated that the present number of members was 229, of whom 116 were life compounders. The Society's losses by death had been: Professor Sylvester, elected June 19th, 1865; Rev. A. Freeman, elected December 14th, 1871; and Lt.-Col. J. R. Campbell, elected December 9th, 1875. He also mentioned that only one additional exchange of *Proceedings* had been made during the Session, viz., with the American Philosophical Society, Philadelphia.

The following communications had been made or received:—

Combinatory Analysis: A Review of the Present State of Knowledge (Presidential Address).

On the Geometrical Calculus (two papers): Herr E. Lasker.

The Calculus of Equivalent Statements (Fifth and Sixth Papers): Mr. H. MacColl.

On a General Integral with some Physical Applications: Mr. G. J. Hurst.

On Ratio: Prof. M. J. M. Hill.

On the Geometrical Construction of Models of Cubic Surfaces: Mr. W. H. Blythe.

The Fluted Vibrations of a Circular Vortex Ring with a Hollow Core: Mr. H. S. Carslaw.

The Differentiation of Spherical Harmonics: Mr. E. G. Gallop.

On the application of Jacobi's Dynamical Method to the General Problem of Three Bodies: Professor E. W. Brown.

On certain Properties of the Mean Motions and the Secular Accelerations of the Principal Arguments used in the Lunar Theory: Professor E. W. Brown.

Note on the Symmetric Group: Professor W. Burnside.

Note on the Electric Capacity of a Conductor in the Form of Two Intersecting Spheres: Mr. W. D. Niven.

Note on the Foregoing Paper by Mr. H. M. Macdonald.

Note on a Discovery in the Theory of Denumeration: Professor Sylvester.

On the Stationary Motion of a System of Equal Elastic Spheres of Finite Diameter: Mr. S. H. Burbury.

On the Influence of Viscosity on Waves and Currents: Mr. S. S. Hough.

Description of a Multiplying Apparatus: Mr. Macfarlane Gray.

On the Connexion of Quadratic Forms: Lt.-Col. Cunningham.

Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holohedrically Isomorphic with the Symmetric and the Alternating Substitution-Groups on $k$ Letters: Professor E. H. Moore.

On a Series of Cotrinodal Quartics: Mr. H. M. Taylor.

On Finite Variations: Mr. E. P. Culverwell.

On the Partition of an Even Number into Two Primes: Professor Sylvester.

Supplementary Note on Matrices: Mr. J. Brill.

Note on some Properties of Bessel's Functions: Dr. E. W. Hobson.

On a Theorem in Non-Euclidean Geometry : Mr. F. S. Macaulay.

Note on High Primes : Lt.-Col. Cunningham.

On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups : Mr. J. E. Campbell.

Some Notes on Symmetric Functions : Mr. W. H. Metzler.

Note on a System of Circles associated with a Triangle : Professor Steggall.

Über verzweigte Potentiale im Raum : Dr. A. Sommerfeld.

Certain Concomitant Determinants : Mr. J. W. Russell.

On the Potentials of Rings : Mr. A. L. Dixon.

On the Deformation of a Plane Closed Polygon, so that a certain Function remains Constant : Mr. F. S. Macaulay.

An Extension of the Theorem

$$\frac{\Pi(\gamma-\alpha-\beta-1)\,.\,\Pi(\gamma-1)}{\Pi(\gamma-\alpha-1)\,.\,\Pi(\gamma-\beta-1)} = F(\alpha,\ \beta,\ \gamma) :$$

Rev. F. H. Jackson.

A Continuation of Gauss's "Dioptrische Untersuchungen" : Professor Sampson.

On Cubic Curves as connected with certain Triangles in Perspective : Mr. S. Roberts.

Determination of certain Primes : Mr. F. W. Lawrence.

An Analogue of Anharmonic Ratio : Mr. J. Brill.

On the Partition of Numbers : Mr. G. B. Mathews.

Notes on Synthetic Geometry : Professor W. Esson.

On the Primitive Substitution-Groups of Degree Fifteen : Dr. G. A. Miller.

A Generalized Form of the Binomial Theorem : Rev. F. H. Jackson.

On the Degeneration of a Cubic Curve : Mr. H. M. Taylor.

In addition to the foregoing papers which were communicated at the monthly meetings the Society had published the "Outline of Lectures on the Partition of Numbers," which were read, by the late Professor Sylvester, at King's College, London, in 1859.

The same journals had been subscribed for, and the same exchanges of *Proceedings*, in addition to the one named above, made, as in the preceding Session.

One present had been made to the Society's Album.

Mr. G. Heppel and Mr. A. E. Western having been appointed Scrutineers, the ballot was taken, with the result that the following gentlemen, nominated by the Council, were elected to serve as the Council for the ensuing Session :—Professor Elliott, F.R.S., President; Dr. Hobson, F.R.S., and Major MacMahon, R.A., F.R.S., Vice-Presidents ; Dr. J. Larmor, F.R.S., Treasurer; Mr. R. Tucker and Mr. A. E. H. Love, F.R.S., Hon. Secretaries. Other members of the Council :—Lt.-Col. Cunningham, R.E., Dr. Glaisher, F.R.S., Professor M. J. M. Hill, F.R.S., Professor W. H. H. Hudson, Mr. M. Jenkins, Mr. A. B. Kempe, F.R.S., Mr. F. S. Macaulay, Mr. D. B. Mair, Mr. G. B. Mathews, F.R.S., and Mr. W. D. Niven, C.B., F.R.S.

Mr. Campbell read a paper "On an Extension of the Exponential Theorem."

Mr. Hargreaves gave an abstract of his paper, entitled "The Integral $\int P_n^2 dx$, and Allied Forms in Legendre's Functions, between Arbitrary Limits."

Mr. Mathews communicated a Note by Professor F. Morley, "On the Poncelet Polygons of a Limaçon."

The following papers were communicated from the chair:—

The Calculus of Equivalent Statements (Paper 7): Mr. H. MacColl.

The Character of the General Integral of Partial Differential Equations: Professor Forsyth.

Note on Bessel Functions: Mr. H. M. Macdonald.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LXII., No. 380.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 10; Leipzig, 1897.

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. 4, No. 186; London, July, 1897.

"Monatshefte für Mathematik und Physik," VIII. Jahrgang, 1897; Wien, 1897.

Lemoine, E.—"Questions relatives à la Géométrie du Triangle, à la Géométrografie et à la Transformation continue," pamphlet, 8vo; Paris, 1896. "Mélanges sur la Géométrie du Triangle," pamphlet, 8vo; Paris, 1895. "Note sur une Construction approchée du Développement de la Circonférence, et Remarques diverses," pamphlet, 8vo; Paris.

"Bulletin de la Société Mathématique de France," Tome XXV., Nos. 6, 7; Paris, 1897.

"Bulletin of the American Mathematical Society," Vol. IV., No. 1; October, 1897, New York.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VI., Fasc. 7, 8; Roma.

"Journal für die reine und angewandte Mathematik," Bd. CXVIII., Heft 4; Berlin, 1897.

"Educational Times," November, 1897.

"Annali di Matematica," Tomo XXVI., Fasc. 2, 3; Milano, 1897.

"Indian Engineering," Vol. XXII., Nos. 13–16; Sept. 25–Oct. 16, 1897.

"Transactions of the Royal Irish Academy," Vol. XXXI., Pt. 4; October, 1897.

## *The Character of the General Integral of Partial Differential Equations.* *By* A. R. Forsyth. Received November 1st, 1897. Read November 11th, 1897.

1. The present investigation is an extension of one due to Ampère,* in which he proved that the number of independent arbitrary functions occurring in the general integral of a partial differential equation of order $m$ in two independent variables is equal to $m$; the integral being supposed finite and without partial quadratures. The same assumption as to the character of the integral will here be made; but it will be supposed that the number of independent variables is now $n$, and it will be shown

(i.) That the number of independent arbitrary functions in the integral is equal to the order of the equation, whatever be the number of independent variables:

(ii.) That, in general, the number of arguments in each such arbitrary function is less by unity than the number of independent variables.

The differential equation of order $m$ in $n$ independent variables will be denoted by $F = 0$.

2. Denote the dependent variable by $z$, and the independent variables by $x_1, x_2, \ldots, x_n$. Suppose that, in addition to these variables, the integral system involves a number, say $k$, of variable quantities $\alpha, \beta, \gamma, \ldots$; these quantities being not necessarily independent of one another. As the integral system is to be equivalent to a single relation which, besides the original variables, shall involve constants and functional forms, it must contain $k+1$ equations from which the $k$ variable quantities are to be eliminated.

Further, suppose that $g$ independent arbitrary functions $\phi, \psi, \ldots$

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* The result referred to is contained in the memoir "Considérations générales sur les intégrales des équations aux différentielles partielles," *Jour. de l'École Polytechnique*, t. x. (1815), pp. 549–611. This memoir was followed by another, "Mémoire contenant l'application de la théorie . . . à l'intégration des équations aux différentielles partielles du premier et du second ordre," *Jour. de l'École Polytechnique*, t. xi. (1820), pp. 1–188.

It is almost superfluous to insist on the importance of these two memoirs by Ampère.

occur in the integral system; and that, in general, each such function has $r$ arguments, though in particular cases the number of arguments may be less than $r$. These arguments may be considered as connected with the $k$ variable quantities $\alpha, \beta, \gamma, \ldots$. Also it may happen that derivatives (some or all up to a specified order) of the arbitrary functions with regard to their arguments occur in the integral system. Let the highest derivatives of $\phi$, which thus occur, be of order $p_\phi$; and so for derivatives of $\psi, \ldots$.

Adopting Ampère's definition of a general integral which is finite and without partial quadratures, we assume that the value of $z$ deduced from the integral equations satisfies the equation $F = 0$, and satisfies also all the equations derived directly from $F = 0$ by differentiation; and further, that all the differential equations of orders not less than $m$ (that is, of the same order as the given equation and of higher orders), which can be obtained from the integral equations, are equivalent to $F = 0$ and equations derived from $F = 0$ by differentiation.

This being so, we form all derivatives of the integral equations up to and including the order $s$, so that the $s^{\text{th}}$ derivatives of $z$ occur in the resulting equations. To deduce differential equations, various quantities must be eliminated from this aggregate.

3. Among the quantities to be eliminated are the derivatives of $\alpha, \beta, \gamma, \ldots$, up to order $s$ with respect to the $n$ variables; the total number of these is

$$\frac{(s+1)(s+2)\ldots(s+n)}{n!}\,k = \frac{(s+n)!}{s!\,n!}\,k.$$

Let $l+1, l'+1, \ldots$ be the orders of those equations deduced from the integral equations which are the earliest, in the succession of increasing orders, to contain derivatives of the arbitrary functions $\phi, \psi, \ldots$ of order higher than $p_\phi, p_\psi, \ldots$, being the highest that occur in the integral system.* Then derivatives of the integral system of all orders after $l+1, l'+1, \ldots$ give further derivatives of $\phi, \psi, \ldots$; so that, as regards derivatives of $\phi$, those which are introduced by differentiation enter through the last $s-l$ derivatives of the integral system. Now the order of the highest derivative of $\phi$ in the integral

* In the simplest case, $l, l', \ldots$ are all zero; but, in general, the values of $l, l', \ldots$ depend upon the mode in which the integral system contains the arbitrary functions. As to this, see Ampère's first memoir, p. 587, and the first section of his second memoir.

system is $p_\phi$; hence, in the system deduced by differentiating the integral system, the order of the highest derivative of $\phi$ is $s-l+p_\phi$. Consequently the total number of derivatives of $\phi$, whether occurring in the integral system or introduced by the differentiations, that can occur in the aggregate of the relations is

$$1+r+\frac{r(r+1)}{2!}+\ldots+\frac{r(r+1)\ldots(r+s-l+p_\phi-1)}{(s-l+p_\phi)!}$$
$$=\frac{(r+1)(r+2)\ldots(r+s-l+p_\phi)}{(s-l+p_\phi)!}$$
$$=\frac{(s-l+p_\phi+1)(s-l+p_\phi+2)\ldots(s-l+p_\phi+r)}{r!}.$$

And so for the derivatives of the other arbitrary functions with regard to their arguments. Hence the total number of derivatives of the arbitrary functions with regard to their arguments, including the functions themselves, can be as great as

$$\frac{(s-l+p_\phi+1)(s-l+p_\phi+2)\ldots(s-l+p_\phi+r)}{r!}, \text{ for } \phi,$$
$$+\frac{(s-l'+p_\psi+1)(s-l'+p_\psi+2)\ldots(s-l+p_\psi+r)}{r!}, \text{ for } \psi,$$
$$+\ldots$$
$$=\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\ldots(s-l+p+r)\},$$

where the summation extends over the $g$ various numbers for the different functions, and the integers $l$ and $p$ may be respectively distinct in these numbers. This sum is an upper limit of the number of derivatives that may occur; the form of the integral equations may be such that not all of these derivatives actually enter into the aggregate of equations.

Consequently, the total number of arguments and their derivatives with regard to the independent variables, and the total number of arbitrary functions and their derivatives with regard to their arguments, are together equal to or less than

$$k\frac{(s+1)(s+2)\ldots(s+n)}{n!}$$
$$+\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\ldots(s-l+p+r)\}.$$

Let $N$ denote the deficiency from this integer of the corporate

aggregate of arguments, functions, and their respective derivatives; so that $N$ may be zero but otherwise is a positive integer. Then the total number of quantities to be eliminated from the integral system and the derived equations is

$$\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\dots(s-l+p+r)\} + k\frac{(s+1)(s+2)\dots(s+n)}{n!} - N;$$

when all the elimination has taken place, the resulting equations involve derivatives of $z$ with regard to the independent variables up to order $s$ inclusive.

4. When the integral system of $k+1$ equations is differentiated so as to give all derivatives of $z$ up to order $s$ inclusive, the total number of equations thus obtained is

$$\frac{(s+1)(s+2)\dots(s+n)}{n!}(k+1).$$

It is from this set that the foregoing number of quantities, both functions and arguments, must be eliminated in order to give partial differential equations. In general, the various quantities will disappear singly during the elimination; in that case, the number of eliminant equations is the excess of the second integer over the first. But it may happen that some of the quantities disappear in combinations, of several together, and one or more in number; in that case, the number of eliminant equations will be greater than the said excess by some integer, say $S$. The number of eliminant equations can therefore be taken as equal to

$$\frac{(s+1)(s+2)\dots(s+n)}{n!}(k+1) - \left[\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\dots(s-l+p+r)\} + \frac{k}{n!}(s+1)(s+2)\dots(s+n) - N\right] + S;$$

that is, the number

$$= \frac{1}{n!}(s+1)(s+2)\dots(s+n) + N + S - \frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\dots(s-l+p+r)\}.$$

Each of the eliminant equations is a differential equation satisfied in virtue of the integral system, and the highest derivatives of $z$ that occur are of order $s$.

5. Now, by the adopted definition of the general integral, these equations should constitute a complete algebraical equivalent of $F=0$, and equations deduced from it by effecting upon $F=0$ differentiations up to order $s-m$ inclusive, so that the deduced equations may involve derivatives of order $s$. The total number of equations in this set is

$$\frac{1}{n!}(s-m+1)(s-m+2)\ldots(s-m+n),$$

which accordingly is equal to the preceding number of eliminant equations; that is,

$$\frac{1}{n!}(s+1)(s+2)\ldots(s+n)+N+S$$

$$-\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\ldots(s-l+p+r)\}$$

$$=\frac{1}{n!}(s-m+1)(s-m+2)\ldots(s-m+n),$$

a relation which must be true for all values of $s$ greater than $m$ and including $m$.

Now each of the integers $l$, and each of the integers $p$, is invariable, that is, no one of them depends upon $s$. On the other hand, $N$ and $S$ may depend upon $s$, and, if different from zero, they will in general do so. But, for appreciably large values of $s$, both $N$ and $S$ are numbers small as compared with the number of equations with which we then have to deal.

Expand each side of the equation, taken in the form

$$\frac{1}{r!}\Sigma\{(s-l+p+1)(s-l+p+2)\ldots(s-l+p+r)\}-N-S$$

$$=\frac{1}{n!}\{(s+1)(s+2)\ldots(s+n)-(s-m+1)(s-m+2)\ldots(s-m+n)\},$$

in descending powers of $s$; we have as the first term on the left-hand side

$$\frac{g}{r!}s^r.$$

On the right-hand side, the term in $s^n$ disappears; the coefficient of $s^{n-1}$ is

$$\frac{1}{n!}\{1+2+\ldots+n+(m-1)+(m-2)+\ldots+(m-n)\} = \frac{m}{(n-1)!};$$

so that we must have

$$\frac{g}{r!}s^r = \frac{m}{(n-1)!}s^{n-1}.$$

This must be true for all values of $s$; hence

$$g = m, \quad r = n-1.$$

The remaining terms in the equation give relations between the numbers $l$, $p$, and the coefficients of powers of $s$ in $N$ and $S$; but they are relatively of less importance.

The equation $g = m$ implies *that the number of independent arbitrary functions in the general integral, supposed finite and without quadratures, of a partial differential equation of any order is equal to the order of the equation.* And the equation $r = n-1$ implies that, *in general, the number of arguments in each such arbitrary function is less by unity than the number of independent variables.* It will be noticed that each of these integers is independent of the other.

Thus the general integral of a partial differential equation of the first order in $n$ independent variables will contain one arbitrary function of $n-1$ arguments; the general integral of a partial differential equation of the second order in two independent variables will contain two arbitrary functions, each of a single argument, these results being well known. And, as a final special case, we infer that the general integral of a partial differential equation of the second order in three independent variables will contain two arbitrary functions, each of two arguments.

6. The results obtained can be connected with an application of Cauchy's existence-theorem,* which, for the case under consideration, can be enunciated as follows:—

Suppose that, for values of $x_1, x_2, \ldots, x_n$ in the vicinity of $x_1 = a_1$, $x_2 = a_2, \ldots, x_n = a_n$, the coefficients of the derivatives of $z$ in the differential equation $F = 0$ of order $m$ in the $n$ independent variables are holomorphic functions; and let $\phi_1, \phi_2, \ldots, \phi_m$ denote $m$ functions of $x_2, x_3, \ldots, x_n$, which are holomorphic for values of those variables in the vicinity of $x_2 = a_2, \ldots, x_n = a_n$, but are otherwise arbitrary. Then

* Jordan, *Cours d'Analyse*, t. iii., p. 306.

a unique solution $z$ of the equation exists, which is a holomorphic function of the variables in the vicinity of $x_1 = a_1$, $x_2 = a_2$, ..., $x_n = a_n$, and is such that the values of $z$, $\frac{\partial z}{\partial x_1}$, $\frac{\partial^2 z}{\partial x_1^2}$, ..., $\frac{\partial^{m-1} z}{\partial x_1^{m-1}}$, for $x_1 = a_1$, are respectively the $m$ assigned arbitrary functions $\phi_1$, $\phi_2$, ..., $\phi_m$.

This proposition shows an origin, from another source, of the $m$ arbitrary functions in the solution. It also indicates a reason for the $n-1$ arguments of the arbitrary functions. For, when $x_1 = a_1$, those $n-1$ arguments are $n-1$ functions of $n-1$ variables, and an arbitrary function of those arguments becomes, in effect, an arbitrary function of the $n-1$ variables simply; and so conversely in the solution, we are entitled to infer that generally an arbitrary function has $n-1$ arguments.

A more general form than that just given can be obtained by effecting appropriate transformations of the independent variables. To show this, consider the particular case of a differential equation of the second order in three independent variables; so that $m = 2$, $n = 3$. Instead of postulating two arbitrary functions of $x_2$ and $x_3$ for a particular value of $x_1$, suppose that, for a general relation

$$\psi(x_1, x_2, x_3) = 0,$$

between $x_1$, $x_2$, and $x_3$, the values of $z$ and $\frac{\partial z}{\partial x_1}$ are to become two assigned arbitrary functions of $x_1$, $x_2$, $x_3$, say $\chi(x_1, x_2, x_3)$ and $\theta(x_1, x_2, x_3)$. The new form of condition can be made symmetrical as regards the derivatives $\frac{\partial z}{\partial x_1}$, $\frac{\partial z}{\partial x_2}$, $\frac{\partial z}{\partial x_3}$. For, taking $\psi = 0$ as the equation of a surface in ordinary space, we have $z = \chi$ everywhere along the surface; and so, for variations on the surface, we have

$$\frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \frac{\partial z}{\partial x_3} dx_3 = dz \text{ always,}$$

$$= \frac{\partial \chi}{\partial x_1} dx_1 + \frac{\partial \chi}{\partial x_2} dx_2 + \frac{\partial \chi}{\partial x_3} dx_3$$

upon the surface. For all such variations, that is, for points upon the surface, we have

$$\frac{\partial z}{\partial x_1} = \theta(x_1, x_2, x_3),$$

so that $\left(\theta - \frac{\partial \chi}{\partial x_1}\right) dx_1 + \left(\frac{\partial z}{\partial x_2} - \frac{\partial \chi}{\partial x_2}\right) dx_2 + \left(\frac{\partial}{\partial x_3} - \frac{\partial \chi}{\partial x_3}\right) dx_3 = 0,$

for variations on the surface. But the sole independent relation between these variations is

$$\frac{\partial \psi}{\partial x_1} dx_1 + \frac{\partial \psi}{\partial x_2} dx_2 + \frac{\partial \psi}{\partial x_3} dx_3 = 0 ;$$

and therefore we have

$$\frac{\dfrac{\partial z}{\partial x_2} - \dfrac{\partial \chi}{\partial x_2}}{\dfrac{\partial \psi}{\partial x_2}} = \frac{\dfrac{\partial z}{\partial x_3} - \dfrac{\partial \chi}{\partial x_3}}{\dfrac{\partial \psi}{\partial x_3}} = \frac{\theta - \dfrac{\partial \chi}{\partial x_1}}{\dfrac{\partial \psi}{\partial x_1}}$$

along the surface, that is, when $\psi = 0$. In other words, the values of the other derivatives can be obtained when the values of $z$ and of one derivative are given subject to a relation

$$\psi (x_1, x_2, x_3) = 0.$$

Effecting now the transformation

$$\left.\begin{aligned} \xi_1 - a_1 &= \psi (x_1, x_2, x_3) \\ \phi_1 (\xi_2, \xi_3) &= \chi (x_1, x_2, x_3) \\ \frac{\partial \psi}{\partial x_1} \phi_2 (\xi_2, \xi_3) &= \theta (x_1, x_2, x_3) - \frac{\partial \chi}{\partial x_1} \end{aligned}\right\},$$

and using the proposition just established, we obtain a special case of the earlier form of the theorem. We therefore see that (except for particular families of relations, corresponding to singularities in the former case) *a holomorphic solution $z$ of the equation of the second order in three variables exists which is uniquely determined by the conditions that, when $\psi (x_1, x_2, x_3) = 0$ is satisfied, the values of $z$ and of one of its derivatives (or of any linear combination of the derivatives) are arbitrarily assigned functions of $x_1, x_2, x_3$.*

The further inference as to the existence of a couple of arguments in the solution in its general form, when no relation is postulated among the independent variables, can be made as before.

7. Of course, the preceding investigation is of a general character, and does not take account of the characteristics of particular equations; the result is that, if arbitrary functions of $n-1$ arguments occur in the integral, then the number of those functions will not exceed the order of the equation.

There exists, however, a different kind of solution in the case of certain classes of differential equations. For instance, in the case of equations of the second order in three independent variables, integrals

can occur containing three arbitrary functions each of a single argument. As an example, let $X$, $Y$, $Z$ denote arbitrary functions of $x$, of $y$, and of $z$, respectively; then it is easy to verify that

$$v = X+Y+Z+(y-z)\,X'+(z-x)\,Y'+(x-y)\,Z'$$

satisfies the equation

$$f+g+h=0,$$

where $$f=\frac{\partial^2 v}{\partial y\partial z},\quad g=\frac{\partial^2 v}{\partial z\partial x},\quad h=\frac{\partial^2 v}{\partial x\partial y};$$

it satisfies no other equation of the second order, and no equation of the first order. It should, however, be remarked that, on the one hand, the integral is not of the most general character possessed by

$$f+g+h=0;$$

and that, on the other hand, it satisfies the equation of the third order, represented in the forms

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial g}{\partial y}=0,\quad \frac{\partial h}{\partial z}=0,$$

or, what is the same thing,

$$\frac{\partial^3 v}{\partial x\partial y\partial z}=0,$$

which is not a derivative of

$$f+g+h=0.$$

Similar remarks apply to the integral equation

$$v = X+Y+Z-(2x+z)\,X'+(x+z)\,Y'+(y-z)\,Z',$$

which satisfies the differential equation

$$n-l=-(2x+z)\,g+(y-z)(f-h)$$

of the second order.

*On a Law of Combination of Operators.* (*Second Paper.**) By J. E. Campbell. Read and Received November 11th, 1897.

1. If $x$ and $y$ are operators which obey the ordinary laws of algebra, we know that

$$e^y e^x = e^{y+x}.$$

I propose to investigate the corresponding theorem when the operators obey the distributive and associative laws, but not the commutative.

Let $y_1$ denote the operator $yx - xy$,

$y_2$ ,, ,, $y_1 x - x y_1$,

$y_r$ ,, ,, $y_{r-1} x - x y_{r-1}$.

Let $(p, q)$ denote $y_p y_q - y_q y_p$,

$(p, q, r)$ ,, $(p, q)\, y_r - y_r\, (p, q)$,

and let $(p, q, r, s)$, &c., have similar meanings.

The theorem to be proved is this :—If $a_1, a_2, a_3, \ldots$ is the series of numerical constants† discussed in a former paper (*Proceedings*,

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* At the suggestion of the referees, and by permission of the Council, the title of this paper has been changed.

† These constants are closely associated with Bernouilli's numbers ; in fact

$$a_{2n} = (-1)^{n-1} \frac{B_{2n-1}}{(2n)!}$$

(Lie, *Transformationsgruppen*, III., § 144). This may easily be verified independently thus :—

$$(e^t - 1)^{-1} = \frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots .$$

Differentiate with respect to $t$ ; then

$$\frac{1}{t^2} - \frac{B_1}{2!} + \frac{3B_3}{4!} t^2 - \ldots = (e^t - 1)^{-1} + (e^t - 1)^{-2}$$

$$= \left( \frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots \right) + \left( \frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots \right)^2.$$

Equating coefficients of $t^{2n}$ on each side, we see that

$$(2n+3) \frac{B_{2n+1}}{(2n+2)!} = \left( \frac{B_1}{2!} \frac{B_{2n-1}}{(2n)!} + \frac{B_3}{4!} \frac{B_{2n-3}}{(2n-2)!} + \ldots \right),$$

from which it can easily be seen that

$$a_{2n} = (-1)^{n-1} \frac{B_{2n-1}}{(2n)!}.$$

Vol. XXVIII., p. 381), and $b_{pq} \ldots b_{pqr} \ldots b_{pqrs} \ldots$ are other series of numerical constants, derived by certain laws from the first series, then

$$e^y e^x = e^w,$$

where

$$w \equiv x+y+a_1y_1+a_2y_2+\ldots+\Sigma b_{pq}(p, q)+\Sigma b_{pqr}(p, q, r)+\ldots,$$

the summation being taken in each case for all positive integral values of $p$, $q$, $r$, ... (zero included).

Having established this result, it will be shown that $x_1$, $x_2$, ... $x_n$ being any $n$ operators, not commutative with respect to one another, and $\lambda_1$, $\lambda_2$, ... $\lambda_n$ and $\mu_1$, $\mu_2$, ... $\mu_n$ two sets of arbitrary constants,

$$e^{\mu_1x_1+\ldots+\mu_nx_n} \cdot e^{\lambda_1x_1+\ldots+\lambda_nx_n} \equiv e^{\Sigma\beta_p x_p+\Sigma\beta_{pq}(p, q)+\Sigma\beta_{pqr}(p, q, r)+\ldots},$$

where $(p, q)$ is now taken to mean $x_px_q-x_qx_p$,

$(p, q, r)$ ,, ,, $(p, q)x_r-x_r(p, q)$,

with similar meanings for the other symbols, and the summation is now made by giving to $p$, $q$, $r$, ... all values from 1 to $n$ inclusive.

The $\beta$'s are now, however, no longer mere numerical constants, but functions of the parameters $\lambda_1 \ldots \lambda_n$ and $\mu_1 \ldots \mu_n$.

It will be seen that the number of indices in a $\beta$ function expresses its degree in $\lambda$, $\mu$; thus $\beta_{pqr}$ is a function of the third degree in $\lambda_1 \ldots \lambda_n$, $\mu_1 \ldots \mu_n$.

In proving these theorems it will be seen that the following more general theorem is also true :—

If $w_1 = a_1x_1+\ldots+a_nx_n+a_{pq}(p, q)+\ldots+a_{pqr}(p, q, r)+\ldots,$

$w_2 = b_1x_1+\ldots+b_nx_n+b_{pq}(p, q)+\ldots+b_{pqr}(p, q, r)+\ldots,$

then $e^{w_1} \cdot e^{w_2} = e^{w_3},$

where $w_3 = c_1x_1+\ldots+c_nx_n+c_{pq}(p, q)+\ldots+c_{pqr}(p, q, r)+\ldots.$

Here the $a$'s and the $b$'s are any constants whatever, and the $c$'s are other constants depending on them.

2. The identity

$$(A) \qquad x^n y = yx^n - ny_1x^{n-1} + \frac{n(n-1)}{2!}y_2x^{n-2} - \ldots,$$

obviously holds when $n = 1$. Assume that it also holds for all values up to $n$; then

$$x^{n+1}y = xyx^n - nxy_1x^{n-1} + \frac{n(n-1)}{2!}xy_2x^{n-2} - \dots .$$

Now $xy = yx - y_1, \quad xy_1 = y_1x - y_2, \quad xy_{r-1} = y_{r-1}x - y_r;$

therefore
$$\begin{aligned} x^{n+1}y &= yx^{n+1} - ny_1x^n + \frac{n(n-1)}{2!}y_2x^{n-1} - \dots \\ &\quad - y_1x^n + ny_2x^{n-1} - \dots \\ &= yx^{n+1} - (n+1)y_1x^n + \frac{(n+1)n}{2!}y_2x^{n-1} - \dots, \end{aligned}$$

so that the identity holds universally, since it holds when $n = 1$.

If $$[yx^r] \equiv yx^r + xyx^{r-1} + x^2yx^{r-2} + \dots + x^ry,$$

we see that

$$\text{(B)} \quad \left[\frac{yx^r}{(r+1)!}\right] = \frac{yx^r}{1!\,r!} - \frac{y_1x^{r-1}}{2!\,(r-1)!} + \frac{y_2x^{r-2}}{3!\,(r-2)!} - \dots + \frac{(-1)^r y_r}{(r+1)!},$$

for the identity holds when $r = 1$; and, assuming that it holds for all values of $r$ up to $n-1$, then

$$\begin{aligned} \left[\frac{yx^n}{n!}\right] &= \left[\frac{yx^{(n-1)}}{n!}\right]x + \frac{x^ny}{n!} \\ &= \frac{yx^n}{1!\,(n-1)!} - \frac{y_1x^{n-1}}{2!\,(n-2)!} + \dots + (-1)^{n-1}\frac{y_{n-1}x}{n!} + \frac{x^ny}{n!}; \end{aligned}$$

but $$x^ny = yx^n - ny_1x^{n-1} + \frac{n(n-1)}{2!}y_2x^{n-2} - \dots,$$

so that, by adding similar terms in the two series, we get

$$\left[\frac{yx^n}{n!}\right] = (n+1)\left\{\frac{yx^n}{1!\,n!} - \frac{y_1x^{n-1}}{2!\,(n-1)!} + \dots + \frac{(-1)^n y_n}{(n+1)!}\right\};$$

that is, the identity also holds when $r = n$.

3. $y_1$ denotes $yx - xy$, which might be written $(yx)$, so $y_2$ denoting $(yx)\,x - x\,(yx)$ may be written $(yxx)$. Similarly $(yxz)$ is taken to denote $(yx)\,z - z\,(yx)$, and so on. From the definition of the symbol,

$$(yx) + (xy) \equiv 0,$$

by expansion we also obtain immediately

$$(xyz)+(yzx)+(zxy) \equiv 0.*$$

The above identity may also be written

$$(yxz) = (yzx)-\{y\,(zx)\},$$

where $\{y\,(zx)\}$ means $y\,(zx)-(zx)\,y$, or

$$(yxz) = (yzx)-(yz_1),$$

$z_1$ denoting $zx-xz$; that is, from the ordinary identity

$$yxz = yzx-yz_1,$$

we deduce the symbolical one

$$(yxz) = (yzx)-(yz_1).$$

It follows in the same way that

$$(yxz_1) = (yz_1x)-(yz_2),$$

and

$$(yxz_{r-1}) = (yz_{r-1}x)-(yz_r),$$

where

$$z_r = z_{r-1}x-xz_{r-1}.$$

Now, by exactly the same reasoning as from the formulæ

$$xy = yx-y_1, \quad xy_1 = y_1x-y_2, \quad xy_{r-1} = y_{r-1}x-y_r,$$

we deduced that

$$yx^r+xyx^{r-1}+\ldots+x^ry \equiv \frac{(r+1)!}{1!\,r!}yx^r-\frac{(r+1)!}{2!\,(r-1)!}y_1x^{r-1}+\ldots+(-1)^r y_r;$$

so from the above symbolical formulæ we deduce that

(B′) $(yzx^{(r)})+(yxzx^{(r-1)})+\ldots+(yx^{(r)}z)$

$$\equiv \frac{(r+1)!}{1!\,r!}(yzx^{(r)})-\frac{(r+1)!}{2!\,(r-1)!}(yz_1x^{(r-1)})+\ldots+(-1)^r(yz_r),$$

where $(yzx^{(r)})$ is a symbol used to denote ($yzx$ ... to $r$ terms).

4. Now

$$\{y\,(u+v)\} = (yu)+(yv),$$

$$\{y\,(u+v)^{(2)}\} = \{y\,(u+v)\}\times(u+v)-(u+v)\times\{y\,(u+v)\}$$

$$= (yu^{(2)})+(yuv)+(yvu)+(yv^{(2)}),$$

* This may easily be deduced as a particular case of the Jacobian identity, and to many of the theorems which follow we have corresponding ones, which are deduced from the Jacobian identity by the same method as the theorems given are deduced from the above formula.

and similarly we see that

$$\{y(u+v)^{(r)}\} = (yu^{(r)}) + (yu^{(r-1)}v) + (yu^{(r-2,}vu) + \ldots + (yv^{(r)}).$$

For the purpose of this paper it will be found necessary to find the coefficient of $\mu$ in $\{y(x+\mu z)^{(r)}\}$, where

$$z = y + a_1 y_1 + a_2 y_2 + \ldots \text{ to infinity};$$

and we see by considering the above that it is

$$(yx^{(r-1)}z) + (yx^{(r-2)}zx) + (yx^{(r-3)}zx^{(2)}) + \ldots,$$

which may also be written

$$\frac{r!}{1!\,(r-1)!}(yzx^{(r-1)}) - \frac{r!}{2!\,(r-2)!}(yz_1x^{(r-2)}) + \frac{r!}{3!\,(r-3)!}(yz_2x^{(r-3)}) - \ldots + (-1)^{r-1}(yz_{r-1}),$$

by (B′).

5. Our object being to express this result in terms of $y$, $y_1$, ... only, we must simplify such an expression as $(yz_px^{(r)})$, or $(yz_p)_r - (z_py)_r$, since

$$(yz_px^{(r)}) = (yz_p)_r - (z_py)_r,$$

where $(uv)_r$ is a symbol used to denote the result of writing $uv$ for $y$ in $y_r$.

It will first be proved that

$$\text{(C)} \quad (uv)_r = u_rv + ru_{r-1}v_1 + \frac{r(r-1)}{2!}u_{r-2}v_2 + \ldots + uv_r.*$$

This theorem is analogous to Leibnitz's, and is proved by the same method, thus

$$(uv)_1 = uvx - xuv = uxv + uv_1 - xuv$$
$$= xuv + u_1v + uv_1 - xuv = u_1v + uv_1.$$

Assuming then that the theorem holds for all values of $r$ up to $n$, it will be proved that it also holds for values up to $n+1$. Since

$$(uv)_n = \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!}u_pv_{n-p},$$

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* For the theorem which bears the same relation to the Jacobian identity which this does to the identity $(xyz) + (yzx) + (zxy) \equiv 0$,
see Lie, II., p. 280.

therefore

$$(uv)_{n+1} = \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!} (u_p v_{n-p})_1$$

$$= \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!} u_{p+1} v_{n-p} + \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!} u_p v_{n-p+1}$$

$$= \sum_{p=0}^{p=n+1} \frac{(n+1)}{p!\,(n+1-p)!} u_p v_{n+1-p};$$

that is, the theorem holds for values of $r$ up to $n+1$, and therefore generally.

6. Using now this result, let us collect all the terms in the series

$$\frac{r!}{1!\,(r-1)!}\{(yz)_{r-1} - (zy)_{r-1}\} - \frac{r!}{2!\,(r-2)!}\{(yz_1)_{r-2} - (z_1 y)_{r-2}\} + \ldots$$

$$\ldots + (-1)^{r-1}\{yz_{r-1} - z_{r-1}y\},$$

which end in $z_q$. We get, by aid of (C),

$$\frac{r!\,q_{r-q-1}\cdot z_q}{(r-q-1)!}\left\{\frac{1}{1!\,q!} - \frac{1}{2!\,(q-1)!} + \ldots + \frac{(-1)^q}{(q+1)!\,0!}\right\}$$

$$= \frac{r!\,y_{r-q-1}z_q}{(q+1)!\,(r-q-1)!};$$

thus the terms in the series which end in $z$, $z_1$, ... or $z_{r-1}$ are given by

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!} y_{r-q-1} z_q$$

(understanding by $z_0$ merely $z$).

Collecting all the terms in the series ending with $y_q$, we get

$$\frac{-r!\,z_{r-q-1}y_q}{q!}\left\{\frac{1}{1!\,(r-q-1)!} - \frac{1}{2!\,(r-q-2)!} + \ldots + \frac{(-1)^{r-q-1}}{(r-q)!\,0!}\right\}$$

$$= \frac{-r!\,z_{r-q-1}y_q}{q!\,(r-q)!},$$

so that the terms in the series which end in $y$, $y_1$, ... or $y_{r-1}$ are given by

$$-\sum_{q=0}^{q=r-1} \frac{r!\,z_{r-q-1}y_q}{q!\,(r-q)!},$$

which may be written (as we see by writing $r-q'-1$ for $q$)

$$-\sum_{q=0}^{q=r-1} \frac{r!\,z_q y_{r-q-1}}{(q+1)!\,(r-q-1)!}.$$

Thus the total series required may be written in the simpler form

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!}\,(y_{r-q-1}z_q).$$

7. Now $z = y + a_1y_1 + a_2y_2 + \ldots$ to infinity;

therefore $z_q = y_q + a_1y_{q+1} + a_2y_{q+2} + \ldots$ to infinity;

therefore $(y_{r-q-1}z_q) = y_{r-q-1}\,(y_q + a_1y_{q-1} + a_2y_{q+2} + \ldots$ to infinity$)$
$$-(y_q + a_1y_{q+1} + a_2y_{q+2} + \ldots \text{ to infinity})\,y_{r-q-1}.$$

We thus see that, $(m, n)$ denoting $y_my_n - y_ny_m$, the series may be written

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!}\,\{(r-q-1,\,q) + a_1\,(r-q-1,\,q+1) + a_2\,(r-q-1,\,q+2) + \ldots \text{ to infinity}\},$$

or in the simpler form

$$\sum_{q=0}^{q=r-1}\sum_{\kappa=0}^{\kappa=\infty} \frac{r!}{(q+1)!\,(r-q-1)!}\,a_\kappa\,(r-q-1,\,q+\kappa);$$

(D) or
$$\sum_{m=0}^{m=r-1}\sum_{n=r-1-m}^{n=\infty} a_{m+n-r+1}\frac{r!}{m!\,(n-m)!}\,(m, n),$$

if we define $a_0$ as unity.

8. We have thus found the coefficient of $\mu$ in the result obtained by substituting for $x$, $x+\mu z$ in any term $y_r$. It will be necessary to find the corresponding coefficient obtained by making the same substitution in any terms such as $(p, q)$ or $(p, q, r)\ldots$. To find this consider a particular term $(p, q, r, s)$, though the reasoning is general.

First, it is clear that the coefficient will be the sum of the coefficients obtained by substituting for each element $y_r$ of the term the series
$$\sum_{m=0}^{m=r-1}\sum_{n=r-1-m}^{n=\infty} a_{m+n-r+1}\frac{r!}{m!\,(r-m)}\,(m, n).$$

Take a particular term of this series $(m, n)$; then it contributes

$$\frac{r!\,a_{m+n-r+1}}{m!\,(r-m)!}\,\{p, q\,(m, n)\,s\},$$

where the part $(m, n)$ is to be taken as one symbol in expanding the complex symbol $\{p, q\,(m, n)\,s\}$.

Now we have seen that

$$(q, m, n)-(q, n, m) \equiv \{q\,(m, n)\};$$

and it is easily seen to follow that

$$\{p, q\,(m, n)\,s\} \equiv (p, q, m, n, s)-(p, q, n, m, s),$$

so that the coefficient derived from $(p, q, r, s)$ will consist of five-lettered symbols.

9. We now know how to obtain the coefficients derived from

$$y_1 y_2 \ldots, \quad (y_p y_q) \ldots, \quad (y_p y_q y_r) \ldots,$$

and must next obtain the coefficient derived from $z$, which might be called the first derived coefficient of $z$; from this we similarly derive a coefficient which might be called the second derived coefficient of $z$, and so on.

10. $$z = y + a_1 y_1 + a_2 y_2 + \ldots \text{ to infinity,}$$

so that the coefficient derived from it will be

$$\Sigma b_{pq}\,(p, q),$$

where $p$ and $q$ may have any positive or zero integral values, and $b_{pq}$ denotes $\sum_{r=p+1}^{r=p+q+1} a_r a_{p+q+1-r} \dfrac{r!}{p!\,(r-p)!}$.

11. What will now be the coefficient which is similarly derived from

$$\Sigma b_{pq}\,(p, q)\ ?$$

From what we have seen it will be of the form

$$\Sigma b_{pqr}\,(p, q, r),$$

where the summation is taken for all positive and zero integral values of $p$, $q$, $r$. Similarly the coefficient derived from this will be of the form

$$\Sigma b_{pqrs}\,(p, q, r, s),$$

where the summation is taken for all positive and zero integral values of $p$, $q$, $r$, $s$; and so generally. We have, however, to show how to calculate the numerical constants $b_{pqr}$, $b_{pqrs}$, ....

12. We saw that the term $(p, q, r, s, t)$ was made up by contributions from the terms $(x, r, s, t)$, $(p, x, s, t)$, $(p, q, x, t)$, and $(p, q, r, x)$.

Take $(x, r, s, t)$ : the term $(p, q, r, s, t)$ was derived from this by

$$a_{p+q-x+1} \frac{x!}{p!\,(x-p)!} (p, q)$$

replacing $x$ in $b_{xrst}(x, r, s, t)$, and by

$$a_{q+p-x+1} \frac{x!}{q!\,(x-q)!} (q, p)$$

replacing $x$ in $b_{xrst}(x, r, s, t)$.

The first of these, then, contributes to the coefficient $b_{pqrst}$ the series

$$\sum_{x=p+1}^{x=p+q+1} a_{p+q-x+1} \frac{x!}{p!\,(x-p)!} b_{xrst};$$

and the second contributes

$$-\sum_{x=q+1}^{x=p+q+1} a_{p+q-x+1} \frac{x!}{q!\,(x-q)!} b_{xrst};$$

or, if we take $x^{(p)}$ as usual to denote $x(x-1) \ldots (x-p+1)$, and remember that $x^{(p)}$ is zero if $x<p$, the two together contribute to $b_{pqrst}$ the series

$$\sum_{x=0}^{x=p+q+1} \left(\frac{x^{(p)}}{p!} - \frac{x^{(q)}}{q!}\right) a_{p+q+1-x} b_{xrst} + a_{p+1} b_{qrst} - a_{q+1} b_{prst}.$$

Similarly, by considering the series which the other terms contribute, we see that

$$\begin{aligned} b_{pqrst} = &\sum_{x=0}^{x=p+q+1} \left(\frac{x^{(p)}}{p!} - \frac{x^{(q)}}{q!}\right) a_{p+q+1-x} b_{xrst} + \sum_{x=0}^{x=q+r+1} \left(\frac{x^{(q)}}{q!} - \frac{x^{(r)}}{r!}\right) a_{q+r+1-x} b_{pxst} \\ &+ \sum_{x=0}^{x=r+s+1} \left(\frac{x^{(r)}}{r!} - \frac{x^{(s)}}{s!}\right) a_{r+s+1-x} b_{pqxt} + \sum_{x=0}^{x=s+t+1} \left(\frac{x^{(s)}}{s!} - \frac{x^{(t)}}{t!}\right) a_{s+t+1-x} b_{pqrx} \\ &+ a_{p+1} b_{qrst} - a_{t+1} b_{pqrs}. \end{aligned}$$

The coefficient of any symbol of $n+1$ letters is deduced by a similar rule from the coefficients of the symbols of $n$ letters, obtained by crossing out two successive letters in the given symbol of $(n+1)$ letters.

13. We now proceed to obtain the theorem $e^y e^x = e^w$. In the paper previously referred to (*Proceedings*, Vol. XXVIII., p. 381) it was proved that, if $\mu$ is a constant, $(1+\mu y)\, e^x$ is equal to $e^{x+\mu z}$ + terms involving powers of $\mu$ of the second and higher orders.

We might express this in the form of the equation

$$(1+\mu y)\, e^x = e^{x+\mu z} + \mu^2 R,$$

where $R$ is some operator formed by combinations of $x$ and $y$.

Let the result of substituting $x+\mu z$ for $x$ in $z$, when $z$ is expressed explicitly in terms of $x$ and $y$, be called $P_1$; and let $P_2$ be obtained from $P_1$ by expressing $P_1$ explicitly in terms of $x$ and $y$, and substituting $x+\mu z$ for $x$ in it, and let $P_3$ be similarly obtained from $P_2$, $P_4$ from $P_3$, and so on.

Defining $x_1$ as $x+\mu z$, $x_2$ as $x_1+\mu P_1$, $x_n$ as $x_{n-1}+\mu P_{n-1}$, we see that the result of substituting $x+\mu z$ for $x$ in $x_1$ (when explicitly expressed in terms of $x$ and $y$) is $x_2$; and generally that the result of substituting $x+\mu z$ for $x$ in $x_{n-1}$ (when explicitly expressed in terms of $x$ and $y$) is $x_n$. We have therefore

$$\begin{aligned}
(1+\mu y)\, e^{x} &= e^{x_1} + \mu^2 R, \\
(1+\mu y)\, e^{x_1} &= e^{x} + \mu^2 R_1, \\
\cdots \quad & \cdots \quad \cdots \quad \cdots \\
(1+\mu y)\, e^{x_{n-1}} &= e^{x_n} + \mu^2 R_{n-1},
\end{aligned}$$

where $R_1$, $R_2$, ... $R_{n-1}$ mean the result of substituting for $x$, $x_1$, $x_2$, ... $x_{n-1}$ respectively in $R$.

Multiplying the first of the above equations by $(1+\mu y)^{n-1}$, the second by $(1+\mu y)^{n-2}$, and so on, and adding, we obtain

$$(1+\mu y)^n\, e^x = e^{x_n} + \mu^2 \left\{(1+\mu y)^{n-1} R + (1+\mu y)^{n+2} R_1 + \ldots + R_{n-1}\right\}.$$

Let us now take $\mu n = 1$, and let $n$ increase indefinitely, then, as in ordinary algebra,

$$(1+\mu y)^n = e^y.$$

The subject on which the operators $x$, $y$ are to act will be supposed such that $R$ operating on it gives a finite result, and the like will be supposed to hold for each of the operators $R_1$, $R_2$, ... . It will follow that the effects of $(1+\mu y)^{n-1} R$, $(1+\mu y)^{n-2} R_1$, ... will also be severally finite, and if $A$ is the greatest effect any one of them can have, then the greatest effect of

$$\mu^2 \left\{(1+\mu y)^{n-1} R + (1+\mu y)^{n-2} R_1 + \ldots + R_{n-1}\right\}$$

will not exceed $\frac{1}{n^2} nA$, that is, their effect will ultimately (as $n$ increases) be zero.

We have thus obtained the equation

$$e^y e^x = e^{x_\infty},$$

where $x_\infty$ denotes the limit to which $x_n$ tends as $n$ increases indefinitely.

14. Now $$x_n = x + \mu z + \mu\,(P_1 + P_2 + \ldots + P_{n-1}),$$

so that to obtain the form of $x_n$ as $n$ approaches infinity it will be necessary to evaluate $P_1, P_2, \ldots P_{n-1}$.

$P_1$ being the result of substituting $x + \mu z$ for $x$ in $z$, we have

$$P_1 = z + \mu z_1 + \mu^2 z_2 + \ldots + \mu^r z_r + \ldots \text{ to infinity.}*$$

So that we obtain

$$\begin{aligned} P_2 = P_1 &+ \mu\, z_1 + \mu^2 \;\; z_{11} + \mu^3 \;\; z_{12} + \ldots \\ &+ \mu^2 z_2 + \mu^3 \;\; z_{21} + \mu^4 \;\; z_{22} + \ldots \\ &+ \ldots \quad \ldots \quad \ldots \quad \ldots \\ &+ \mu^r z_r + \mu^{r+1} z_{r1} + \mu^{r+2} z_{r2} + \ldots \\ &+ \ldots \quad \ldots \quad \ldots \quad \ldots .\dagger \end{aligned}$$

We notice that in $P_2$ the coefficient of $\mu^r$ is

$$2z_r + z_{1,r-1} + z_{2,r-2} + \ldots + z_{r-1,1},$$

that is, the $z$'s which have the single suffix are counted twice, those which have the double suffix are counted once.

Expressing $P_3$ similarly, we notice that the coefficient of $\mu^r$ is

$$3z_r + 3\,(z_{1,r-1} + z_{2,r-} + \ldots + z_{r-1,1}) + \Sigma z_{p_1 \ldots},$$

where the summation is taken for all positive integral values (excluding zeros) of $p_1, p_2, p_3$ which make

$$p_1 + p_2 + p_3 = r.$$

Let $\Sigma z_{p_1 p_2 \ldots p_\lambda}$, where the summation is taken for all positive integral values of $p_1, p_2, \ldots p_\lambda$ which make

$$p_1 + p_2 + \ldots + p_\lambda = r,$$

be written $\overset{\lambda\,.\,r}{C}$.

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* The notation $z_r$ is here employed for temporary convenience to denote the coeffieient of $\mu^r$, and has no connexion with the notation $z_r$ meaning $z_{r-1}x - xz_{r-1}$.

† Here $z_{pq}$ is taken to denote the coefficient $\mu^q$ when $x + \mu z$ is substituted for $x$ in $z_p$ (explicitly expressed in terms of $x$ and $y$), and the result expanded in powers of $\mu$.

Then it is suggested by the above two particular cases that the coefficient of $\mu^r$ in $P_n$ is

$$nz_r + \frac{n(n-1)}{2!}\overset{2.r}{C} + \ldots \frac{n!}{\lambda!\,(n-\lambda)!}\overset{\lambda.r}{C} + \ldots + \overset{n.r}{C}.$$

The summation is for $n$ terms, but if $r<n$ it need not be taken to more than $r$ terms, for from the definition

$$\overset{r+\kappa.r}{C} \equiv 0.$$

Assuming that this law holds for $P_n$ (we have seen its truth when $n = 1$, 2, or 3), we shall prove that it holds also for $P_{n+1}$.

15. We can write $P_n$ in the following form :—

$$P_n = z + \mu n z_1 + \mu^2 \left\{ n z_2 + \frac{n(n-1)}{2!} z_{11} \right\}$$
$$+ \mu^3 \left\{ n z_3 + \frac{n(n-1)}{2!}(z_{12}+z_{21}) + \frac{n(n-1)(n-2)}{3!} z_{111} \right\}$$
$$+ \ldots .$$

Consider now how the separate terms of this series contribute to the generation of the coefficient of $\mu^r$ in $P_{n+1}$. A term such as $\mu^\kappa z_{p_1 p_2 \ldots p_m}$ becomes, in generating $P_{n+1}$,

$$\mu^\kappa z_{p_1 p_2 \ldots p_m} + \mu^{\kappa+1} z_{p_1 p_2 \ldots p_m 1} + \mu^{\kappa+2} z_{p_1 p_2 \ldots p_m 2} + \ldots \mu^{\kappa+s} z_{p_1 p_2 \ldots p_m s} + \ldots .$$

If, then, we use the symbol $\underset{s}{\overset{\lambda.m}{C}}$ to denote the result of changing every term such as $z_{p_1 p_2 \ldots p_m}$ in $\overset{\lambda.m}{C}$ into $z_{p_1 p_2 \ldots p_m s}$, we see that to the coefficient of $\mu^r$ in $P_{n+1}$,

$z$ has contributed $z_r$,

$\mu n z_1$ ,, ,, $n \underset{r-1}{\overset{1.1}{C}}$,

$\mu^2 \left\{ n z_2 + \frac{n(n-1)}{2!} z_{11} \right\}$ ,, ,, $n \underset{r-2}{\overset{1.2}{C}} + \frac{n(n-1)}{2!} \underset{r-2}{\overset{2.2}{C}}$,

the $\kappa^{\text{th}}$ term ,, ,, $n \underset{r-\kappa+1}{\overset{1.\kappa-1}{C}} + \frac{n(n-1)}{2!} \underset{r-\kappa+1}{\overset{2.\kappa-1}{C}} + \ldots$,

the summation going till either the numerical coefficient or $\underset{r-\kappa+1}{\overset{\lambda.\kappa-1}{C}}$ vanishes; that is, either for $n$ terms or $\kappa-1$ terms, whichever may be the lesser number.

The limiting value of $\kappa$ will be $r+1$, if we understand by ${}^{p.q}_{0}C$ merely ${}^{p.q}C$.

Hence the coefficient of $\mu^r$ in $P_{n+1}$ is

$$z_r + n\left({}^{1.1}_{r-1}C + {}^{1.2}_{r-2}C + \ldots + {}^{1.r-1}_{1}C\right) + \frac{n(n-1)}{2!}\left({}^{2.2}_{r-2}C + {}^{2.3}_{r-3}C + \ldots + {}^{2.r-1}_{1}C\right) + \ldots$$

$$\ldots + \frac{n!}{\lambda!\,(n-\lambda)}\left({}^{\lambda.\lambda}_{r-\lambda}C + {}^{\lambda.\lambda+1}_{r-\lambda-1}C + \ldots + {}^{\lambda.r-1}_{1}C\right) + \ldots$$

(the limit of $\lambda$ being $n$ or $r-1$, whichever may be the lower integer),

$$+ n\,{}^{1.r}_{0}C + \frac{n(n-1)}{2!}\,{}^{2.r}_{0}C + \ldots + \frac{n!}{\lambda!\,(n-\lambda)}\,{}^{\lambda.r}_{0}C + \ldots,$$

(the limit of $\lambda$ being $n$ or $r$, whichever may be the lower integer).

Now it is at once seen from the definition of the symbol ${}^{\lambda.r}C$ that

$${}^{\lambda.r}C = {}^{\lambda-1.r-1}_{1}C + {}^{\lambda-1.r-2}_{2}C + \ldots + {}^{\lambda-1.\lambda-1}_{r-\lambda+1}C;$$

therefore

$${}^{2.r}C = {}^{1.1}_{r-1}C + {}^{1.2}_{r-2}C + \ldots + {}^{1.r-1}_{1}C,$$

$${}^{3.r}C = {}^{2.2}_{r-2}C + {}^{2.3}_{r-3}C + \ldots + {}^{2.r-1}_{1}C,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$${}^{\lambda+1.r}C = {}^{\lambda.\lambda}_{r-\lambda}C + {}^{\lambda.\lambda+1}_{r-\lambda+1}C + \ldots + {}^{\lambda.r-1}_{1}C;$$

the coefficient of $\mu^r$ in $P_{n+1}$ is therefore

$$z_r + n\,{}^{2.r}C + \frac{n(n-1)}{2!}\,{}^{3.r}C + \ldots + \frac{n!}{(\lambda-1)!\,(n+1-\lambda)}\,{}^{\lambda.r}C + \ldots$$

(the limit of $\lambda$ being the smaller of the two, $n+1$ and $r$)

$$+ n\,{}^{r}C + \frac{n(n-1)}{2!}\,{}^{2.r}C + \frac{n(n-1)(n-2)}{3!}\,{}^{3.r}C + \ldots \frac{n!}{\lambda!\,(n-\lambda)}\,{}^{\lambda.r}C + \ldots$$

(the limit of $\lambda$ being the smaller of the two, $n$ and $r$)

$$= (n+1)\,z_r + \frac{(n+1)\,n}{2!}\,{}^{2.r}C + \ldots + \frac{(n+1)!}{\lambda!\,(n+1-\lambda)!}\,{}^{\lambda.r}C + \ldots$$

(the limit of $\lambda$ being the smaller of the two, $n+1$ and $r$);

that is, the law holds also for $P_{n+1}$, and therefore universally.

16. We have thus found the form of $P_n$. Our object is, however, to find the form of $x_n$; this may easily be deduced. First, let us find the coefficient of $\mu^r$ $(r>0)$ in $P_1+P_2+\ldots+P_{n-1}$; this is made up of contributions

from $P_1$, $\quad z_r$,

„ $P_2$, $\quad 2z_r+\overset{2.r}{C}$,

„ $P_3$, $\quad 3z_r+3\overset{2.r}{C}+\overset{3.r}{C}$,

„ $P_{n-1}$, $\quad (n-1)\,z_r+\dfrac{(n-1)(n-2)}{2!}\overset{2.r}{C}+\ldots+\dfrac{(n-1)!}{\lambda!\,(n-1-\lambda)}\overset{\lambda.r}{C}+\ldots$

(the limit of $\lambda$ being the smaller of the two, $n-1$ and $r$).

Adding, we see that the coefficient of $\mu^r$ is

$$\frac{n\,(n-1)}{2!}z_r+\frac{n\,(n-1)(n-2)}{3!}\overset{2.r}{C}+\ldots+\frac{n!}{(\lambda+1)!\,(n-\lambda-1)!}\overset{\lambda.r}{C}+\ldots$$

(the limit of $\lambda$ being the smaller of the two, $n-1$ and $r$).

Call this series $D_{r+1}$.

Now the term independent of $\mu$ in $P_1+P_2+\ldots+P_{n-1}$ is obviously $(n-1)\,z$; therefore

$$\begin{aligned} x_n &= x+\mu z+\mu\,(P_1+P_2+\ldots+P_{n-1}) \\ &= x+n\mu z+\mu^2D_2+\mu^3D_3+\ldots \text{ to infinity.} \end{aligned}$$

17. What, then, does this approach when we take $\mu n=1$, and let $n$ increase indefinitely?

The only coefficient in the series $\mu^rD_r$ which does not vanish when $\mu=\dfrac{1}{n}$ and $n$ is indefinitely increased, is the coefficient of $\overset{r-1.r-1}{C}$; for

$$\frac{n!\left(\frac{1}{n}\right)^r}{(\lambda+1)!\,(n-\lambda-1)!}=\frac{n\,(n-1)\ldots(n-\lambda)}{n^r\,(\lambda+1)!},$$

and

$$r-1\geqq\lambda;$$

therefore the expression vanishes when $n$ increases indefinitely, unless

$$r=\lambda+1,$$

in which case $\mu^rD_r$ becomes $\dfrac{\overset{r-1.r-1}{C}}{r!}$.

The limiting value of $x_n$, as $n$ approaches infinity, is therefore

$$x+z+\frac{\overset{1.1}{C}}{2!}+\frac{\overset{2.2}{C}}{3!}+\frac{\overset{3.3}{C}}{4!}+\ldots \text{ to infinity.}$$

But the only term in $\overset{1.1}{C}$ is $z_1$, the only term in $\overset{2.2}{C}$ is $z_{11}$, the only term in $\overset{r.r}{C}$ is $z_{1,1,\ldots}$ to $r$ suffixes.

Now $z_1$ denotes the coefficient of $\mu$ in the result of substituting $x+\mu z$ for $x$ in $z$, and we have seen that it is equal to $\Sigma b_{pq}(p, q)$.

$z_{11}$, then, denoting the coefficient of $\mu$ when for $x$ we write $x+\mu z$ in $\Sigma b_{pq}(p, q)$, has been shown to be $\Sigma b_{pqr}(p, q, r)$, and so on; so that we see finally that

$$x_\infty = x+y+a_1y_1+a_2y_2+\ldots+\frac{1}{2!}\Sigma b_{bq}(p, q)+\frac{1}{3!}\Sigma b_{pqr}(p, q, r)+\ldots,$$

and

$$e^y e^x = e^{x_\infty}.$$

It will be noticed that the terms in $b_{pq}$ are of the second degree in the set $a_1a_2a_3\ldots$, and of weight $p+q+1$ in the suffixes of that set; and we easily see the more general fact that the weight of every term is constant; *e.g.*, $b_{pqrst}$ has weight

$$p+q+r+s+t+4;$$

thus the functions are isobaric in the set $a_1, a_2, a_3, \ldots$.

Also we see that the number of letters in a symbol $(pqrst)$ expresses the degree of the symbol in $y$, and $p+q+r+s+t$ expresses the degree in $x$.

18. We have seen how to express $e^y e^x$ in the form $e^w$, and have obtained an expression for $w$, but for development of the result it will be found useful to change the form in which $w$ is expressed.

$y_p$ means the same as $(yx^{(p)})$; it will be advantageous to express $(p, q)$ also in the form of a sum of terms $(yx^{(m)}\ldots yx^{(n)})$, and similarly to express $(p, q, r)\ldots$.

We can easily see that

$$-(y_py_q) = (y_{p+1}y_{q-1})-(y_py_{q-1}x),$$

and, by applying the same transformation a second time, we see that

$$(y_py_q) = (y_{p+2}y_{q-2})-2(y_{p+1}y_{q-2}x)+(y_py_{q-2}xx).$$

Assume, then,

$$(-1)^r (y_p y_q)$$

$$= (y_{p+r} y_{q-r}) - r (y_{p+r-1} y_{q-2} x) + \frac{r(r-1)}{2!} (y_{p+r-2} y_{q-r} x^{(2)}) - \dots .$$

We shall prove that the expression on the right is unaltered in value when for $r$ we write $r+1$; that is, if the identity holds for the value $r$, it also holds for the value $r+1$, and, holding for $r=1$, it will then be seen to hold generally.

Since $-(y_p y_q) = (y_{p+1} y_{q-1}) - (y_p y_{q-1} x)$,

we see that $-(y_p y_q x^{(r)}) = (y_{p+1} y_{q-1} x^{(r)}) - (y_p y_{q-1} x^{(r+1)})$;

therefore $-(y_{p+r} y_{q-r}) = (y_{p+r+1} y_{q-r-1}) - (y_{p+r} y_{q-r-1} x)$

$$+r (y_{p+r-1} y_{q-r} x) = -r (y_{p+r} y_{q-r-1} x) + r (y_{p+r-1} y_{q-r-1} x^{(2)})$$

$$-\frac{r(r-1)}{2!} (y_{p+r-2} y_{q-r} x^{(2)}) = \frac{r(r-1)}{2!} (y_{p+r-1} y_{q-r-1} x^{(2)})$$

$$-\frac{r(r-1)}{2!} (y_{p+r-2} y_{q-r-1} x^{(3)})$$

$$+ \quad \dots \quad \dots \quad \dots \quad = \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Adding, we see that the sum of the members on the left, which we know, is

$$(-1)^{r+1} (y_p y_q)$$

$$= (y_{p+r+1} y_{q-r-1}) - (r+1)(y_{p+r} y_{q-r-1} x) + \frac{(r+1) r}{2!} (y_{p+r-1} y_{q-r-1} x^{(2)}) - \dots,$$

and this proves the theorem required.

19. We can therefore write $(-1)^q (y_p y_q)$ in the form

$$(y_{p+q} y) - q (y_{p+q-1} yx) + \frac{q(q-1)}{2!} (y_{p+q-2} yx^{(2)}) - \dots$$

$$= (yx^{(p+q)} y) - q (yx^{(p+q-1)} yx) + \frac{q(q-1)}{2!} (yx^{(p+q-2)} yx^{(2)}) - \dots,$$

the general term being

$$(-1)^s \frac{q!}{s!\,(q-s)!} (yx^{(p+q-s)} yx^{(s)}),$$

where $s$ is any integer $\leq q$.

By taking the separate terms of this series we see in the same way that the general term in $(y_p y_q y_r)$ is

$$\frac{(-1)^{s+t+q+r} q!\, r!}{s!\, t!\, (q-s)!\, (r-t)!} (yx^{(p+q-s)} yx^{(r+s-t)} yx^{(t)}),$$

and the summation is to be taken for all integral values

$$s \leqq q \quad \text{and} \quad t \leqq r.$$

Any other term in $w$ may be similarly expressed.

If, now, instead of $y$ we write $x_1$, and instead of $x$ write $x_2$, and if we denote such a term as $(x_1 x_2^{(p)} x_1^{(q)} x_2^{(r)})$ by $(12^{(p)}1^{(q)}2^{(r)})$, we see that

$$e^{x_1} e^{x_2} = e^{w},$$

where $$w = x_1 + x_2 + \beta_{12}(12) + \beta_{122}(122) + \beta_{121}(121) + \ldots,$$

where the $\beta$'s are a new set of numerical constants.

20. How are these constants to be determined? By expressing the old symbols $(y_p y_q y_r)$ in terms of the new; *e.g.*, because

$$\Sigma b_{pqr}(y_p y_q y_r) = \Sigma (-1)^{s+t+q+r} b_{pqr} \frac{q!\, r!\, (12^{(p+q-s)} 12^{(r+s-t)} 12^{(t)})}{s!\, t!\, (q-s)!\, (r-t)!},$$

writing $$p+q-s = \lambda, \quad r+s-t = \mu, \quad t = \nu,$$

we see that

$$\beta_{1\,2^{(\lambda)}1\,2^{(\mu)}1\,2^{(\nu)}} = \Sigma (-1)^{\mu+q} \frac{b_{pqr}(\lambda+\mu+\nu-q)!\, r!}{(\lambda-p)!\,(r-\nu)!\,(\mu+\nu-r)!\,\nu!},$$

where the summation on the right is to be taken for all possible (positive integral or zero) values of $p$, $q$, and $r$ which make no factorial $(\lambda-p)!$, &c., negative.

21. Let $x_1, x_2, \ldots x_n$ be $n$ operators which obey the distributive and associative laws, but not the commutative.

$(1, 2)$ will be used to denote $x_1 x_2 - x_2 x_1$,

$(1, 2, 3)$ ,, ,, $(1, 2)\, x_3 - x_3 (1, 2)$,

and so generally. We thus get a series of symbols; $X_1$ will be used to denote any symbol of the first order, *i.e.*, $x_1, x_2, \ldots$ or $x_n$, $X_2$ any symbol of the form $(p, q)$, and so on.

Suppose that from these symbols $X_1, X_2, \ldots$ we were to try and form others, such as

$$(X_p X_q) = X_p X_q - X_q X_p;$$

it will be proved that we get no new symbols, but merely a linear combination of symbols of the form $X_{p+q}$, so that the symbols themselves may be said to form a linear group.

If one of the suffixes $p$ or $q$ in $(X_p X_q)$ is unity, this is evident, for $X_p x_r - x_r X_p$ is by definition a symbol of form $X_{p+1}$.

Suppose, now, that we have proved that, if one of the suffixes $p$ or $q$ does not exceed $r$, $(X_p X_q)$ can be expressed as a linear combination of symbols of the form $X_{p+q}$; then it will be shown that the same result holds if one of the suffixes does not exceed $r+1$; and the theorem, having been proved when $r = 1$, will then hold generally.

By definition $\quad X_{r+1} \equiv X_r x_s - x_s X_r;$

therefore

$$\begin{aligned} X_n X_{r+1} - X_{r+1} X_n &\equiv X_n (X_r x_s - x_s X_r) - (X_r x_s - x_s X_r) X_n \\ &\equiv (X_n X_r - X_r X_n) x_s - x_s (X_n X_r - X_r X_n) \\ &\quad + X_r (X_n x_s - x_s X_n) - (X_n x_s - x_s X_n) X_r. \end{aligned}$$

Now, by hypothesis, $X_n X_r - X_r X_n$ is expressible in terms of symbols of the form $X_{n+r}$; therefore

$$(X_n X_r - X_r X_n) x_s - x_s (X_n X_r - X_r X_n)$$

is expressible as a linear combination of symbols of the form $X_{n+r+1}$.

Again, $\quad X_r (X_n x_s - x_s X_n) - (X_n x_s - x_s X_n) X_r$

is of the form $X_r X_{n+1} - X_{n+1} X_r$; and therefore (since one suffix does not exceed $r$) in terms of symbols of the form $X_{n-r+1}$; that is, $(X_n X_{r+1})$ is expressible as a linear combination of symbols of the form $X_{n+r+1}$.

22. There are $n$ independent symbols of the first order; there are $n^2$ symbols of the second order, but only $\frac{n(n-1)}{2}$ of these will be independent, owing to the identities

$$(p, q) + (q, p) \equiv 0, \quad (p, p) \equiv 0.$$

The $n^3$ symbols of the third order will be still further reduced by reason of the additional identity

$$(p, q, r) + (q, r, p) + (r, p, q) \equiv 0;$$

and there are doubtless additional identities for symbols of the fourth and higher orders. Without entering on the question of the number of independent symbols of any given order, we shall suppose that a system of independent symbols has been obtained.

23. If, now,
$$x_1' = a_{11}x_1 + a_{21}x_2 + \ldots + a_{n1}x_n,$$
$$\vdots$$
$$x_n' = a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{nn}x_n,$$

where the set of letters $a_{p,q}$ denote any arbitrary constants, we can express the symbols $X_1'X_2' \ldots$, which correspond to $X_1 X_2 \ldots$, as linear combinations of the latter, the coefficients in the identities being functions of the set of constants $a_{pq}$.

In particular, if
$$x_1' = \lambda_1 x_1 + \ldots + \lambda_n x_n,$$
$$x_2' = \mu_1 x_1 + \ldots + \mu_n x_n,$$

we can express any such symbol as $(1, 2, 2)'$ in terms of the complete set of independent symbols of the third order, and the parameters
$$\lambda_1 \ldots \lambda_n, \quad \mu_1 \ldots \mu_n;$$
but we have proved that $e^{x_1'} e^{x_2'} = e^{w}$,
where $w = x_1' + x_2' + \beta_{12}(12)' + \beta_{122}(122)' + \beta_{121}(121)'$,
and the $\beta$'s are numerical constants whose values we know how to determine; therefore
$$e^{\lambda_1 x_1 + \ldots + \lambda_n x_n} e^{\mu_1 x_1 + \ldots + \mu_n x_n} = e^{\Sigma a_p x_p + \Sigma a_{pq}(p, q) + \Sigma a_{pqr}(p, q, r) + \ldots},$$

where the $a$'s are now functions of the parameters $\lambda_1 \ldots \lambda_n$, $\mu_1 \ldots \mu_n$, and known numerical constants; thus $a_{p,q,r}$ is the numerical constant $\beta_{122} \times$ coefficient of the term $(p, q, r)$ which occurs in expressing $(122)'$ in terms of symbols of the third order derived from $x_1 \ldots x_n + \beta_{121} \times$ coefficient similarly obtained from $(121)' + \ldots$.

$a_{p,q,r}$ is clearly of the third degree in $\lambda_1 \ldots \lambda_n$, $\mu_1 \ldots \mu_n$, and so generally.

24. The more general theorem which has been enunciated is seen to be implicitly proved in what has gone before; for we have proved that
$$e^{w_1} e^{w_2} = e^{w_3},$$
where
$$w_3 = w_1 + w_2 + a_1(w_1 w_2) + a_2(w_1 w_2 w_2) + \ldots$$
$$+ \tfrac{1}{2}\Sigma b_{pq}\left\{(w_1 w_2^{(p)})(w_1 w_2^{(q)})\right\} + \ldots,$$

and we see that every term on the right is of the form
$$c_1 x_1 + \ldots + c_n x_n + c_{pq}(p, q) + \ldots + c_{pqr}(p, q, r) + \ldots.$$

## *A Continuation of Gauss's "Dioptrische Untersuchungen."*

*By* Professor R. A. Sampson. Read April 8th, 1897. Received, in recast form, September 18th, 1897.

Of all methods by which we can discuss the most general case of repeated asymmetrical refraction of a beam of light, that which Gauss invented and applied to the symmetrical instrument appears to be most appropriate; for it rests upon the correspondence of ray to ray—the simplest correspondence which no refraction can disturb. The deeper physical significance of the method of the characteristic function appears to have diverted attention from what may be done by direct consideration of the rays.

How complete a solution Gauss's method can furnish will be seen below, where a simple and elegant algebraical correspondence expresses the limits of the modification which any narrow beam can experience in passing in any manner through any singly refracting media which vary continuously or discontinuously.

In this scheme sixteen coefficients express the correspondence of ray to ray: six of these are absorbed in satisfaction of six identical relations; four more relate to an irrelevant choice of axes, and six constants are left to express the properties peculiar to any, the most general, given optical system. To exhibit these constants free from irrelevant quantities and identical relations requires a somewhat elaborate theory; which, however, yields an immediate test of the equivalence of two given systems, and, among other things, proves that direct incidence on a thick lens whose curvatures are unequal in each face and in different planes in the two faces is equivalent to any, the most general, given system.

The method of the characteristic function has been made by Larmor to yield the same result—in fact, to furnish an implicit, though unelaborated, solution of the whole problem (*Proceedings Lond. Math. Soc.*, Vol. xx., p. 192, and Vol. xxiii., p. 172). But a question of so much generality and importance will justify a second, radically different, discussion; and I trust that the new results of the following pages, which include an account of the geometrical properties and a sketch classification of the most general systems, will show that Gauss's method, applied freely and according to the spirit of it, is a capable rival of even so powerful, elegant, and well tried a method as the characteristic function.

## I.

Let a narrow beam of light be composed of discrete rays, not related to one another in any manner, except that the squares of their mutual inclinations are negligible. Let this beam be refracted at any surfaces of discontinuity, arbitrary in number, in situation, and in curvatures. Let an arbitrary ray be called the axis of the beam, and at each surface let the origin of coordinates be placed where this ray impinges on the surface, with the axis of $x$ along the normal and the axis of $y$ along the tangent line which is perpendicular to the axis of the beam. Then the equation of the surface may be written

$$2x = A_0y^2+2H_0yz+B_0z^2,$$

and those of the ray,

before refraction :

$$\frac{x}{\cos\theta} = \frac{y}{0} = \frac{z}{\sin\theta},$$

after refraction :

$$\frac{x}{\cos\theta'} = \frac{y}{0} = \frac{z}{\sin\theta'},$$

where $$\mu\sin\theta-\mu'\sin\theta' = 0.$$

If we write the equations of any other ray of the beam,

before refraction : $$\frac{x}{\cos(\theta+\delta)} = \frac{y-b_0}{\epsilon} = \frac{z-c_0}{\sin(\theta+\delta)},$$

after refraction : $$\frac{x}{\cos(\theta'+\delta')} = \frac{y-b_0}{\epsilon'} = \frac{z-c_0}{\sin(\theta'+\delta')},$$

where the quantities $b_0$, $c_0$, $\delta$, $\epsilon$, $\delta'$, $\epsilon'$ are all small, we may derive the relations between these quantities from the formula

$$\mu l-\mu'l' = (\mu\cos\Theta-\mu'\cos\Theta')\,p,$$

where $l$, $l'$, $p$ are direction cosines with respect to any axis of original and refracted ray and normal to the surface respectively, and $\Theta$, $\Theta'$ are the inclinations of the rays to the normal. We get thus

$$\mu\epsilon-\mu'\epsilon' = (A_0b_0+H_0c_0)(\mu\cos\theta-\mu'\cos\theta'),$$
$$\mu\cos\theta\,\delta-\mu'\cos\theta'\,\delta' = (H_0b_0+B_0c_0)(\mu\cos\theta-\mu'\cos\theta').$$

Now refer the equations of the rays to new systems of coordinates in which their own axes are the axes of $x$. Rotate the axes of $x$, $z$

about that of $y$ through an angle $-\theta$; the equation of the original ray becomes

$$\frac{x-c_0\sin\theta}{\cos\delta}=\frac{y-b_0}{\epsilon}=\frac{z-c_0\cos\theta}{\sin\delta},$$

or, ignoring squares of small quantities,

$$x=\frac{y-b_0}{\epsilon}=\frac{z-c_0\cos\theta}{\delta},$$

which we shall write in the form

$$y=\frac{\beta}{\mu}x+b,\quad z=\frac{\gamma}{\mu}x+c,$$

where $$b=b_0,\quad \beta=\mu\epsilon,\quad c=c_0\cos\theta,\quad \gamma=\mu\delta.$$

To find corresponding equations for the emergent ray we must shift the origin to that arbitrary point where the axis of the beam next impinges upon a refracting surface and rotate the axes of $y$, $z$ until the axis of $y$ is a tangent line to that surface. The equations of the ray then become

$$y'=\frac{\beta'}{\mu'}x'+b',\quad z'=\frac{\gamma'}{\mu'}x'+c',$$

where

$$\begin{aligned}\beta' &= \mu'\epsilon'\cos\phi'+\mu'\delta'\sin\phi',\\ \gamma' &= -\mu'\epsilon'\sin\phi'+\mu'\delta'\cos\phi',\\ b' &= b_0\cos\phi'+c_0\cos\theta'\sin\phi'+\alpha'\beta',\\ c' &= -b_0\sin\phi'+c_0\cos\theta'\cos\phi'+\alpha'\gamma',\end{aligned}$$

where $\mu'\alpha'$ is the shift of the origin along the ray, and $\phi'$ is the angle of rotation. Eliminate $\epsilon$, $\epsilon'$, $\delta$, $\delta'$, $b_0$, $c_0$, and we get

$$b=(b'-\alpha'\beta')\cos\phi'-(c'-\alpha'\gamma')\sin\phi',$$

$$\beta=\beta'\cos\phi'-\gamma'\sin\phi'+Ab+Hc\frac{\cos\theta'}{\cos\theta},$$

$$c=\left[(b'-\alpha'\beta')\sin\phi'+(c'-\alpha'\gamma')\cos\phi'\right]\frac{\cos\theta}{\cos\theta'},$$

$$\gamma=\left[\beta'\sin\phi'+\gamma'\cos\phi'+Hb+Bc\frac{\cos\theta'}{\cos\theta}\right]\frac{\cos\theta'}{\cos\theta}$$

where we have written

$$\frac{A_0}{A}=\frac{H_0}{H\cos\theta'}=\frac{B_0}{B\cos^2\theta'}=\frac{1}{\mu\cos\theta-\mu'\cos\theta'}.$$

If we write

$$b = g_1 b' + h_1 \beta' + p_1 c' + q_1 \gamma',$$
$$\beta = k_1 b' + l_1 \beta' + r_1 c' + s_1 \gamma',$$
$$c = p_2 b' + q_2 \beta' + g_2 c' + h_2 \gamma',$$
$$\gamma = r_2 b' + s_2 \beta' + k_2 c' + l_2 \gamma',$$

we get the following scheme of values, in which $\varpi$ has been written for $\cos\theta/\cos\theta'$ :—

| | | | |
|---|---|---|---|
| $g_1 = \cos\phi'$ | $h_1 = -a'\cos\phi'$ | $p_1 = -\sin\phi'$ | $q_1 = a'\sin\phi'$ |
| $k_1 = A\cos\phi' + H\sin\phi'$ | $l_1 = -Aa'\cos\phi' - Ha'\sin\phi' + \cos\phi'$ | $r_1 = -A\sin\phi' + H\cos\phi'$ | $s_1 = Aa'\sin\phi' - Ha'\cos\phi' - \sin\phi'$ |
| $p_2 = \varpi\sin\phi'$ | $q_2 = -\varpi a'\sin\phi'$ | $g_2 = \varpi\cos\phi'$ | $h_2 = -\varpi a'\cos\phi'$ |
| $r_2 = \varpi^{-1}[H\cos\phi' + B\sin\phi']$ | $s_2 = \varpi^{-1}[-Ha'\cos\phi' - Ba'\sin\phi' + \sin\phi']$ | $k_2 = \varpi^{-1}[-H\sin\phi' + B\cos\phi']$ | $l_2 = \varpi^{-1}[Ha'\sin\phi' - Ba'\cos\phi' + \cos\phi']$ |

Regarding this scheme of coefficients it may be verified by inspection that they obey the following twelve relations, namely :—

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} = 1, \tag{1}$$

$$\begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix} + \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} = 0, \tag{2}$$

$$\begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix} + \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} = 0, \tag{3}$$

$$\begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix} + \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix} = 0, \tag{4}$$

$$\begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix} + \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} = 0, \tag{5}$$

$$\begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} = 1, \tag{6}$$

and

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} = 1, \tag{1'}$$

$$\begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} = 0, \tag{2'}$$

$$\begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} = 0, \tag{3'}$$

$$\begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix} + \begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} = 0, \tag{4'}$$

$$\begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} = 0, \tag{5'}$$

$$\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} = 1. \tag{6'}$$

The structure of these equations is readily seen; each consists of two minors of the determinant formed with the sixteen coefficients as elements. The first six are composed of one minor made out of the first and second rows, together with a minor formed similarly from the third and fourth rows, and the second six are formed in the same way from the first and second, third and fourth columns.

Now these same relations hold when any number of successive refractions separate in the most general manner the rays $(b, \beta, c, \gamma)$, $(b', \beta', c', \gamma')$. For, if we have the three schemes of relation

$$b = g_1 b' + h_1 \beta' + p_1 c' + q_1 \gamma', \qquad b' = \gamma_1 b'' + \eta_1 \beta'' + \pi_1 c'' + \chi_1 \gamma'',$$

$$\beta = k_1 b' + l_1 \beta' + r_1 c' + s_1 \gamma', \qquad \beta' = \kappa_1 b'' + \lambda_1 \beta'' + \rho_1 c'' + \sigma_1 \gamma'',$$

$$c = p_2 b' + q_2 \beta' + g_2 c' + h_2 \gamma', \qquad c' = \pi_2 b'' + \chi_2 \beta'' + \gamma_2 c'' + \eta_2 \gamma'',$$

$$\gamma = r_2 b' + s_2 \beta' + k_2 c' + l_2 \gamma', \qquad \gamma = \rho_2 b'' + \sigma_2 \beta'' + \kappa_2 c'' + \lambda_2 \gamma'',$$

$$b = G_1 b'' + H_1 \beta'' + P_1 c'' + Q_1 \gamma'',$$

$$\beta = K_1 b'' + L_1 \beta'' + R_1 c'' + S_1 \gamma'',$$

$$c = P_2 b'' + Q_2 \beta'' + G_2 c'' + H_2 \gamma'',$$

$$\gamma = R_2 b'' + S_2 \beta'' + K_2 c'' + L_2 \gamma'',$$

and if the first two obey the twelve relations just written, then the third also obeys the same relations. For we have

$$\begin{aligned} G_1 &= g_1\gamma_1+h_1\kappa_1+p_1\pi_2+q_1\rho_2, \\ H_1 &= g_1\eta_1+h_1\lambda_1+p_1\chi_2+q_1\sigma_2, \\ P_1 &= g_1\pi_1+h_1\rho_1+p_1\gamma_2+q_1\kappa_2, \\ Q_1 &= g_1\chi_1+h_1\sigma_1+p_1\eta_2+q_1\lambda_2, \\ K_1 &= k_1\gamma_1+l_1\kappa_1+r_1\pi_2+s_1\rho_2, \\ &\quad \&c., \qquad \&c. \end{aligned}$$

Now form, for example, the minors which make up (3′) :—

$$\begin{vmatrix} G_1 & H_1 \\ R_2 & S_2 \end{vmatrix} = \begin{vmatrix} g_1\gamma_1+h_1\kappa_1+p_1\pi_2+q_1\rho_2, & g_1\eta_1+h_1\lambda_1+p_1\chi_2+q_1\sigma_2 \\ r_2\gamma_1+s_2\kappa_1+k_2\pi_2+l_2\rho_2, & r_2\eta_1+s_2\lambda_1+k_2\chi_2+l_2\sigma_2 \end{vmatrix}$$

$$= \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} \begin{vmatrix} \gamma_1 & \kappa_1 \\ \eta_1 & \lambda_1 \end{vmatrix} + \begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix} \begin{vmatrix} \gamma_1 & \pi_2 \\ \eta_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix} \begin{vmatrix} \gamma_1 & \rho_2 \\ \eta_1 & \sigma_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix} \begin{vmatrix} \kappa_1 & \pi_2 \\ \lambda_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix} \begin{vmatrix} \kappa_1 & \rho_2 \\ \lambda_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} \begin{vmatrix} \pi_2 & \rho_2 \\ \chi_2 & \sigma_2 \end{vmatrix}.$$

It will be seen that the minors of the $(gh)$ system are all formed out of the first and fourth rows, while those of the $(\gamma\eta)$ system are formed out of the first and second columns ; in the same way, we have

$$\begin{vmatrix} P_1 & Q_1 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} \begin{vmatrix} \pi_1 & \rho_1 \\ \chi_1 & \sigma_1 \end{vmatrix} + \begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix} \begin{vmatrix} \pi_1 & \gamma_2 \\ \chi_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix} \begin{vmatrix} \pi_1 & \kappa_2 \\ \chi_1 & \lambda_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix} \begin{vmatrix} \rho_1 & \gamma_2 \\ \sigma_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix} \begin{vmatrix} \rho_1 & \kappa_2 \\ \sigma_1 & \lambda_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} \begin{vmatrix} \gamma_2 & \kappa_2 \\ \eta_2 & \lambda_2 \end{vmatrix}.$$

Thus, in virtue of the relations

$$\begin{vmatrix} \gamma_1 & \kappa_1 \\ \eta_1 & \lambda_1 \end{vmatrix} + \begin{vmatrix} \pi_1 & \rho_1 \\ \chi_1 & \sigma_1 \end{vmatrix} = 1, \tag{1′}$$

$$\begin{vmatrix} \gamma_1 & \pi_2 \\ \eta_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} \pi_1 & \gamma_2 \\ \chi_1 & \eta_2 \end{vmatrix} = 0, \tag{2′}$$

$$\begin{vmatrix} \gamma_1 & \rho_2 \\ \eta_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \pi_1 & \kappa_2 \\ \chi_1 & \lambda_2 \end{vmatrix} = 0, \tag{3′}$$

$$\begin{vmatrix} \kappa_1 & \pi_2 \\ \lambda_1 & \chi_2 \end{vmatrix} + \begin{vmatrix} \rho_1 & \gamma_2 \\ \sigma_1 & \eta_2 \end{vmatrix} = 0, \tag{4'}$$

$$\begin{vmatrix} \kappa_1 & \rho_2 \\ \lambda_1 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \rho_1 & \kappa_2 \\ \sigma_1 & \lambda_2 \end{vmatrix} = 0, \tag{5'}$$

$$\begin{vmatrix} \pi_2 & \rho_2 \\ \chi_2 & \sigma_2 \end{vmatrix} + \begin{vmatrix} \gamma_2 & \kappa_2 \\ \eta_2 & \lambda_2 \end{vmatrix} = 1, \tag{6'}$$

we have

$$\begin{vmatrix} G_1 & H_1 \\ R_2 & S_2 \end{vmatrix} + \begin{vmatrix} P_1 & Q_1 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} = 0. \tag{3'}$$

In the same way, we get

$$\begin{vmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{vmatrix} + \begin{vmatrix} G_2 & H_2 \\ K_2 & L_2 \end{vmatrix} = \begin{vmatrix} \pi_1 & \chi_1 \\ \rho_1 & \sigma_1 \end{vmatrix} + \begin{vmatrix} \gamma_2 & \eta_2 \\ \kappa_2 & \lambda_2 \end{vmatrix} = 1, \tag{6}$$

and the remainder follow on exactly the same lines.

Let us now remove two unnecessary restrictions. We have supposed the origin to lie in the first surface of discontinuity, and the axis of $y$ to be a tangent line. But introduce at an arbitrary point of the beam before the first surface is reached a hypothetical surface with zero change of refractive index, and the origin will be transferred to an arbitrary point, and the axis of $y$ to an arbitrary direction. And the same applies to the emergent beam.

Secondly, we have supposed density to vary discontinuously; but, since there is no restriction upon the number of surfaces of discontinuity, the results apply equally to continuous variation.

Thus we have proved the following theorem:—

*If the equations of an original ray and of the same ray emergent after passing through any singly refracting continuous or discontinuous media whatever in any manner be respectively*

$$\left.\begin{aligned} y &= \frac{\beta}{\mu}x + b \\ z &= \frac{\gamma}{\mu}x + c \end{aligned}\right\} \quad \textit{and} \quad \left.\begin{aligned} y' &= \frac{\beta'}{\mu'}x' + b', \\ z' &= \frac{\gamma'}{\mu'}x' + c', \end{aligned}\right\}$$

*where* $b, \beta, c, \gamma, b', \beta', c', \gamma'$ *are all small, so that the relations between them are linear, viz.,*

$$b = G_1 b' + H_1 \beta' + P_1 c' + Q_1 \gamma',$$
$$\beta = K_1 b' + L_1 \beta' + R_1 c' + S_1 \gamma',$$
$$c = P_2 b' + Q_2 \beta' + G_2 c' + H_2 \gamma',$$
$$\gamma = R_2 b' + S_2 \beta' + K_2 c' + L_2 \gamma',$$

*then the sixteen coefficients obey the relations*

$$G_1 L_1 - H_1 K_1 + P_2 S_2 - Q_2 R_2 = 1, \quad (1)$$
$$G_1 R_1 - P_1 K_1 + P_2 K_2 - G_2 R_2 = 0, \quad (2)$$
$$G_1 S_1 - Q_1 K_1 + P_2 L_2 - H_2 R_2 = 0, \quad (3)$$
$$H_1 R_1 - P_1 L_1 + Q_2 K_2 - G_2 S_2 = 0, \quad (4)$$
$$H_1 S_1 - Q_1 L_1 + Q_2 L_2 - H_2 S_2 = 0, \quad (5)$$
$$P_1 S_1 - Q_1 R_1 + G_2 L_2 - H_2 K_2 = 1, \quad (6)$$

$$G_1 L_1 - H_1 K_1 + P_1 S_1 - Q_1 R_1 = 1, \quad (1')$$
$$G_1 Q_2 - H_1 P_2 + P_1 H_2 - Q_1 G_2 = 0, \quad (2')$$
$$G_1 S_2 - H_1 R_2 + P_1 L_2 - Q_1 K_2 = 0, \quad (3')$$
$$K_1 Q_2 - L_1 P_2 + R_1 H_2 - S_1 G_2 = 0, \quad (4')$$
$$K_1 S_2 - L_1 R_2 + R_1 L_2 - S_1 K_2 = 0, \quad (5')$$
$$P_2 S_2 - Q_2 R_2 + G_2 L_2 - H_2 K_2 = 1. \quad (6')$$

Let us now consider further the passage from one scheme of coefficients to another, as written at length on p. 37, or, as we shall write it hereafter,

$$(b\beta c\gamma) = \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} (b'\beta' c'\gamma'),$$

$$(b'\beta' c'\gamma') = \begin{Bmatrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{Bmatrix} (b''\beta'' c''\gamma''),$$

$$(b\beta c\gamma) = \begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix} (b''\beta''c''\gamma'),$$

where it is postulated that each group of coefficients satisfies the relations (1) ... (6), (1′) ... (6′).

First let us prove the following theorem:—

*If we have*

$$(b\beta c\gamma) = \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} (b'\beta'c'\gamma'),$$

*then inversely*

$$(b'\beta'c'\gamma') = \begin{Bmatrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{Bmatrix} (b\beta c\gamma).$$

For $$l_1 b - h_1\beta = \begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} b' + 0 \,.\, \beta' + \begin{vmatrix} p_1 & h_1 \\ r_1 & l_1 \end{vmatrix} c' + \begin{vmatrix} q_1 & h_1 \\ s_1 & l_1 \end{vmatrix} \gamma',$$

$$s_2 c - q_2\gamma = \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} b' + 0 \,.\, \beta' + \begin{vmatrix} g_2 & q_2 \\ k_2 & s_2 \end{vmatrix} c' + \begin{vmatrix} h_2 & q_2 \\ l_2 & s_2 \end{vmatrix} \gamma'.$$

Add these; then in virtue of (1), (4), (5), we get the first of the above relations, and the others follow similarly.

Now it is physically and also algebraically evident that the ray is reversible. Hence the scheme $\{l_1, -h_1, \text{&c.}\}$ must satisfy the twelve fundamental relations. Make the necessary substitutions, $l_1$ for $g_1$, $-h_1$ for $h_1$, $s_2$ for $p_1$, $-q_2$ for $q_1$, &c., and we find that (1) becomes (1′), (2) becomes (5′), (3) becomes (4′), (4) becomes (3′), (5) becomes (2′), (6) becomes (6′).

Hence, from the reversibility of the ray, *if the relations* (1)...(6) *are satisfied, it follows that the relations* (1′)...(6′) *are also satisfied, and conversely.*

This may be proved without introducing the principle of reversibility. For we have

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix}\begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - \begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix}\begin{vmatrix} g_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + \begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix}\begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix}$$

$$+ \begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix}\begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - \begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix}\begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} + \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix}\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{vmatrix} \equiv \Delta, \text{ say,}$$

and

$$\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}\begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix}\begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + \begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix}\begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix}$$

$$+ \begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix}\begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - \begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix}\begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix} + \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix}\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{vmatrix} \equiv 0.$$

Add these:

$$(1)\begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} - (2)\begin{vmatrix} q_2 & h_2 \\ s_2 & l_2 \end{vmatrix} + (3)\begin{vmatrix} q_2 & g_2 \\ s_2 & k_2 \end{vmatrix} + (4)\begin{vmatrix} p_2 & h_2 \\ r_2 & l_2 \end{vmatrix} - (5)\begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix}$$

$$+ (6)\begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} \equiv \Delta,$$

where (1), (2), ... denote the left-hand members of the equations so named.

In the same way,

$$(1)\begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} - (2)\begin{vmatrix} h_1 & q_1 \\ l_1 & s_1 \end{vmatrix} + (3)\begin{vmatrix} h_1 & p_1 \\ l_1 & r_1 \end{vmatrix} + (4)\begin{vmatrix} g_1 & q_1 \\ k_1 & s_1 \end{vmatrix} - (5)\begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix}$$

$$+ (6)\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} \equiv \Delta.$$

Add these: we get

$$\begin{aligned}\Delta &\equiv (1)(6)-(2)(5)+(3)(4)\\ &= 1,\end{aligned}$$

in virtue of the values of the expressions (1)...(6); hence, substituting the values of $\Delta$, (1) ... (6) above, we get

$$\begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix} = 1, \qquad (6')$$

$$\begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix} = 1. \qquad (1')$$

Again, in like manner, we have

$$\begin{aligned}0 \equiv &\begin{vmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ g_1 & h_1 & p_1 & q_1 \\ p_2 & q_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \\ g_1 & h_1 & p_1 & q_1 \\ p_2 & q_2 & g_2 & h_2 \end{vmatrix}\\ \equiv (1)&\begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & q_1 \\ q_2 & h_2 \end{vmatrix} + (3)\begin{vmatrix} h_1 & p_1 \\ q_2 & g_2 \end{vmatrix} + (4)\begin{vmatrix} g_1 & q_1 \\ p_2 & h_2 \end{vmatrix}\\ &- (5)\begin{vmatrix} g_1 & p_1 \\ p_2 & g_2 \end{vmatrix} + (6)\begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix}\end{aligned}$$

$$\begin{aligned}0 \equiv (1)&\begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & q_1 \\ s_2 & l_2 \end{vmatrix} + (3)\begin{vmatrix} h_1 & p_1 \\ s_2 & k_2 \end{vmatrix} + (4)\begin{vmatrix} g_1 & q_1 \\ r_2 & l_2 \end{vmatrix}\\ &- (5)\begin{vmatrix} g_1 & p_1 \\ r_2 & k_2 \end{vmatrix} + (6)\begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix}\end{aligned}$$

$$\begin{aligned}0 \equiv (1)&\begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} - (2)\begin{vmatrix} h_1 & s_1 \\ q_2 & h_2 \end{vmatrix} + (3)\begin{vmatrix} l_1 & r_1 \\ q_2 & g_2 \end{vmatrix} + (4)\begin{vmatrix} k_1 & s_1 \\ p_2 & h_2 \end{vmatrix}\\ &- (5)\begin{vmatrix} k_1 & r_1 \\ p_2 & g_2 \end{vmatrix} + (6)\begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix}\end{aligned}$$

$$\begin{aligned}0 \equiv (1)&\begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} - (2)\begin{vmatrix} l_1 & s_1 \\ s_2 & l_2 \end{vmatrix} + (3)\begin{vmatrix} l_1 & r_1 \\ s_2 & k_2 \end{vmatrix} + (4)\begin{vmatrix} k_1 & s_1 \\ r_2 & l_2 \end{vmatrix}\\ &- (5)\begin{vmatrix} k_1 & r_1 \\ r_2 & k_2 \end{vmatrix} + (6)\begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix},\end{aligned}$$

and these, in virtue of the values of the expressions (1) ... (6), give respectively

$$\begin{vmatrix} p_1 & q_1 \\ g_2 & h_2 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ p_2 & q_2 \end{vmatrix} = 0, \tag{2'}$$

$$\begin{vmatrix} p_1 & q_1 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} g_1 & h_1 \\ r_2 & s_2 \end{vmatrix} = 0, \tag{3'}$$

$$\begin{vmatrix} r_1 & s_1 \\ g_2 & h_2 \end{vmatrix} + \begin{vmatrix} k_1 & l_1 \\ p_2 & q_2 \end{vmatrix} = 0, \tag{4'}$$

$$\begin{vmatrix} r_1 & s_1 \\ k_2 & l_2 \end{vmatrix} + \begin{vmatrix} k_1 & l_1 \\ r_2 & s_2 \end{vmatrix} = 0. \tag{5'}$$

Hence the twelve fundamental relations contain only six independent statements.

Let us write the relation between the three schemes of p. 37, as follows:—

$$\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix} = \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} \begin{Bmatrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{Bmatrix}$$

where in fact the value of any element on the left is found by treating the schemes on the right as if they were determinants, but multiplying the rows of the first into the columns of the second; *e.g.*,

$$Q_2 = p_2\eta_1 + q_2\lambda_1 + g_2\chi_2 + k_2\sigma_2,$$

that is, we take the third row of $\{g_1 h_1\}$ into the second column of $\{\gamma_1 \eta_1 \ldots\}$, since $Q_2$ stands in the third row and second column on the left. It is clear that this multiplication is not in general commutative —a limitation that has a direct and obvious physical meaning. Inversely we get

$$\begin{Bmatrix} L_1 & -H_1 & S_2 & -Q_2 \\ -K_1 & G_1 & -R_2 & P_2 \\ S_1 & -Q_1 & L_2 & -H_2 \\ -R_1 & P_1 & -K_2 & G_2 \end{Bmatrix} = \begin{Bmatrix} \lambda_1 & -\eta_1 & \sigma_2 & -\chi_2 \\ -\kappa_1 & \gamma_1 & -\rho_2 & \pi_2 \\ \sigma_1 & -\chi_1 & \lambda_2 & -\eta_2 \\ -\rho_1 & \pi_1 & -\kappa_2 & \gamma_2 \end{Bmatrix} \begin{Bmatrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{Bmatrix},$$

and also

$$\begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} = \begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & -\eta_1 & \sigma_2 & -\chi_2 \\ -\kappa_1 & \gamma_1 & -\rho_2 & \pi_2 \\ \sigma_1 & -\chi_1 & \lambda_2 & -\eta_2 \\ -\rho_1 & \pi_1 & -\kappa_2 & \gamma_2 \end{Bmatrix},$$

$$\begin{Bmatrix} \gamma_1 & \eta_1 & \pi_1 & \chi_1 \\ \kappa_1 & \lambda_1 & \rho_1 & \sigma_1 \\ \pi_2 & \chi_2 & \gamma_2 & \eta_2 \\ \rho_2 & \sigma_2 & \kappa_2 & \lambda_2 \end{Bmatrix} = \begin{Bmatrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{Bmatrix} \cdot \begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}.$$

It is to be observed that a scheme $\{g_1\, h_1 \ldots\}$ obeying the fundamental relations may represent a mere shift of origin; thus, if

$$(b\beta c\gamma) = \begin{Bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{Bmatrix} (b'\beta' c'\gamma'),$$

we have

$$b = b' + \lambda\beta', \quad c = c' + \lambda\gamma',$$

$$\beta = \beta', \qquad \gamma = \gamma',$$

or the origin of the $(b, c)$ system has been shifted a distance $+\mu\lambda$ along the axis of $x$ from the origin of the $(b', c')$ system.

Or, again, a mere rotation; thus, if

$$(b\beta c\gamma) = \begin{Bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{Bmatrix} (b'\beta' c'\gamma'),$$

we have

$$b = b'\cos\phi + c'\sin\phi, \quad c = -b'\sin\phi + c'\cos\phi,$$

$$\beta = \beta'\cos\phi + c'\sin\phi, \quad \gamma = -\beta'\sin\phi + \gamma'\cos\phi,$$

or the axes of $(b, c)$ have been rotated through an angle $+\phi$ from the positions for $(b', c')$.

It is possible that, if the fundamental relations are taken as definition

of a scheme of sixteen coefficients, $g_1, h_1$, &c., then the scheme so defined includes other systems than can be produced by pure optical means; but I cannot adduce an example.

The foregoing results permit us to construct with mechanical facility the scheme of coefficients that belong to any given optical system.*

*Example I.*—Multiply together the schemes

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ A & 1 & H & 0 \\ 0 & 0 & \varpi & 0 \\ H/\varpi & 0 & B/\varpi & 1/\varpi \end{matrix}\right\} \left\{\begin{matrix} 1 & -a' & 0 & 0 \\ 0 & 1 & 0 \cdot 0 & \\ 0 & 0 & 1 & -a' \\ 0 & 0 & 0 & 1 \end{matrix}\right\}$$

$$\left\{\begin{matrix} \cos\phi' & 0 & -\sin\phi' & 0 \\ 0 & \cos\phi' & 0 & -\sin\phi' \\ \sin\phi' & 0 & \cos\phi' & 0 \\ 0 & \sin\phi' & 0 & \cos\phi' \end{matrix}\right\},$$

which correspond respectively to refraction at a surface of discontinuity, shift of the origin of the emergent ray, a distance $+\mu' a'$ along the axis of the beam, and rotation of the axes for the same through an angle $+\phi'$. We obtain the original scheme of p. 36.

*Example II. Thick Lens.*—Let a beam impinge directly upon a lens of finite thickness whose faces are of arbitrary curvatures. Let the thickness be $-\mu c$, the equation of the first surface

$$2x = a_1 y^2 + b_1 z^2,$$

and that of the second with parallel axes

$$2x = A_2 y^2 + 2H_2 yz + B_2 z^2,$$

or with axes rotated through an angle $+\theta$ from these,

$$2x = a_2 y^2 + b_2 z^2.$$

* It is worth remembering that the same method applies to surfaces symmetrically disposed along an axis when each scheme contains only two rows.

First multiply together the schemes

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_1 & 1 \end{matrix}\right\} \left\{\begin{matrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} 1 & 0 & 0 & 0 \\ A_2 & 1 & H_2 & 0 \\ 0 & 0 & 1 & 0 \\ H_2 & 0 & B_2 & 1 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 1 & c & 0 & 0 \\ a_1 & 1+a_1c & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & b_1 & 1+b_1c \end{matrix}\right\} \left\{\begin{matrix} 1 & 0 & 0 & 0 \\ A_2 & 1 & H_2 & 0 \\ 0 & 0 & 1 & 0 \\ H_2 & 0 & B_2 & 1 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 1+A_2c & c & cH_2 & 0 \\ a_1+A_2+a_1A_2c & 1+a_1c & H_2(1+a_1c) & 0 \\ cH_2 & 0 & 1+B_2c & c \\ H_2(1+b_1c) & 0 & b_1+B_2+b_1B_2c & 1+b_1c \end{matrix}\right\},$$

or we may form this scheme otherwise by rotating the axes between the two refractions until the equation of the second surface assumes its simplest form. Thus

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_1 & 1 \end{matrix}\right\} \left\{\begin{matrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{matrix}\right\} \left\{\begin{matrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_2 & 1 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \cos\theta & c\cos\theta & -\sin\theta & -c\sin\theta \\ a_1\cos\theta & (1+a_1c)\cos\theta & -a_1\sin\theta & -(1+a_1c)\sin\theta \\ \sin\theta & c\sin\theta & \cos\theta & c\cos\theta \\ b_1\sin\theta & (1+b_1c)\sin\theta & b_1\cos\theta & (1+b_1c)\cos\theta \end{matrix}\right\} \left\{\begin{matrix} 1 & 0 & 0 & 0 \\ a_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_2 & 1 \end{matrix}\right\}.$$

$$= \left\{\begin{matrix} (1+a_2c)\cos\theta & c\cos\theta & -(1+b_2c)\sin\theta & -c\sin\theta \\ (a_1+a_2+a_1a_2c)\cos\theta & (1+a_1c)\cos\theta & -(a_1+b_2+a_1b_2c)\sin\theta & -(1+a_1c)\sin\theta \\ (1+a_2c)\sin\theta & c\sin\theta & (1+b_2c)\cos\theta & c\cos\theta \\ (b_1+a_2+b_1a_2c)\sin\theta & (1+b_1c)\sin\theta & (b_1+b_2+b_1b_2c)\cos\theta & (1+b_1c)\cos\theta \end{matrix}\right\}.$$

This scheme becomes identical with the one already obtained if we further multiply by

$$\begin{Bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{Bmatrix},$$

that is, if we rotate the axes back to their original positions.

*Example III. Curved Prism.*—Let the two faces of a prism of any angle be of arbitrary curvatures, the edge being formed by a common tangent line. Let a beam impinge at the edge, perpendicular to this line. Let this line be taken as axis of $y$ and the equations of the surfaces be

$$2x = ay^2+2hyz+bz^2, \quad 2x = Ay^2+2Hyz+Bz^2,$$

respectively. The resulting coefficients are given by

$$\begin{Bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & h & 0 \\ 0 & 0 & \varpi & 0 \\ \frac{h}{\varpi} & 0 & \frac{b}{\varpi} & \frac{1}{\varpi} \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 & 0 \\ A & 1 & H & 0 \\ 0 & 0 & \Pi & 0 \\ \frac{H}{\Pi} & 0 & \frac{B}{\Pi} & \frac{1}{\Pi} \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 0 & 0 & 0 \\ a+A & 1 & H+h\Pi & 0 \\ 0 & 0 & \varpi\Pi & 0 \\ \frac{H+h\Pi}{\varpi\Pi} & 0 & \frac{B+b\Pi^2}{\varpi\Pi} & \frac{1}{\varpi\Pi} \end{Bmatrix};$$

that is to say, the prism is equivalent to a single surface.

## II.

1. It is evident that, if there are two systems of rays in correspondence, so that the relations between the parameters that define individual pairs are expressed by linear equations, then a scheme of sixteen coefficients suffices to define the correspondence, and that irrespective of the means, whether optical or other, by which one beam is related to the other.

To introduce these sixteen coefficients the axes of $x, x'$ are identified

with one corresponding pair of rays while the situations of the origins and the directions of the axes of $y$, $y'$ are undefined. But we must consider that two systems are equivalent which differ merely in respect to these undefined quantities, because the beams in two such systems might be identified ray with ray by mere screw shifts along the axes of $x$, $x'$, and such shifts are not significant in relation to the intrinsic representation of geometrical relationship afforded by the sixteen coefficients.

Thus, out of sixteen quantities contained in the coefficients, four (called $\lambda$, $\rho$, $\phi$, $\psi$ below) must be allotted to describe irrelevant features, and this suggests a conception of invariant forms which are independent of irrelevant quantities, and are related only to the geometrical modification produced in the beam. When we fully understand how far a scheme of coefficients may be modified by changes of axes, we shall find it easy to detect expressions which are invariant in form and value in spite of such changes. These are the quantities we require; it will be evident that their number is limited only by their degree, that their degree is always even, and that in the most general case not more than twelve can be independent.

Associated with the invariants we shall observe forms which vary with one only of the quantities $\lambda$, $\rho$, $\phi$, or $\psi$, which permit us to read off immediately the change of axes which will reduce a scheme of coefficients from one given shape to any other, its equivalent. These may be called seminvariants.

The following discussion, up to the introduction of the optical equations, applies to the modification of a narrow beam of rays by any means, *e.g.*, by repeated reciprocation at quadric surfaces, though its plan and scope have been determined by the optical problem.

2. Let $\lambda$, $\rho$ define shifts of the origins, so that

$$\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} \begin{Bmatrix} 1 & \rho & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$$= \begin{Vmatrix} g_1+\lambda k_1 & h_1+\rho g_1+\lambda l_1+\lambda\rho k_1 & p_1+\lambda r_1 & q_1+\rho p_1+\lambda s_1+\lambda\rho r_1 \\ k_1 & l_1+\rho k_1 & r_1 & s_1+\rho r_1 \\ p_2+\lambda r_2 & q_2+\rho p_2+\lambda s_2+\lambda\rho r_2 & g_2+\lambda k_2 & h_2+\rho g_2+\lambda l_2+\lambda\rho k_2 \\ r_2 & s_2+\rho r_2 & k_2 & l_2+\rho k_2 \end{Vmatrix}.$$

The forms which are independent of $\lambda$ and $\rho$ and not higher than the second degree are

$$k_1, \quad r_1, \quad r_2, \quad k_2,$$

$$\begin{vmatrix} g_1 & p_1 \\ k_1 & r_1 \end{vmatrix}, \quad \begin{vmatrix} g_1 & p_2 \\ k_1 & r_2 \end{vmatrix}, \quad \begin{vmatrix} g_1 & g_2 \\ k_1 & k_2 \end{vmatrix}, \quad \begin{vmatrix} p_1 & p_2 \\ r_1 & r_2 \end{vmatrix}, \quad \begin{vmatrix} p_1 & g_2 \\ r_1 & k_2 \end{vmatrix}, \quad \begin{vmatrix} p_2 & g_2 \\ r_2 & k_2 \end{vmatrix},$$

$$\begin{vmatrix} k_1 & r_2 \\ l_1 & s_2 \end{vmatrix}, \quad \begin{vmatrix} k_1 & r_1 \\ l_1 & s_1 \end{vmatrix}, \quad \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix}, \quad \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix}, \quad \begin{vmatrix} r_1 & k_2 \\ s_1 & l_2 \end{vmatrix}, \quad \begin{vmatrix} r_2 & k_2 \\ s_2 & l_2 \end{vmatrix},$$

$$\begin{vmatrix} g_1 & h_1 \\ k_1 & l_1 \end{vmatrix}, \quad \begin{vmatrix} p_1 & q_1 \\ r_1 & s_1 \end{vmatrix}, \quad \begin{vmatrix} p_2 & q_2 \\ r_2 & s_2 \end{vmatrix}, \quad \begin{vmatrix} g_2 & h_2 \\ k_2 & l_2 \end{vmatrix}.$$

These are each equal to the similarly formed quantity with letters $G_1$, $H_1$, &c.

Next consider a rotation of the axis of $y$ through an angle $\phi$,

$$\begin{Vmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Vmatrix}$$

$$= \begin{Vmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{Vmatrix} \begin{Vmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Vmatrix}$$

$$= \begin{Vmatrix} g_1\cos\phi+p_2\sin\phi & h_1\cos\phi+q_2\sin\phi & p_1\cos\phi+g_2\sin\phi & q_1\cos\phi+h_2\sin \\ k_1\cos\phi+r_2\sin\phi & l_1\cos\phi+s_2\sin\phi & r_1\cos\phi+k_2\sin\phi & s_1\cos\phi+l_2\sin \\ -g_1\sin\phi+p_2\cos\phi & -h_1\sin\phi+q_2\cos\phi & -p_1\sin\phi+g_2\cos\phi & -q_1\sin\phi+h_2\cos \\ -k_1\sin\phi+r_2\cos\phi & -l_1\sin\phi+s_2\cos\phi & -r_1\sin\phi+k_2\cos\phi & -s_1\sin\phi+l_2\cos \end{Vmatrix}$$

Confine attention to the terms in $k_1, r_1, r_2, k_2$. We have

$$K_1^2+R_2^2 = k_1^2+r_2^2,$$

$$R_1^2+K_2^2 = r_1^2+k_2^2,$$

$$K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$K_1R_1+K_2R_2 = k_1r_1+k_2r_2,$$

$$K_1^2-K_2^2+R_1^2-R_2^2 = (k_1^2-k_2^2+r_1^2-r_2^2)\cos 2\phi+2(k_1r_2+k_2r_1)\sin 2\phi,$$

$$2(K_1R_2+K_2R_1) = -(k_1^2-k_2^2+r_1^2-r_2^2)\sin 2\phi+2(k_1r_2+k_2r_1)\cos 2\phi.$$

Now consider a rotation $(\psi)$ of the axis of $y'$,

$$\begin{Bmatrix} G_1 & H_1 & P_1 & Q_1 \\ K_1 & L_1 & R_1 & S_1 \\ P_2 & Q_2 & G_2 & H_2 \\ R_2 & S_2 & K_2 & L_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} g_1 & h_1 & p_1 & q_1 \\ k_1 & l_1 & r_1 & s_1 \\ p_2 & q_2 & g_2 & h_2 \\ r_2 & s_2 & k_2 & l_2 \end{Bmatrix} \begin{Bmatrix} \cos\psi & 0 & -\sin\psi & 0 \\ 0 & \cos\psi & 0 & -\sin\psi \\ \sin\psi & 0 & \cos\psi & 0 \\ 0 & \sin\psi & 0 & \cos\psi \end{Bmatrix}$$

$$= \begin{Bmatrix} g_1\cos\psi+p_1\sin\psi & h_1\cos\psi+q_1\sin\psi & -g_1\sin\psi+p_1\cos\psi & -h_1\sin\psi+q_1\cos\psi \\ k_1\cos\psi+r_1\sin\psi & l_1\cos\psi+s_1\sin\psi & -k_1\sin\psi+r_1\cos\psi & -l_1\sin\psi+s_1\cos\psi \\ p_2\cos\psi+g_2\sin\psi & q_2\cos\psi+h_2\sin\psi & -p_2\sin\psi+g_2\cos\psi & -q_2\sin\psi+h_2\cos\psi \\ r_2\cos\psi+k_2\sin\psi & s_2\cos\psi+l_2\sin\psi & -r_2\sin\psi+k_2\cos\psi & -s_2\sin\psi+l_2\cos\psi \end{Bmatrix}$$

When we observe

$$K_1^2+R_1^2 = k_1^2+r_1^2,$$

$$K_2^2+R_2^2 = k_2^2+r_2^2,$$

$$K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$K_1R_2+K_2R_1 = k_1r_2+k_2r_1,$$

$$K_1^2-K_2^2-R_1^2+R_2^2 = (k_1^2-k^2-r_1^2+r_2^2)\cos 2\psi+2(k_1r_1+k_2r_2)\sin 2\psi,$$

$$2(K_1R_1+K_2R_2) = -(k_1^2-k_2^2-r_1^2+r_2^2)\sin 2\psi+2(k_1r_1+k_2r_2)\cos 2\psi.$$

3. Compare these with the former, and we find the following equations true in spite of all changes in axes, namely :—

$$(a)\quad K_1^2+K_2^2+R_1^2+R_2^2 = k_1^2+k^2+r_1^2+r_2^2,$$

$$(b)\quad K_1K_2-R_1R_2 = k_1k_2-r_1r_2,$$

$$(\alpha)\quad K_1^2-K_2^2+R_1^2-R_2^2 = (k_1^2-k_2^2+r_1^2-r_2^2)\cos 2\phi + 2(k_1r_2+k_2r_1)\sin 2\phi,$$

$$(\alpha')\quad 2(K_1R_2+K_2R_1) = -(k_1^2-k_2^2+r_1^2-r_2^2)\sin 2\phi + 2(k_1r_2+k_2r_1)\cos 2\phi,$$

$$(\beta)\quad K^2-K-R_1^2+R_2^2 = (k_1^2-k_2^2-r_1^2+r_2^2)\cos 2\psi + 2(k_1r_1+k_2r_2)\sin 2\psi,$$

$$(\beta')\quad 2(K_1R_1+K_2R_2) = -(k_1^2-k_2^2-r_1^2+r_2^2)\sin 2\psi + 2(k_1r_1+k_2r_2)\cos 2\psi.$$

The forms $(a)$, $(b)$ are two of the invariants we are seeking; $(\alpha)$, $(\alpha')$ are seminvariants with respect to $\phi$; $(\beta)$, $(\beta')$ are seminvariants with respect to $\psi$.

It would be tedious to follow the details in all subsequent cases; it may be verified without difficulty that we arrive at the following results, true in spite of all changes of axes :—

$$(c)\quad G_1R_1-K_1P_1-G_2R_2+K_2P_2 = g_1r_1-k_1p_1-g_2r_2+k_2p_2,$$

$$(d)\quad G_1R_2-K_1P_2-G_2R_1+K_2P_1 = g_1r_2-k_1p_2-g_2r_1+k_2p_1,$$

$$(\gamma)\quad G_1R_1-K_1P_1+G_2R_2-K_2P_2 = (g_1r_1-k_1p_1+g_2r_2-k_2p_2)\cos 2\phi + (g_1k_2-g_2k_1-p_1r_2+p_2r_1)\sin 2\phi,$$

$$(\gamma')\quad G_1K_2-G_2K_1-P_1R_2+P_2R_1 = -(g_1r_1-k_1p_1+g_2r_2-k_2p_2)\sin 2\phi + (g_1k_2-g_2k_1-p_1r_2+p_2r_1)\cos 2\phi,$$

$$(\delta)\quad G_1R_2-K_1P_2+G_2R_1-K_2P_1 = (g_1r_2-k_1p_2+g_2r_1-k_2p_1)\cos 2\psi + (g_1k_2-g_2k_1+p_1r_2-p_2r_1)\sin 2\psi,$$

$$(\delta')\quad G_1K_2-G_2K_1+P_1R_2-P_2R_1 = -(g_1r_2-k_1p_2+g_2r_1-k_2p_1)\sin 2\psi + (g_1k_2-g_2k_1+p_1r_2-p_2r_1)\cos 2\psi;$$

and also

$$(e)\quad K_1S_2-L_1R_2-K_2S_1+L_2R_1 = k_1s_2-l_1r_2-k_2s_1+l_2r_1,$$

$$(f)\quad K_1S_1-L_1R_1-K_2S_2+L_2R_2 = k_1s_1-l_1r_1-k_2s_2+l_2r_2,$$

$$(\epsilon)\quad K_1S_2-L_1R_2+K_2S_1-L_2R_1 = (k_1s_2-l_1r_2+k_2s_1-l_2r_1)\cos 2\psi + (k_1l_2-k_2l_1+r_1s_2-r_2s_1)\sin 2\psi,$$

$$(\epsilon')\quad K_1L_2-K_2L_1+R_1S_2-R_2S_1 = -(k_1s_2-l_1r_2+k_2s_1-l_2r_1)\sin 2\psi + (k_1l_2-k_2l_1+r_1s_2-r_2s_1)\cos 2\psi,$$

$$(\phi)\quad K_1S_1-L_1R_1+K_2S_2-L_2R_2 = (k_1s_1-l_1r_1+k_2s_2-l_2r_2)\cos 2\phi + (k_1l_2-k_2l_1-r_1s_2+r_2s_1)\sin 2\phi,$$

$$(\phi')\quad K_1L_2-K_2L_1-R_1S_2+R_2S_1 = -(k_1s_1-l_1r_1+k_2s_2-l_2r_2)\sin 2\phi + (k_1l_2-k_2l_1-r_1s_2+r_2s_1)\cos 2\phi\ ;$$

and, finally,

$$(g)\quad G_1L_1-H_1K_1+P_1S_1-Q_1R_1+P_2S_2-Q_2R_2+G_2L_2-H_2K_2 = g_1l_1-h_1k_1+p_1s_1-q_1r_1+p_2s_2-q_2r_2+g_2l_2-h_2k_2.$$

There are also seminvariants with respect to $\lambda$ and $\rho$, namely,

$$(\alpha_1)\quad G_1K_1+G_2K_2+P_1R_1+P_2R_2 = g_1k_1+g_2k_2+p_1r_1+p_2r_2+\lambda\,(k_1^2+k_2^2+r_1^2+r_2^2),$$

$$(\alpha_2)\quad L_1K_1+L_2K_2+S_1R_1+S_2R_2 = l_1k_1+l_2k_2+s_1r_1+s_2r_2+\rho\,(k_1^2+k_2^2+r_1^2+r_2^2),$$

$$(\beta_1)\quad G_1K_2+G_2K_1-P_1R_2-P_2R_1 = g_1k_2+g_2k_1-p_1r_2-p_2r_1+2\lambda\,(k_1k_2-r_1r_2),$$

$$(\beta_2)\quad L_1K_2+L_2K_1-S_1R_2-S_2R_1 = l_1k_2+l_2k_1-s_1r_2-s_2r_1+2\rho\,(k_1k_2-r_1r_2),$$

$$(\phi_1)\quad G_1S_1-L_1P_1-G_2S_2+L_2P_2 = g_1s_1-l_1p_1-g_2s_2+l_2p_2+\lambda\,(k_1s_1-l_1r_1-k_2s_2+l_2r_2) +\rho\,(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\phi_2)\quad K_1Q_1-R_1H_1-K_2Q_2+R_2H_2 = k_1q_1-r_1h_1-k_2q_2+r_2h_2+\lambda\,(k_1s_1-l_1r_1-k_2s_2+l_2r_2) -\rho\,(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\delta_1)\quad G_1S_2-L_1P_2-G_2S_1+L_2P_1 = g_1s_2-l_1p_2-g_2s_1+l_2p_1+\lambda\,(k_1s_2-l_1r_2-k_2s_1+l_2r_1) +\rho\,(g_1r_2-k_1p_2-g_2r_1+k_2p_1),$$

$$(\delta_2)\quad K_1Q_2-H_1R_2-K_2Q_1+H_2R_1 = k_1q_2-h_1r_2-k_2q_1+h_2r_1+\lambda\,(k_1s_2-l_1r_2-k_2s_1+l_2r_1) -\rho\,(g_1r_2-k_1p_2-g_2r_1+k_2p_1),$$

$$(\omega_1)\quad G_1Q_1-H_1P_1-G_2Q_2+H_2P_2$$
$$= g_1q_1-h_1p_1-g_2q_2+h_2p_2$$
$$+\lambda\,(g_1s_1-l_1p_1-g_2s_2+l_2p_2+k_1q_1-h_1r_1-k_2q_2+h_2r_2)$$
$$+\lambda^2(k_1s_1-l_1r_1-k_2s_2+l_2r_2),$$

$$(\omega_2)\quad H_1S_1-L_1Q_1-H_2S_2+L_2Q_2$$
$$= h_1s_1-l_1q_1-h_2s_2+l_2q_2$$
$$+\rho\,(g_1s_1-l_1p_1-g_2s_2+l_2p_2-k_1q_1+h_1r_1+k_2q_2-h_2r_2)$$
$$+\rho^2(g_1r_1-k_1p_1-g_2r_2+k_2p_2),$$

$$(\zeta_1)\quad G_1Q_2-H_1P_2-G_2Q_1+H_2P_1$$
$$= g_1q_2-h_1p_2-g_2q_1+h_2p_1$$
$$+\lambda\,(g_1s_2-l_1p_2-g_2s_1+l_2p_1+k_1q_2-h_1r_2-k_2q_1+h_2r_1)$$
$$+\lambda^2(k_1s_2-l_1r_2-k_2s_1+l_2r_1),$$

$$(\zeta_2)\quad H_1S_2-L_1Q_2-H_2S_1+L_2Q_1$$
$$= h_1s_2-l_1q_2-h_2s_1+l_2q_1$$
$$+\rho\,(g_1s_2-l_1p_2-g_2s_1+l_2p_1-k_1q_2+h_1r_2+k_2q_1-h_2r_1)$$
$$+\rho^2(g_1r_2-k_1p_2-g_2r_1+k_2p_1).$$

4. If two schemes are equivalent, all the above relations hold; inversely, let us investigate what conditions must hold in order that $\lambda$, $\rho$, $\phi$, $\psi$ determined from any of the above should be consistent.

First, consider the equations in $\varphi$, $\psi$. If there are four simultaneous equations

$$a_1 = \quad a_2\cos\theta+b_2\sin\theta,$$
$$b_1 = -\,a_2\sin\theta+b_2\cos\theta,$$
$$c_1 = \quad c_2\cos\theta+d_2\sin\theta,$$
$$d_1 = -\,c_2\sin\theta+d_2\cos\theta,$$

then we must have

$$a_1^2+b_1^2 = a_2^2+b_2^2, \qquad c_1^2+d_1^2 = c_2^2+d_2^2,$$
$$a_1c_1+b_1d_1 = a_2c_2+b_2d_2, \quad a_1d_1-b_1c_1 = a_2d_2-b_2c_2,$$

between which an identity subsists

$$(a_1^2+b_1^2)(c_1^2+d_1^2) = (a_1c_1+b_1d_1)^2+(a_1d_1-b_1c_1)^2.$$

We must find the corresponding forms for the equations in $\phi$, $\psi$.

We shall make repeated use of the identity

$$\begin{aligned}(a_1b_1c_3d_4) \equiv\ & \big[(a_1b_2)+(a_3b_4)\big]\big[(c_1d_2)+(c_3d_4)\big] \\ & -\big[(a_1c_2)+(a_3c_4)\big]\big[(b_1d_2)+(b_3d_4)\big] \\ & +\big[(a_1d_2)+(a_3d_4)\big]\big[(b_1c_2)+(b_3c_4)\big] \\ \equiv & -\big[(a_1b_2)-(a_3b_4)\big]\big[(c_1d_2)-(c_3d_4)\big] \\ & +\big[(a_1c_2)-(a_3c_4)\big]\big[(b_1d_2)-(b_3d_4)\big] \\ & -\big[(a_1d_2)-(a_3d_4)\big]\big[(b_1c_2)-(b_3c_4)\big],\end{aligned}$$

where $(a_1b_2)$ stands for $a_1b_2-a_2b_1$, and $(a_1b_2c_3d_4)$ is any determinant of four rows and columns. This has already been proved (p. 43 *ante*) under the form

$$\Delta \equiv (1)(6)-(2)(5)+(3)(4).$$

Now for the consistency of the equations $(\alpha)$, $(\alpha')$, we must have the following form invariant, viz.,

$$\begin{aligned}(A) &\equiv (K_1^2-K_2^2+R_1^2-R_2^2)^2+4\,(K_1R_2+K_2R_1)^2 \\ &\equiv (a)^2-4\,(b)^2 \\ &\equiv \begin{vmatrix} K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ K_2 & K_1 & -R_2 & -R_1 \end{vmatrix}.\end{aligned}$$

Again, $(\gamma)$, $(\gamma')$ give

$$\begin{aligned}(C) &\equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)^2+(G_1K_2-G_2K_1-P_1R_2+P_2R_1)^2 \\ &\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ -P_2 & P_1 & -G_2 & G_1 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix},\end{aligned}$$

and, taking $(\alpha)$, $(\alpha')$, $(\gamma)$, $(\gamma')$ together,

$$\begin{aligned}(ac) &\equiv (K_1^2-K_2^2+R_1^2-R_2^2)(G_1R_1-K_1P_1+G_2R_2-K_2P_2)\\ &\quad +2\,(K_1R_2+K_2R_1)(G_1K_2-G_2K_1-P_1R_2+P_2R_1)\\ &\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2\\ K_1 & K_2 & R_1 & R_2\\ K_2 & K_1 & -R_2 & -R_1\\ R_2 & -R_1 & K_2 & -K_1\end{vmatrix} \equiv (a)(c)+2\,(b)(d),\end{aligned}$$

$$\begin{aligned}(ac') &\equiv (K_1^2-K_2^2+R_1^2-R_2^2)(G_1K_2-G_2K_1-P_1R_2+P_2R_1)\\ &\quad -2\,(K_1R_2+K_2R_1)(G_1R_1-K_1P_1+G_2R_2-K_2P_2)\\ &\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2\\ K_1 & K_2 & R_1 & R_2\\ R_1 & -R_2 & -K_1 & K_2\\ R_2 & -R_1 & K_2 & -K_1\end{vmatrix},\end{aligned}$$

and among these four forms we have the relation

$$(A)(C) \equiv (ac)^2+(ac')^2,$$

so that these four forms actually introduce only one new one.

Next, take $(\alpha)$, $(\alpha')$, $(\phi)$, $(\phi')$. We get

$$\begin{aligned}(F) &\equiv (K_1S_1-L_1R_1+K_2S_2-L_2R_2)^2+(K_1L_2-K_2L_1-R_1S_2+R_2S_1)^2\\ &\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2\\ -S_2 & S_1 & -L_2 & L_1\\ K_1 & K_2 & R_1 & R_2\\ R_2 & -R_1 & K_2 & -K_1\end{vmatrix}.\end{aligned}$$

Also

$$\begin{aligned}(af) &\equiv (K_1^2-K_2^2+R_1^2-R_2^2)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)\\ &\quad +2\,(K_1R_2+K_2R_1)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)\\ &\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2\\ K_1 & K_2 & R_1 & R_2\\ K_2 & K_1 & -R_2 & -R_1\\ -R_2 & R_1 & -K_2 & K_1\end{vmatrix} \equiv (a)(f)+2\,(b)(e),\end{aligned}$$

and

$$(af') \equiv (K_1^2-K_2^2+R_1^2-R_2^2)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$-2\,(K_1R_2+K_2R_1)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

and we have the relation

$$(A)(F) \equiv (af)^2+(af')^2;$$

these three, then, introduce one new form.

Lastly, take $(\gamma)$, $(\gamma')$, $(\phi)$, $(\phi')$, which, of course, can introduce no new form.

$$(cf) \equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$-(G_1K_2-G_2K_1-P_1R_2+P_2R_1)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ -S_2 & S_1 & -L_2 & L_1 \\ K_1 & K_2 & R_1 & R_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

$$(cf') \equiv (G_1R_1-K_1P_1+G_2R_2-K_2P_2)(K_1L_2-K_2L_1-R_1S_2+R_2S_1)$$
$$-(G_1K_2-G_2K_1-P_1R_2+P_2R_1)(K_1S_1-L_1R_1+K_2S_2-L_2R_2)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ -R_2 & R_1 & -K_2 & K_1 \end{vmatrix},$$

and we have the relations

$$(A)(cf) \equiv (ac)(af)+(ac')(af'),$$
$$(A)(cf') \equiv (ac)(af')-(ac')(af),$$

leading to $$(C)(F) \equiv (cf)^2+(cf')^2.$$

5. Next, take the equations in $\psi$ beginning with $(\beta)$, $(\beta')$, $(\delta)$, $(\delta')$; we get

$$(B) \equiv (K_1^2-K_2^2-R_1^2+R_2^2)^2+4(K_1R_1+K_2R_2)^2 \equiv (A),$$

$$(D) \equiv (G_1R_2-K_1P_2+G_2R_1-K_2P_1)^2+(G_1K_2-G_2K_1+P_1R_2-P_2R_1)^2.$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ P_1 & -P_2 & -G_1 & G_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix},$$

$$(ad) \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(G_1R_2-K_1P_2+G_2R_1-K_2P_1) + 2(K_1R_1+K_2R_2)(G_1K_2-G_2K_1+P_1R_2-P_2R_1)$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix} \equiv (a)(d)+2(b)(c),$$

$$(ad') \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(G_1K_2-G_2K_1+P_1R_2-P_2R_1) - 2(K_1R_1+K_2R_2)(G_1R_2-K_1P_2+G_2R_1-K_2P_1)$$

$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix} \equiv (ac'),$$

and among these $(A)(D) \equiv (ad)^2+(ad')^2$.

And we also have the identity, easily proved,

$$(C)-(c)^2 \equiv (D)-(d)^2.$$

These, then, introduce no new form.

In the same way

$$(E) \equiv (K_1S_2-L_1R_2+K_2S_1-L_2R_1)^2+(K_1L_2-K_2L_1+R_1S_2-R_2S_1)^2$$

$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ S_1 & -S_2 & -L_1 & L_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix},$$

$$(ae) \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(K_1S_2-L_1R_2+K_2S_1-L_2R_1)$$
$$+2\,(K_1R_1+K_2R_2)(K_1L_2-K_2L_1+R_1S_2-R_2S_1)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix} \equiv (a)(e)+2\,(b)(f),$$

$$(ae') \equiv (K_1^2-K_2^2-R_1^2+R_2^2)(K_1L_2-K_2L_1+R_1S_2-R_2S_1)$$
$$-2\,(K_1R_1+K_2R_2)(K_1S_2-L_1R_2+K_2S_1-L_2R_1)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix} \equiv (af'),$$

and we have $(A)(E) \equiv (ae)^2+(ae')^2$,

and also $(E)-(e)^2 \equiv (F)-(f)^2$.

No new form is introduced.

Again take $(\delta)$, $(\delta')$; $(\epsilon)$, $(\epsilon')$; we have

$$(de) \equiv (G_1R_2-K_1P_2+G_2R_1-K_2P_1)(K_1S_2-L_1R_2+K_2S_1-L_2R_1)$$
$$+(G_1K_2-G_2K_1+P_1R_2-P_2R_1)(K_1L_2-K_2L_1+R_1S_2-R_2S_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ S_1 & -S_2 & -L_1 & L_2 \\ K_1 & K_2 & R_1 & R_2 \\ -R_1 & R_2 & K_1 & -K_2 \end{vmatrix},$$

$$(de') \equiv (G_1R_2-K_1P_2+G_2R_1-K_2P_1)(K_1L_2-K_2L_1+R_1S_2-R_2S_1)$$
$$-(G_1K_2-G_2K_1+P_1R_2-P_2R_1)(K_1S_2-L_1R_2+K_2S_1-L_2R_1)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \end{vmatrix}.$$

These are related to the earlier forms by the equations

$$(A)(de) \equiv (ad)(ae)+(ad')(ae'),$$
$$(A)(de') \equiv (ad)(ae')-(ad')(ae),$$
$$(D)(E) \equiv (de)^2+(de')^2,$$

and also
$$(cf)-(c)(f) \equiv (de)-(d)(e),$$
$$(a)(cf')-2(b)(de')+(f)(ac')+(c)(af') \equiv 0.$$

6. Thus the consistency of all these equations requires the existence of only two new invariant forms. Further the same forms imply the consistency of several of the equations in $\lambda$, $\rho$. Thus take the equations $(a_1)$, $(\beta_1)$; $(a_2)$, $(\beta_2)$,

$$(ab_1) \equiv (G_1K_1+G_2K_2+P_1R_1+P_2R_2)\,2\,(K_1K_2-R_1R_2)$$
$$+(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(K_1^2+K^2+R_1^2+R_2^2)$$
$$\equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix} \equiv (ac') \equiv (ad'),$$

$$(ab_2) \equiv (L_1K_1+L_2K_2+S_1R_1+S_2R_2)\,2(K_1K_2-R_1R_2)$$
$$+(L_1K_2+L_2K_1-S_1R_2-S_2R_1)(K_1^2+K_2^2+R_1^2+R_2^2)$$
$$\equiv \begin{vmatrix} L_1 & L_2 & S_1 & S_2 \\ K_1 & K_2 & R_1 & R_2 \\ R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \end{vmatrix} \equiv (af') \equiv (ae').$$

Again eliminate $\lambda$, $\rho$ from $(a_1)$, $(a_2)$, $(\phi_1)$; we get

$$-(G_1S_1-L_1P_1-G_2S_2+L_2P_2)(K^2+K_2^2+R_1^2+R_2^2)$$
$$+(G_1K_1+G_2K_2+P_1R_1+P_2R_2)(K_1S_1-L_1R_1-K_2S_2+L_1R_2)$$
$$+(L_1K_1+L_2K_2+S_1R_1+S_2R_2)(G_1R_1-K_1P_1-G_2R_2+K_2P_2) \equiv (de').$$

In the same way, from $(\beta_1)$, $(\beta_2)$, $(\phi_1)$,

$$-(G_1S_1-L_1P_1-G_2S_2+L_2P_2)\,2\,(K_1K_2-R_1R_2)$$
$$+(K_1S_1-L_1R_1-K_2S_2+L_2R_2)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)$$
$$+(G_1R_1-G_1K_1-G_2R_2+P_2K_2)(K_1L_2+K_2L_1-S_1R_2-S_2R_1) \equiv (cf'),$$

and again $(\alpha_1)$, $(\alpha_2)$, $(\delta_1)$, lead to $(cf')$ and $(\beta_1)$, $(\beta_2)$, $(\delta_1)$ lead to $(de')$, with the equations :

$$-(G_1S_2-L_1P_2-G_2S_1+L_2P_1)(K_1^2+K_2^2+R_1^2+R_2^2)$$
$$+(K_1S_2-L_1R_2-K_2S_1+L_2R_1)(G_1K_1+G_2K_2+P_1R_1+P_2R_2)$$
$$+(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(L_1K_1+L_2K_2+S_1R_1+S_2R_2)\equiv(cf'),$$

and

$$-(G_1S_2-L_1P_2-G_2S_1+L_2P_1)\,2\,(K_1K_2-R_1R_2)$$
$$+(K_1S_2-L_1R_2-K_2S_1+L_2R_1)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)$$
$$+(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(L_1K_2+L_2K_1-S_1R_2-S_2R_1)\equiv(de').$$

Now consider the forms $(\omega_1)$, $(\omega_2)$ in conjunction with $(\varphi_1)$, $(\varphi_2)$. Eliminate $\lambda$ from $\omega_1$. We get the invariant

$$(G_1Q_1-H_1P_1-G_2Q_2+H_2P_2)(K_1S_1-L_1R_1-K_2S_2+L_2R_2)$$
$$-\tfrac{1}{4}(G_1S_1-L_1P_1-G_2S_2+L_2P_2+K_1Q_1-H_1R_1-K_2Q_2+H_2R_2)^2.$$

Eliminate $\rho$ from $\omega_2$. We get

$$(G_1R_1-K_1P_1-G_2R_2+K_2P_2)(H_1S_1-L_1Q_1-H_2S_2+L_2Q_2)$$
$$-\tfrac{1}{4}(G_1S_1-L_1P_1-G_2S_2+L_2P_2-K_1Q_1+H_1R_1+K_2Q_2-H_2R_2)^2.$$

The difference of these forms gives

$$\begin{vmatrix} G_1 & H_1 & K_1 & L_1 \\ P_1 & Q_1 & R_1 & S_1 \\ P_2 & Q_2 & R_2 & S_2 \\ G_2 & H_2 & K_2 & L_2 \end{vmatrix} \equiv (K).$$

The form $(K)$ is an addition to those already found. It may be found in the same way, using $(\zeta_1)$, $(\zeta_2)$, in place of $(\omega_1)$ $(\omega_2)$. We have not yet exhausted the forms that may be derived from these equations, but we shall stop here, because the discussion has already grown somewhat elaborate, and all that remain become ciphers in the optical problem.

7. Let us now introduce the six fundamental optical equations.

These render nugatory $(c)$, $(e)$, $(g)$, $(\omega_2)$, $(\zeta_1)$, and make $(\varphi_2)$, $(\delta_2)$ repetitions of $(\varphi_1)$, $(\delta_1)$. Moreover, they reduce the number of independent quantities contained in the coefficients, other than $\lambda$, $\rho$, $\phi$, $\psi$, to six, so that only six of the above-found invariant forms can

be independent. We have already seen that two forms, in addition to $(a)$, $(b)$, $(d)$, $(f)$, will reconcile all the above equations, excepting $(\omega_1)$, $(\zeta_2)$, and these are reconciled by a seventh form $(K)$. We must then express $(K)$ in terms of the other six.

We have

$$(K)(A) \equiv \begin{vmatrix} G_1 & G_2 & P_1 & P_2 \\ H_1 & H_2 & Q_1 & Q_2 \\ K_1 & K_2 & R_1 & R_2 \\ L_1 & L_2 & S_1 & S_2 \end{vmatrix} \begin{vmatrix} R_1 & -R_2 & -K_1 & K_2 \\ R_2 & -R_1 & K_2 & -K_1 \\ K_1 & K_2 & R_1 & R_2 \\ K_2 & K_1 & -R_2 & -R_1 \end{vmatrix}$$

$$\equiv \begin{vmatrix} 0 & (d) & (G_1K_1) & (G_1K_2) \\ -(K_1Q_1) & -(K_1Q_2) & (H_1K_1) & (H_1K_2) \\ 0 & 0 & (a) & 2\,(b) \\ -(f) & 0 & (L_1K_1) & (L_1K_2) \end{vmatrix},$$

where we have written

$$(K_1Q_1) \quad \text{for} \quad K_1Q_1 - H_1R_1 - K_2Q_2 + H_2R_2,$$
$$(K_1Q_2) \quad \text{for} \quad K_1Q_2 - H_1R_2 - K_2Q_1 + H_2R_1,$$
$$\&c.\ ;$$

but
$$(K_1Q_1) = G_1S_1 - L_1P_1 - G_2S_2 + L_2P_2, \qquad (3), (4)$$
$$-(K_1Q_2) = G_1S_2 - L_1P_2 - G_2S_1 + L_2P_1. \qquad (3'), (4')$$

Hence
$$(A)(K) = (G_1S_1)(d)(ae') + (G_1S_2)(f)(ac') + (d)(f)\{2\,(b)(H_1K_1) - (a)(H_1K_2)\}.$$

But
$$(cf') = -2\,(b)(G_1S_1) + (f)(G_1K_2),$$
$$(de') = -2\,(b)(G_1S_2) + (d)(L_1K_2).$$

Therefore

$$2\,(b)(A)(K) + (d)(ae')(cf') + (f)(ac')(de')$$
$$= (d)(f)\{(ae')(G_1K_2) + (ac')(L_1K_2) + 4\,(b)^2\,(H_1K_1) - 2\,(a)(b)(H_1K_2)\}$$
$$= (d)(f)\{(G_1K_2)\big[(a)(L_1K_2) - 2\,(b)(L_1K_1)\big] + (L_1K_2)\big[(a)(G_1K_2) - 2\,(b)(G_1K_1)\big] + 4\,(b)^2\,(H_1K_1) - 2\,(a)(b)(H_1K_2)\}.$$

Consider the terms in $\{\ \}$.

The term

$$-2(a)(b)(H_1K_2+H_2K_1-Q_1R_2-Q_2R_1)$$
$$=-2(a)\{(K_2^2+R_1^2)(H_1K_1+Q_2R_2)+(K_1^2+R_2^2)(H_2K_2+Q_1R_1)\}$$
$$+2(a)(K_1R_1+K_2R_2)(G_1S_1+L_1P_1+G_2S_2+L_2P_2),$$

using (3) and (4).

The term

$$2(a)(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(L_1K_2+L_2K_1-S_1R_2-S_2R_1)$$
$$=\ 2(a)\{(K_2^2+R_1^2)(G_1L_1+P_2S_2)+(K_1^2+R_2^2)(G_2L_2+P_1S_1)\}$$
$$-2(a)(K_1R_1+K_2R_2)(G_1S_1+L_1P_1+G_2S_2+L_2P_2)$$
$$+2(a)(b)(G_1L_2+G_2L_1-P_1S_2-P_2S_1).$$

Together these terms give

$$2(a)^2+2(a)(b)(G_1L_2+G_2L_1-P_1S_2-P_2S_1).$$

To the terms

$$-2(b)\big[(G_1K_2+G_2K_1-P_1R_2-P_2R_1)(L_1K_1+L_2K_2+S_1R_1+S_2R_2)$$
$$+(L_1K_2+L_2K_1-S_1R_2-S_2R_1)(G_1K_1+G_2K_2+P_1R_1+P_2R_2)\big]$$

add the term

$$-2(b)\big[(G_1R_1-K_1P_1-G_2R_2+K_2P_2)(K_1S_2-L_1R_2-K_2S_1+L_2R_1)$$
$$+(G_1R_2-P_1K_2-G_2R_1+P_2K_1)(K_1S_1-L_1R_1-K_2S_2+L_2R_2)\big]$$
$$=-2(b)(d)(f).$$

The sum is

$$-2(b)\{2(b)\big[G_1L_1+G_2L_2+P_1S_1+P_2S_2\big]$$
$$+(a)\big[G_1L_2+G_2L_1-P_1S_2-P_2S_1\big]\}.$$

Add in the terms already considered, and also the remaining term

$$2(b).2(b)\big[H_1K_1+H_2K_2+Q_1R_1+Q_2R_2\big].$$

Altogether we get $\quad 2(a)^2-8(b)^2=2(A).$

Thus the required expression for $(K)$ is

$$2(b)(A)(K)+(d)(ae')(cf')+(f)(ac')(de')$$
$$=2(d)(f)(A)+2(b)(d)^2(f)^2.$$

8. Let us adopt the following definition :—

*Two optical systems are called equivalent when the original beams in each and also the corresponding emergent beams may be identified ray with ray by mere screw displacements.*

Thus, if the systems are defined by means of schemes of coefficients $G_1H_1$, ..., $g_1'h_1'$, ..., these schemes must differ only in respect to the quantities $\lambda, \rho, \phi, \psi$. We may then consider it almost self-evident that *two optical systems are equivalent when they give the same values to the invariants* $(a)$, $(b)$, $(d)$, $(f)$, $(ac')$, $(af')$, *or any other six implying these;* but, on account of the importance of the result, we shall add a direct proof of it.

Determine two angles $\phi, \psi$ by means of the equations

$$K_1^2-K_2^2+R_1^2-R_2^2 = (k_1'^2-k_2'^2+r_1'^2-r_2'^2)\cos 2\phi+2(k_1'r_2'+k_2'r_1')\sin 2\phi,$$

$$\text{or } 2(K_1R_2+K_2R_1) = -(k_1'^2-k_2'^2+r_1'^2-r_2'^2)\sin 2\phi+2(k_1'r_2'+k_2'r_1')\cos 2\phi,$$

and

$$K_1^2-K_2^2-R_1^2+R_2^2 = (k_1'^2-k_2'^2-r_1'^2+r_2'^2)\cos 2\psi+2(k_1'r_1'+k_2'r_2')\sin 2\psi,$$

$$\text{or } 2(K_1R_1+K_2R_2) = -(k_1'^2-k_2'^2-r_1'^2+r_2'^2)\sin 2\psi+2(k_1'r_1'+k_2'r_2')\cos 2\psi.$$

It is immaterial which of these we use, because, $(a)$, $(b)$, and therefore (A), having the same value for each, their consistency is implied. Then, by rotations of the axes $(\varphi, \psi)$ in the scheme $G_1H_1\ldots$, transform it to a scheme $g_1h_1\ldots$. The equations that effect this transformation are $(\alpha)$, $(\alpha')$, $(\beta)$, $(\beta')$, &c., above. Compare these with the equations above in $k_1'$, &c., and we see that

$$k_1 = k_1', \quad r = r_1', \quad r_2 = r_2', \quad k_2 = k_2',$$

with alternatives that are of no importance.

In exactly the same way,

$$g_1r_1-k_1p_1-g_2r_2+k_2p_2 = g_1'r_1'-k_1'p_1'-g_2'r_2'+k_2'p_2',$$

$$g_1r_2-k_1p_2-g_2r_1+k_2p_1 = g_1'r_2'-k_1'p_2'-g_2'r_1'+k_2'p_1',$$

and, since $(ac')$ has the same value in the two systems,

$$g_1r_1-k_1p_1+g_2r_2-k_2p_2 = g_1'r_1'-k_1'p_1'+g_2'r_2'-k_2'p_2',$$

$$g_1k_2-g_2k_1-p_1r_2+p_2r_1 = g_1'k_2'-g_2'k_1'-p_1'r_2'+p_2'r_1';$$

whence

$$\frac{g_1-g_1'}{k_1} = \frac{p_1-p_1'}{r_1} = \frac{p_2-p_2'}{r_2} = \frac{g_2-g_2'}{k_2},$$

or $\quad g_1' = g_1 + \lambda k_1, \quad p_1' = p_1 + \lambda r_1, \quad p_2' = p_2 + \lambda r_2, \quad g_2' = g_2 + \lambda k_2,$

and, in the same way,

$$l_1' = l_1 + \rho k_1, \quad s_1' = s_1 + \rho r_1, \quad s_2' = s_2 + \rho r_2, \quad l_2' = l_2 + \rho k_2.$$

Therefore shifts of the origins complete the identification, since the equality of $h_1 q_1 q_2 h_2$ with $h_1' q_1' q_2' h_2'$ is implied by the fundamental equations (1), (6), (1'), (6').

9. *Any optical transformation is equivalent to direct passage through a thick lens whose curvatures are unequal in each face and in different planes in the two faces.* This is equivalent to a theorem given by Larmor (*Proceedings*, Vol. xx.). We shall show by actual determination of them that the six features that determine such a lens appear in such a manner as to give independent values to six of the invariants. We shall use the invariants $(b)$, $(d)$, $(f)$, $(D)$, $(F)$, $(K)$. In the scheme of Part I., p. 47, write

$$1 + a_1 c = cX, \quad 1 + b_1 c = cY, \quad 1 + a_2 c = cx, \quad 1 + b_2 c = cy,$$

and we have

$$\left\{ \begin{array}{cccc} Xc\cos\theta & c\cos\theta & Yc\sin\theta & c\sin\theta \\ \left(cXx - \frac{1}{c}\right)\cos\theta & cx\cos\theta & \left(cYx - \frac{1}{c}\right)\sin\theta & cx\sin\theta \\ -Xc\sin\theta & -c\sin\theta & Yc\cos\theta & c\cos\theta \\ -\left(cXy - \frac{1}{c}\right)\sin\theta & -cy\sin\theta & \left(cYy - \frac{1}{c}\right)\cos\theta & cy\cos\theta \end{array} \right\};$$

whence

$$(b) = c^2 XYxy - \tfrac{1}{2}(X+Y)(x+y) + \frac{1}{c^2} - \tfrac{1}{2}\cos 2\theta\,(X-Y)(x-y),$$

$$(d) = \tfrac{1}{2}c^2 \sin 2\theta\,(X-Y)(x-y)(X+Y),$$

$$(f) = \tfrac{1}{2}c^3 \sin 2\theta\,(X-Y)(x-y)(x+y),$$

$$(D) = (X-Y)^2 + 2c^2 \cos 2\theta\,(X-Y)(x-y)\,XY + c^4 (x-y)^2 X^2 Y^2,$$

$$(F) = (x-y)^2 + 2c^2 \cos 2\theta\,(X-Y)(x-y)\,xy + c^4 (X-Y)^2 x^2 y^2,$$

$$(K) = -\tfrac{1}{4}c^4 \sin^2 2\theta\,(X-Y)^2 (x-y)^2.$$

These equations admit of algebraic solution, for we have

$$X+Y=(d)/\sqrt{-(K)},$$

$$x+y=(f)/\sqrt{-(K)},$$

$$(b)-\tfrac{1}{2}\frac{(d)(f)}{(K)}=c^2XYxy+\frac{1}{c^2}-\frac{1}{c^2}\cot 2\theta\sqrt{-(K)},$$

$$(D)=-\frac{(d)^2}{(K)}-4c^2XY\left\{\frac{1}{c^2}-\frac{1}{c^2}\cot 2\theta\sqrt{-(K)}\right\}+c^4(x-y)^2X^2Y^2.$$

Eliminate $\cot 2\theta$ from between these;

$$(f)^2c^4X^2Y^2+2c^2XY\{2(b)(K)-(d)(f)\}+(d)^2+(K)(D)=0.$$

This gives $c^2XY$; in the same way, $c^2xy$ is found from

$$(d)^2c^4x^2y^2+2c^2xy\{2(b)(K)-(d)(f)\}+(f)^2+(K)(F)=0,$$

and, eliminating $\cos 2\theta$ between $(D)$ and $(F)$, we get, to determine $c$,

$$\frac{4}{c^2}(\xi-\Xi)=(D)\xi-(F)\Xi-\frac{1}{(K)}\{(d)^2(\xi^2\Xi-1)-(f)^2(\xi\Xi^2-1)\},$$

where, for clearness, $\xi$, $\Xi$ are written in place of $c^2xy$, $c^2XY$. $\theta$ is then easily found.

It appears that there are four such lenses, all real if any are;* but under what circumstances they are real we cannot usefully discuss without knowing what, besides real optical systems, is contained by a scheme of coefficients defined only by obedience to the fundamental equations; it is the less important to discover, because the equivalence is illusory in a very important class of cases. Namely, if $(d)$, $(f)$ are finite, while $(K)=0$, all the curvatures are infinite, and this is the class which possesses a single pair of planes whereon point corresponds to point, as we shall immediately prove.

10. Introduce the scheme

$$b=(gb'+h\beta'+c')\frac{\sigma}{\varpi_1},$$

$$\beta=(kb'+l\beta'+\gamma')\varpi_1,$$

$$c=(b'-lc'+h\gamma')\frac{\sigma}{\varpi_2},$$

$$\gamma=(\beta'+kc'-g\gamma')\varpi_2,$$

where $$\sigma^{-1}=gl-hk+1.$$

* Generally one only is equivalent to the given system; $(D)$ and $(F)$ above would give $\pm(ac')$, $\pm(af')$ in § 8.

This scheme satisfies the fundamental equations, involves six symbols only, and is capable of representing any system whatever; as we may see by noticing that the forms

$$(a) = k^2(\varpi_1^2+\varpi_2^2),$$

$$(b) = k^2\varpi_1\varpi_2,$$

$$(d) = k\sigma(\varpi_2^2-\varpi_1^2)/\varpi_1\varpi_2,$$

$$(f) = k(\varpi_1^2-\varpi_2^2),$$

$$(ac') = k^3\sigma(\varpi_1-\varpi_2^2)\left(g\frac{\varpi_2}{\varpi_1}+l\frac{\varpi_1}{\varpi_2}\right),$$

$$(af') = k^3\varpi_1\varpi_2(\varpi_2^2-\varpi_1^2)(g+l),$$

are independent; in such a scheme every letter is an invariant; if a name is wanted, it may be called canonical. Now here

$$(K) = hk\sigma^2(\varpi_1^2-\varpi_2^2)^2/\varpi_1^2\varpi_2^2,$$

and, if $(K) = 0$, while $(d)$, $(f)$ are finite, we must have $h = 0$, which implies

$$b = (gb'+c')\sigma/\varpi_1,$$

$$c = (b'-lc')\sigma/\varpi_2,$$

or points in the planes $Oyz$, $O'y'z'$ correspond one with another. When such a pair of planes in correspondence exist, we shall call them principal planes, because, as we shall see in Part III., they form the extreme generalization of principal planes of Gauss.

## III.

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1. It is our next task to elicit the properties of an optical system from the foregoing mass of algebra.

In a system symmetrical about an axis there exists a complete one-to-one correspondence of point to point, and from this fact alone all properties of such a system, excepting the relation of linear and angular magnification, can, and I should say ought to, be deduced; the excepted relation, generally known as Helmholtz's theorem, is

the interpretation of Gauss's equation

$$gl-hk=1.$$

The want of a one-to-one correspondence has been so far the main and effectual bar to any detailed theory of an unsymmetrical system, and therefore it is of the first importance to remark that such a correspondence not only exists completely, but exists twice over. To enunciate this statement formally I shall take a slight liberty with established language.

*In any two narrow beams related linearly ray to ray, there exists a system of lines perpendicular to the original axis, corresponding one to one with a system perpendicular to the emergent axis, and such that the triply infinite complex of rays which meet one line emerge and meet its conjugate also. Let such a pair of lines be called conjugate focal lines. It follows that, out of the doubly infinite complex issuing from any point upon a focal line, a singly infinite complex passes through each point upon its conjugate line; this singly infinite complex lies in a plane in the original and also in the emergent system. Let such planes be called "planes of correspondence."* Through any point two focal lines pass in directions which the position of the point and the constants of the system determine.*

*In particular in an optical system every plane of correspondence is perpendicular to the associated focal line, and every two focal lines that intersect are conjugate to two which are at right angles.*

2. To prove these statements let us begin by considering the conditions under which two rays which meet once will meet again. Let one of the rays be the axis, and let the other meet it at the origin; then

$$0=b=G_1b'+H_1\beta'+P_1c'+Q_1\gamma',$$

$$0=c=P_2b'+Q_2\beta'+G_2c'+H_2\gamma'.$$

If the emergent ray meets the axis at a distance $\mu'a'$ from the origin, then these are consistent with

$$b'+a'\beta'=0,\quad c'+a'\gamma'=0;$$

or

$$a'(G_1\beta'+P_1\gamma')-H_1\beta'-Q_1\gamma'=0,$$

$$a'(P_2\beta'+G_2\gamma')-Q_2\beta'-H_2\gamma'=0.$$

* The existence of such planes has been remarked by Larmor.

Eliminate $\alpha'$; then

$$\beta'^2 (G_1 Q_2 - H_1 P_2) + \beta'\gamma' (G_1 H_2 - G_2 H_1 + P_1 Q_2 - P_2 Q_1) + \gamma'^2 (P_1 H_2 - G_2 Q_1) = 0.$$

The two solutions of this $(\beta' : \gamma' = c_1', c_2')$ give two emergent planes of correspondence, associated, as we shall presently see, with the two focal lines that meet at $O$. They are at right angles in virtue of (2'); associated with each is a single point $(\mu' a_1', 0, 0;\ \mu' a_2', 0, 0)$ and a single original plane of correspondence $(\beta : \gamma = c_1, c_2)$.

3. Now, let us simplify, by a rotation of axes $(\psi)$ such as to make

$$G_1 Q_2 - H_1 P_2 = G_2 Q_1 - H_2 P_1 = 0,$$

so that the coordinate planes are the emergent planes of correspondence, and

$$\frac{G_1}{H_1} = \frac{P_2}{Q_2} = \frac{1}{a_2'}, \qquad \frac{G_2}{H_2} = \frac{P_1}{Q_1} = \frac{1}{a_1'};$$

then the equations connecting any two rays $(B\beta C\gamma)$, $(B'\beta'C'\gamma')$ are

$$\begin{aligned} B' &= \ \ L_1 B + S_2 C - a_2' (G_1 \beta + P_2 \gamma), \\ \beta' &= -K_1 B - R_2 C + \ \ G_1 \beta + P_2 \gamma, \\ C' &= \ \ S_1 B + L_2 C - a_1' (P_1 \beta + G_2 \gamma), \\ \gamma' &= -R_1 B - K_2 C + \ \ P_1 \beta + G_2 \gamma; \end{aligned}$$

and the planes of correspondence in the original system are

(1) associated with $\beta' = 0$, $\mu' a_1'$, $\quad 0 = G_1 \beta + P_2 \gamma$,

(2) „ „ $\gamma' = 0$, $\mu' a_2'$, $\quad 0 = P_1 \beta + G_2 \gamma$.

Now, eliminate $\beta, \gamma$ from between $C', \gamma'$, and we see that, if any ray meets the line

$$x' = \mu' a_1', \quad z' = C_1' = C' + a_1' \gamma',$$

then it also meets the line

$$\begin{aligned} x = 0, \quad C_1' &= \ \ (S_1 - a_1' R_1)\ B + \ \ (L_2 - a_1' K_2)\ C \\ &= (P_1 S_1 - Q_1 R_1) \frac{B}{P_1} + (G_2 L_2 - H_2 K_2) \frac{C}{G_2}. \end{aligned}$$

But consider $\quad G_1 Q_2 - H_1 P_2 = 0, \quad G_2 Q_1 - H_2 P_1 = 0, \qquad (2')$

$$G_1 S_2 - H_1 R_2 + P_1 L_2 - Q_1 K_2 = 0; \qquad (3')$$

from these $\quad \frac{G_1}{P_2} (P_2 S_2 - Q_2 R_2) + \frac{P_1}{G_2} (G_2 L_2 - H_2 K_2) = 0;$

hence, in virtue of (1), (6), (1′), (6′),

$$\frac{G_2L_2-H_2K_2}{G_1G_2}=\frac{P_1S_1-Q_1R_1}{-P_1P_2}=\frac{1}{G_1G_2-P_1P_2},$$

or

$$C_1'=\frac{-P_2B+G_1C}{G_1G_2-P_1P_2},$$

and in the same way, if an emergent ray meets

$$x'=\mu'a_2', \quad y'=B_2'=B'+a_2'\beta'$$

then it follows that the original ray meets

$$x=0, \quad B_2'=\frac{G_2B-P_1C}{G_1G_2-P_1P_2}.$$

These results contain the statements of p. 68; indeed, they contain more, namely, referring to the figure:—

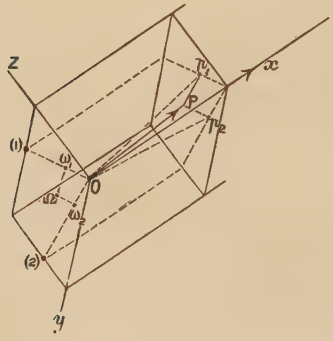
*If* $\Omega\omega_1$ *be a focal line passing through* $\Omega$, *and* $\Omega_1'\omega_1'$ *its conjugate, and similarly a pair of conjugates parallel to these pass through* $O$, $A_1'$, *then the distance* $O\omega_1$ *between the lines at* $O$, $\Omega$ *is related to the distance* $A_1'\Omega_1'$ *between their conjugates by the equation*

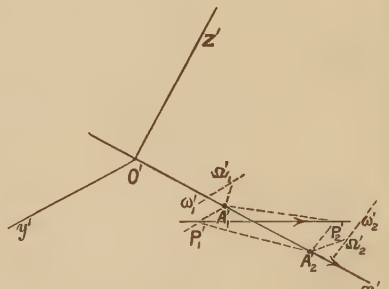
$$A_1'\Omega_1'=\frac{(G_1^2+P_2^2)^{\frac{1}{2}}}{G_1G_2-P_1P_2}\,O\omega_1,$$

*and, in the same way, if* $\Omega\omega_2$, $\Omega_2'\omega_2'$ *be conjugates of the other system,*

$$A_2'\Omega_2'=\frac{(G_2^2+P_1^2)^{\frac{1}{2}}}{G_1G_2-P_1P_2}\,O\omega_2.$$

These two multipliers shall be called the two linear magnifications for the point $O$; they are determined by the position of $O$ and the constants of the instrument, and their values for any point, together with the position and directions of the focal lines, are given below.



4. Now consider the magnification of angles for any ray through $O$. If $B$, $C$ vanish,

$$\beta' = G_1\beta + P_2\gamma,$$

$$\gamma' = P_1\beta + G_2\gamma;$$

these quantities vanish respectively when the ray $OP$ is in the first or second plane of correspondence through $O$, and when this is not the case it may be seen with little calculation that

$$P_1\beta + G_2\gamma = \mu \frac{G_1G_2 - P_1P_2}{(G_1^2 + P_2^2)^{\frac{1}{2}}} \angle p_1Ox,$$

where $Pp_1$ is parallel to $O$ (2); also

$$\gamma' = \mu \angle P_2'A_1'A_2',$$

if $P_1'P_2'$ be the emergent ray corresponding to $OP$; hence we have a generalization of Helmholtz's theorem

$$\mu \,.\, O\omega_1 \angle p_1Ox = \mu' \,.\, A_1'\Omega_1' \angle P_2'A_1'A_2',$$

$$\mu \,.\, O\omega_2 \angle p_2Ox = \mu' \,.\, A_1'\Omega_2' \angle P_1'A_2'A_1'.$$

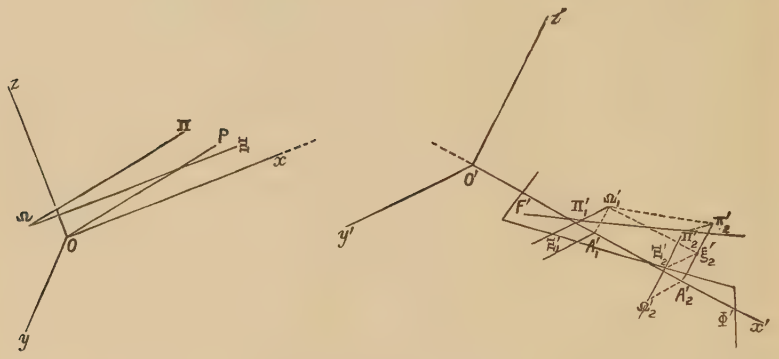
We notice that, if a ray be projected thus upon either plane of correspondence through any point of itself and upon the associated emergent plane, then its projections are optically related like original and emergent rays of the beam.

5. The foregoing results supply several alternative constructions for the emergent ray corresponding to any incident ray.

i. *Determination of Emergent Ray by means of two Points on Original Ray.*—Let $\Omega$, $Z$, be the two points; find the focal lines $\Omega_1'$, $\Omega_2'$, $Z_1'$, $Z_2'$ whose conjugates pass through $\Omega$, $Z$; the emergent ray is the ray meeting these four lines.*

ii. *Determination by means of Point and Direction.*—Let the ray be $\Omega\Pi$ parallel to $OP$. First find the emergent ray corresponding to $\Omega\Xi$, a parallel to $Ox$; this is the ray which meets the focal lines at $\Omega_1'$, $\Omega_2'$, and also the principal focal lines at $F'$, $\Phi'$ whose conjugates are at infinity. This is the ray ($B$, 0, $C$, 0), and

* Observe that two lines meet any four, but in this case only one is eligible; for the lines are parallel to a plane, so that three are generators of a paraboloid, and the fourth parallel to a generator; hence one intersection with this paraboloid is at infinity and must be rejected.

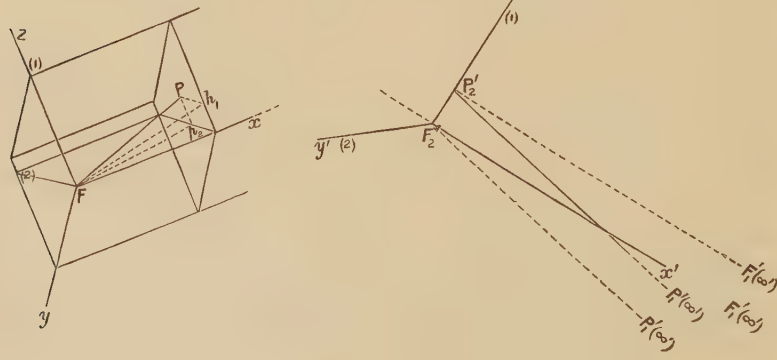



we see, from the equations of p. 69, that the projections of the emergent ray $\Pi_1'\Pi_2'$ make with the projections of $\Xi_1'\Xi_2'$ angles equal to $\angle P_2'A_1'A_2'$, $\angle P_1'A_2'A_1'$ respectively; evidently this determines the emergent ray.

6. We must now consider the problem :—*Given the position of a point O, to express the directions of the focal lines that meet in O, the positions and directions of their conjugates, and the magnification associated with each pair in terms of fixed geometrical features or algebraic invariants of the system.*

The chief fixed features of a system are the principal focal lines at $F$, $\Phi$ whose conjugates are at $\infty'$, and those at $F'$, $\Phi'$ whose conjugates are at $\infty$.

Consider $F$ as an origin of rays. The focal lines conjugate to those at $F$ are at $\infty'$ and another point $F_2'$. All rays from $F$ in the first plane of correspondence emerge parallel to $O'x'$ in the plane



$z'O'x'$; all rays in the second plane of correspondence lie in the plane $y'O'x'$, and meet the foot of this line $(F_2)$; all other rays meet the line at other points and are parallel to the plane $y'O'x'$. Let us form the scheme of coefficients with $F$ as origin, $Ox$ as the first plane of correspondence, and $F_2'$ as emergent origin. Then, if the scheme is

$$(b'c'\beta'\gamma') = \begin{Bmatrix} l_1 & -h_1 & s_2 & -q_2 \\ -k_1 & g_1 & -r_2 & p_2 \\ s_1 & -q_1 & l_2 & -h_2 \\ -r_1 & p_1 & -k_2 & g_2 \end{Bmatrix} (bc\beta\gamma),$$

we must have, when $b$, $c$ vanish, $\gamma' = 0$, $b' = 0$, and $\beta' = 0$, implying $\beta = 0$; hence

$$p_1 = g_2 = h_1 = q_2 = p_2 = 0;$$

hence, from (2), $r_1 = 0$; from (3'), $q_1 = ng_1$, $s_2 = nk_2$; from (1), $g_1 l_1 = 1$; from (6), $h_2 k_2 = -1$; from (3), $g_1 s_1 - q_1 k_1 - h_2 r_2 = 0$; hence the scheme may be written

$$(bc\beta\gamma) = \begin{Bmatrix} g_1 & 0 & 0 & ng_1 \\ k_1 & \frac{1}{g_1} & 0 & s_1 \\ 0 & 0 & 0 & -\frac{1}{k_2} \\ r_2 & nk_2 & k_2 & l_2 \end{Bmatrix} (b'c'\beta'\gamma'),$$

where $$s_1 = nk_1 - r_2/g_1 k_2.$$

This is a reduction of the general scheme to expression by six independent letters—what we have called a canonical reduction on p. 67.

Now let us change the origin and axes until they occupy the corresponding position with respect to $\Phi$ which they now occupy with respect to $F$. Let these changes be $\lambda = f$, $\rho = d$, $\phi = \frac{\pi}{2}$, $\psi = \alpha$; $\phi$ is a right angle because the planes of correspondence in question are both associated with a single point $\infty'$; the distance $F\Phi = \mu f$, $F_2'\Phi_2' = \mu' d$; let Greek letters denote the coefficients arrived at for $\Phi$. Then, in the first place, $\rho_1 = 0$, which gives

$$0 = -r_2 \sin\alpha + k_2 \cos\alpha,$$

while $$\kappa_1 = r_2 \cos\alpha + k_2 \sin\alpha;$$

hence $$r_2 = \kappa_1 \cos\alpha, \quad k_2 = \kappa_1 \sin\alpha;$$

in future we shall employ $\alpha$, $\kappa$, and drop subscript marks. Then the result of the change $\phi = \frac{\pi}{2}$ is

$$\begin{Bmatrix} 0 & 0 & 0 & -\frac{1}{\kappa \sin\alpha} \\ \kappa\cos\alpha & n\kappa\sin\alpha & \kappa\sin\alpha & l \\ -g & 0 & 0 & -ng \\ -k & -\frac{1}{g} & 0 & -s \end{Bmatrix},$$

and the further change $\psi = \alpha$ gives

$$\begin{Bmatrix} 0 & -\frac{1}{\kappa} & 0 & -\frac{1}{\kappa}\cot\alpha \\ \kappa & n\kappa\sin\alpha\cos\alpha + l\sin\alpha & 0 & -n\kappa\sin^2\alpha + l\cos\alpha \\ -g\cos\alpha & -ng\sin\alpha & g\sin\alpha & -ng\cos\alpha \\ -k\cos\alpha & -\frac{1}{g}\cos\alpha - s\sin\alpha & k\sin\alpha & \frac{1}{g}\sin\alpha - s\cos\alpha \end{Bmatrix}.$$

Further changes $\lambda = f$, $\rho = d$ make

$$h_1 = p_1 = p_2 = q_2 = g_2 = 0,$$

and $$g_1 = \gamma, \quad q_1 = \nu\gamma, \quad l_1 = \frac{1}{\gamma}, \quad s_1 = \nu\kappa + \frac{1}{\gamma}\cot\alpha,$$

&c., &c.

Hence $$\gamma = f\kappa,$$

$$0 = p_2 = -\cos\alpha\,(g + fk);$$

therefore $$g = -fk.$$

$$l_1 = \frac{1}{\gamma} = n\kappa\sin\alpha\cos\alpha + l\sin\alpha + \kappa d,$$

$$s_1 = \nu\kappa + \frac{1}{\gamma}\cot\alpha = -n\kappa\sin^2\alpha + l\cos\alpha.$$

Eliminate $l$: $$-\nu\kappa\sin\alpha = n\kappa\sin\alpha + \kappa d\cos\alpha,$$

or $$n + \nu + d\cot\alpha = 0.$$

Solve for $l$:
$$l = \nu\kappa\cos\alpha + \frac{1}{\gamma\sin\alpha} - d\kappa\sin\alpha$$
$$= (n\sin^2\alpha+\nu)\frac{\kappa}{\cos\alpha} + \frac{1}{f\kappa\sin\alpha}.$$

It may be verified that these results are got over and over again by taking other of the coefficients $g, h_1$, &c. .... . Hence, if the system is specified by the six constants $k$, $\kappa$, $\alpha$, $f$, $n$, $\nu$, the scheme of coefficients reads :—$F$ as origin:

$$\left\{\begin{array}{cccc} -kf & 0 & 0 & nkf \\ k & -1/kf & 0 & nk+\cot\alpha/kf \\ 0 & 0 & 0 & -1/\kappa\sin\alpha \\ \kappa\cos\alpha & n\kappa\sin\alpha & \kappa\sin\alpha & (n\sin^2\alpha+\nu)\kappa/\cos\alpha+1/f\kappa\sin\alpha \end{array}\right\},$$

$\Phi$ as origin:

$$\left\{\begin{array}{cccc} \kappa f & 0 & 0 & -\nu\kappa f \\ \kappa & 1/\kappa f & 0 & \nu\kappa+\cot\alpha/\kappa f \\ 0 & 0 & 0 & -1/k\sin\alpha \\ -k\cos\alpha & \nu k\sin\alpha & k\sin\alpha & -(\nu\sin^2\alpha+n)k/\cos\alpha-1/fk\sin\alpha \end{array}\right\}.$$

7. The quantities we have employed are easily expressed in terms of algebraic invariants of the system

$$(a) = k^2+\kappa^2,$$
$$(b) = k\kappa\sin\alpha,$$
$$(d) = -k\kappa f\cos\alpha,$$
$$(f) = nk^2+\nu\kappa^2,$$
$$(ac') = -k^3\kappa f\sin\alpha\,(k^2+\kappa^2) = -(a)(b)\,k^2 f,$$
$$(cf') = -k\kappa f\sin\alpha\left[\nu\kappa^2-nk^2+\frac{2}{f}\cot\alpha\right],$$
$$(K) = -n\nu k^2\kappa^2 f^2-\cot^2\alpha$$
$$\text{&c.,} \qquad \text{&c.}$$

The solution of these is easy; we shall employ them to find the schemes of coefficients when the instrument is reversed, and $F'$ or $\Phi'$ is origin; let $k'$, $\kappa'$, &c., denote quantities corresponding to $k$, $\kappa$, &c.

Now when the instrument is reversed, out of the above invariants, $(a)$, $(b)$, $(K)$, $(cf')$ are unchanged in value, while $(d') = (f)$ and $(f') = (d)$. Hence we must solve

$$k'^2+\kappa'^2 = k^2+\kappa^2, \quad k'\kappa' \sin \alpha' = k\kappa \sin \alpha,$$

$$n'k'^3+\nu'\kappa'^2 = -k\kappa f \cos \alpha, \quad -k'\kappa' f' \cos \alpha' = nk^3+\nu\kappa^2,$$

$$n'\nu' k'^2\kappa'^2 f'^2+\cot^2 \alpha' = n\nu k^2\kappa^2 f^2+\cot^2 \alpha,$$

$$k'\kappa' f' \sin \alpha' \left[\nu'\kappa'^2-nk^2+\frac{2}{f'}\cot \alpha'\right] = k\kappa f \sin \alpha \left[\nu\kappa^2-nk^2+\frac{2}{f}\cot \alpha\right].$$

These are satisfied by the six results

$$k' = k, \quad \kappa' = \kappa, \quad \alpha' = \alpha,$$

$$f'n' = fn, \quad f'\nu' = f\nu,$$

$$k^2fn+\kappa^2f\nu+k\kappa ff' \cos \alpha = 0,$$

which also imply $$f'd' = fd.$$

We may use $f'$, and $d$ or $d'$, to express $n$, $\nu$, and then the four quantities $f$, $f'$, $\alpha$, $d$ have clear geometrical meanings, and may be found for any system by the simplest direct observations. The quantities $k$, $\kappa$ may also be found from observation in a variety of simple ways by help of the results of p. 78 below.

8. We shall have occasion to use the angle between a plane of correspondence at $F'$ and one at $F_2'$. Let us say, the first plane of correspondence for each; it may be found as follows:—

The scheme for $F'$ is equivalent to that for $F_2'$, and this latter is simply the scheme for $F$, inverted; consider the terms in $k_1$, $r_1$, $r_2$, $k_2$ in each. They are respectively,

for $F'$: $\quad k \quad 0 \quad \kappa \cos \alpha \quad \kappa \sin \alpha,$

for $F_2'$: $\quad -k \quad -\kappa \cos \alpha \quad 0 \quad -\kappa \sin \alpha.$

The rotation $(\phi)$ which is required to reconcile these equivalent schemes is given by equation $(\alpha)$ or $(\alpha')$ of p. 52;

$$(\alpha) \quad k^2+\kappa^2 \cos 2\alpha = (k^2-\kappa^2) \cos 2\phi+2k\kappa \cos \alpha \sin 2\phi,$$

$$(\alpha') \quad 2\kappa^2 \sin \alpha \cos \alpha = -(k^2-\kappa^2) \sin 2\phi+2k\kappa \cos \alpha \cos 2\phi,$$

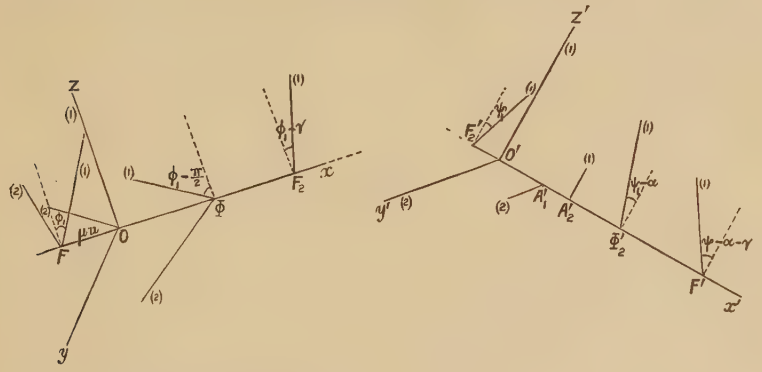
which are consistent. If we write

$$(k^4+\kappa^4+2k^2\kappa^2 \cos 2\alpha)^{\frac{1}{2}} = (k^2-\kappa^2)/\cos \delta = (k^2+\kappa^2 \cos 2\alpha)/\cos \epsilon,$$

then $$2\phi = \delta+\epsilon;$$

denote this value of $\phi$ by a single symbol $\gamma$. It may be easily shown that the corresponding angle upon the other side of the system is also equal to $\gamma$.

9. We are now in a position to answer the questions of p. 72; but first let us simplify the expressions by choosing the plane of correspondence of which we are speaking as the plane of $zOx$. Thus, for




the first plane of correspondence,

$$P_2 = 0,$$

$$A_1'\Omega_1' = \frac{1}{G_2} O\omega_1;$$

for the second plane of correspondence,

$$G_2 = 0,$$

$$A_2'\Omega_2' = -\frac{1}{P_2} O\omega_2.$$

Now suppose displacements from the origin $F$, $\lambda = u$, $\phi = \phi_1$, $\psi = \psi_1$, convert the scheme for $F$ into the scheme for another point $O$, distant $\mu u$ from $F$.

Then, for the point $O$,

$$\begin{aligned} G_2 = &-\sin\psi_1\left[-\sin\phi_1\,(-kf+ku)+\cos\phi_1\,.\,u\kappa\cos\alpha\right] \\ &+\cos\psi_1\left[\qquad\qquad\qquad\cos\phi_1\,.\,u\kappa\sin\alpha\right], \\ 0 = P_2 = &\cos\psi_1\left[-\sin\phi_1\,(\;ku-kf)+\cos\phi_1\,.\,u\kappa\cos\alpha\right] \\ &+\sin\psi_1\left[\qquad\qquad\qquad\cos\phi_1\,.\,u\kappa\sin\alpha\right]. \end{aligned}$$

Hence $$G_2 = u\kappa \sin \alpha \cos \phi_1 / \cos \psi_1,$$

and $$\tan \phi_1 = \frac{\kappa}{k} \frac{u}{u-f} \frac{\cos (\psi_1 - \alpha)}{\cos \psi_1}.$$

It may be verified that the same equations are found by starting from $\Phi$ as origin in place of $F$.

Again, if we take $F'$ as origin and proceed to $A_1'$, the angles $\phi$, $\psi$ interchange their *rôles*, and each is diminished by the angle $\gamma$. Hence if $\mu' u_1'$ be the distance $F'A_1'$, we have

$$\frac{1}{G_2} = u_1' \kappa \sin \alpha \cos (\psi_1 - \gamma) / \cos (\phi_1 - \gamma),$$

$$\tan (\psi_1 - \gamma) = \frac{\kappa}{k} \frac{u_1'}{u_1' - f'} \frac{\cos (\phi_1 - \gamma - \alpha)}{\cos (\phi_1 - \gamma)}.$$

These equations give $G_2$, $\phi_1$, $\psi_1$, $u_1'$, in terms of $u$, which solves the problem proposed. In the same way $\phi_2$, $\psi_2$, $u_2'$, and $P_2$ when $G_2 = 0$, may be found. Including the theorem on angular magnification already proved, we have thus a complete outline of the geometrical theory of a general optical system. It will be observed that most of its properties, with slight modifications, obtain in all systems, whether optical or not, where there is a linear relation between the rays.

10. Let us now compare our results with the corresponding relations in a symmetrical instrument, and, observing the successive steps by which the properties of the latter become generalized or disappear, and the associated values of the invariants, let us attempt a classification of optical systems in general.

I. When six invariants, say $(b)$, $(d)$, $(f)$, $(D)$, $(F)$, $(K)$, are all independent and finite, the system is completely general.

II. If $(K) = 0$, while $(d)$ and $(f)$ are finite, the system possesses one pair of planes whereon point corresponds to point.

III. If, in addition to $(K) = 0$, $(d)$, $(f)$ finite, we have a further relation, namely,

$$(cf')(de') = (a)(d)(f) + (b)\{(d)^2 + (f)^2\},$$

the system is equivalent to oblique refraction at a single surface. This may be seen by forming the invariants of the scheme which belongs to a single surface, and effecting the elimination (other than

$(K) = 0$) of the four quantities on which they depend; if we write the scheme

$$\left\{\begin{matrix} 1 & 0 & 0 & 0 \\ x & 1 & \varpi z & 0 \\ 0 & 0 & \varpi & 0 \\ z & 0 & y & 1/\varpi \end{matrix}\right\},$$

we find $(a) = x^2+y^2+z^2(1+\varpi^2)$, $(b) = xy-\varpi z^2$,

$(d) = z(1-\varpi^2)$, $(f) = z(1-\varpi^2)/\varpi$,

$(cf') = (f)(\varpi x+y)$, $(de') = (d)(x/\varpi+y)$,

$(D) = (x\varpi-y)^2+z^2(1+\varpi^2)^2$,

$(F) = (x/\varpi-y)^2+z^2(\varpi+1/\varpi)^2$.

Eliminating, we get the relation above. The limitation which this imposes upon the general scheme of p. 67 is worth notice; there

$$(cf') = k^2\sigma(\varpi_1^2-\varpi_2^2)\left(g\frac{\varpi_2}{\varpi_1}-l\frac{\varpi_1}{\varpi_2}\right),$$

$$(de') = k^2\sigma(\varpi_1^2-\varpi_2^2)(g-l);$$

hence, removing $k^4(\varpi_1^2-\varpi_2^2)^2$, the relation is

$$(g-l)\left(g\frac{\varpi_2}{\varpi_1}-l\frac{\varpi_1}{\varpi_2}\right) = \left(\frac{1}{\varpi_1}-\frac{\varpi_1}{\sigma}\right)\left(\frac{1}{\varpi_2}-\frac{\varpi_2}{\sigma}\right),$$

which may be written

$$(g\varpi_2^2-l\varpi_1^2)_2 = \left(g^2\varpi_2^2+\varpi_1^2-\frac{\varpi_1^2\varpi_2^2}{\sigma^2}\right)\left(l\varpi+\varpi-\frac{\varpi_1^2\varpi_2^2}{\sigma^2}\right),$$

and this is the condition that the circle $x^2+y^2 = c$ and the ellipse

$$(gx+y)^2\sigma^2/\varpi_1^2+(x-ly)^2\sigma^2/\varpi_2^2 = c$$

have double contact. Now in a single surface the planes where point corresponds to point are evidently situated at the surface itself, and corresponding points are the projections of any, the same, point of the surface on planes perpendicular to the original and emergent axes respectively; and the traces of such points are evidently related as above, namely, one may be a circle and the other an ellipse, which, if in the same plane, would be concentric and in double contact.

IV. $(d)$ cannot vanish without $(f)$ in ordinary systems produced by finite curvatures. If $(d)$, $(f)$ both vanish, we see that, in the equivalent lens of p. 65, $\theta = 0$, or the principal curvatures of the two faces of the lens are parallel; also, from p. 75, $\alpha = \frac{\pi}{2}$, and this makes $n$, $\nu$ vanish, and hence $(K)$. Turning to the expressions for magnification, we see that

$$\gamma = 0, \quad \phi_1 = \psi_1 = 0,$$

$$\kappa^2 u u_1' = 1;$$

and, in the same way, $k^2 (u-f)(u_2'-f') = 1$.

That is to say, all the first planes of correspondence coincide, and all the second planes coincide, and are at right angles to the first. Thus the system treats the projections of any ray upon these planes exactly as two separate symmetrical instruments would treat them. Such a system possesses two pairs of planes whereon point corresponds to point. For we see by p. 47 that the relations are

$$\left.\begin{aligned} b &= g_1 b' + h_1 \beta', \\ \beta &= k_1 b' + l_1 \beta', \end{aligned}\right\} \qquad \left.\begin{aligned} c &= g_2 c' + h_2 \gamma', \\ \gamma &= k_2 c' + l_2 \gamma', \end{aligned}\right\}$$

and, if $b + u\beta = 0, \quad c + u\gamma = 0,$

then $(g_1 + uk_1)\, b' + (h_1 + ul_1)\, \beta' = 0,$

$$(g_2 + uk_2)\, b' + (h_2 + ul_2)\, \beta' = 0,$$

which give $b' + u'\beta' = 0, \quad c' + u'\gamma' = 0,$

when $u$, $u'$ have the two pairs of values satisfying

$$u' = \frac{h_1 + ul_1}{g_1 + uk_1} = \frac{h_2 + ul_2}{g_2 + uk_2}.$$

The existence of systems possessing two pairs of planes of conjugate foci has been noticed by Larmor.

V. A sub-class of IV. is given by the scheme on p. 79, in which $z = 0$, that is to say, the equivalent of oblique refraction at a surface of double curvature, one of whose lines of curvature is perpendicular to the axis of the beam. We see that

$$(d) = (f) = 0,$$

and, in addition,

$$\sqrt{(D)(F)} + \{\sqrt{(D)} + \sqrt{(F)}\}\{\sqrt{(a)+2(b)} \pm \sqrt{(a)-2(b)}\} - \sqrt{(A)} = 0.$$

This class possesses two pairs of planes where point corresponds to point: one of these coincides with the surface $(u = 0)$; the other is at distance

$$u = \frac{\varpi^2 - 1}{x - \varpi y} = \frac{\{(D)(F) - 2(a)\sqrt{(D)(F)} + (A)\}^{\frac{1}{2}}}{(b)\sqrt{(F)}},$$

which also coincides with the surface in the Case VII. below.

VI. The two pairs of planes in IV. coincide, and give a single pair of planes of conjugate foci, provided that, in addition to $(d) = (f) = 0$, giving $\theta = 0$ in the equivalent lens, we have $cx = cy$, which, expressed in invariants, requires

$$\{(a) - 2(b)\}^2 = (D)(F).$$

VII. If, in addition,

$$(a) - 2(b) = (D) = (F),$$

then, disregarding irrelevant alternatives,

$$c^2xy = c^2XY = 1,$$

and this implies $c = 0$, or the equivalent lens is a thin lens; the single pair of planes of conjugate foci coincide with the surface of the lens. Such a system is equivalent to direct incidence upon a single surface; for then, on p. 79, $\varpi = 1$, and this makes

$$(d) = (f) = 0, \qquad (a) - 2(b) = (D) = (F).$$

VIII. Lastly, if

$$(a) - 2(b) = (d) = (f) = (D) = (F) = 0,$$

the system reduces to the ordinary symmetrical instrument which Gauss discussed. It has but one invariant, the focal length, and the value of $(a)$ or $(b)$ defines it.

11. It is interesting to observe the gradual disappearance of the features of a symmetrical instrument.

The ordinary symmetrical instrument is, simply, a single spherical surface with direct incidence, duplicated, and the duplicates separated. The duplicates of the surface itself are Gauss's principal planes; the duplicates of its centres are the nodal points. Such a system has an infinite number of pairs of planes of conjugate foci. (VIII.)

Preserve direct incidence, but make the curvatures unequal; all planes of conjugate foci disappear except the principal planes, and

we are left with a single pair of planes of conjugate foci, coinciding with the duplicates of the surface, and possessing unit magnification in all directions. (VII.)

Introduce a spherical surface perpendicular to the rays at a finite distance from the other surface; a pair of planes of conjugate foci remain, but do not coincide with either surface; magnification there is equal in all directions, but is no longer unity. (VI.)

Replace the spherical surface by a surface of double curvature whose principal curvatures are parallel to those of the first surface; the single pair of planes of conjugate foci separates into two, in which the magnification is unequal in different directions, but in two fixed directions is directly proportional to the distance of the point from the two foci corresponding to infinity. (IV.)

Rotate either of the surfaces with respect to the other, and the system becomes completely general; both pairs of planes of conjugate foci disappear together. (I.)

Or, again, take oblique incidence. Oblique incidence upon a single surface one of whose lines of curvature is perpendicular to the axes of the beam gives rise to two pairs of planes where point and point correspond; one of these always coincides with the surface, and the other degenerates into coincidence with the first if the incidence be made direct; the magnification at these planes is not the same in all directions, and at that one which coincides with the surface its maximum or minimum is unity. All the planes of correspondence of each series coincide with one or other of two planes at right angles, and generally the instrument behaves towards rays in these two planes exactly as two different symmetrical intruments would do. (V.)

If we introduce a spherical surface normal to the rays as they emerge, the two pairs of planes of conjugate foci remain, but the condition of unit magnification as a maximum or minimum disappears. (IV.)

If the beam falls upon any single surface whose lines of curvature are oblique to it, we have one and only one pair of planes of conjugate foci, situated at the surface itself; magnification in this plane is unequal in different directions, but its maximum or minimum is unity. (III.)

If we introduce a spherical surface normal to the rays as they emerge, the pair of planes of conjugate foci remain, though no longer coinciding with either surface, and the maximum or minimum magnification is no longer unity. (II.)

Any other change makes the planes of conjugate foci disappear, and the system becomes completely general. (I.)

12. Again, consider the relations between the positions of conjugate focal lines. All systems of the type IV., and onwards, are, to put it briefly, resolvable into two symmetrical systems acting in two planes at right angles, both as regards position of conjugate lines and magnification; systems of the types I., II., III. are not so resolvable, and the simplest expressions for the relations between conjugate focal lines or for the magnification are involved with the directions of those lines. Nevertheless, the change of form is not very great, and we can even see that, were the systems not related optically at all, it would not be very much greater.

13. This classification of optical systems is probably very imperfect, for the few properties which I have examined and used as a base for it must have omitted many notable cases; but they seem sufficient to show that no analytical obstacle remains against examining every corner of the problem, whether in its broadest or its finest features.

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## *On the Poncelet Polygons of a Limaçon.* *By* F. Morley.

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### 1. *The Limaçon.*

If we take $n$ elements, points or lines, in a determinate order, say they are numbered $1, 2, \ldots, n, 1$, and also the elements, lines or points $12, 23, \ldots, n1$, we have the polygon. The problem of determining the condition under which the points are elements of one conic and the lines are elements of a second conic is a famous one, and it is sufficient to make the single reference to Halphen's *Fonctions Elliptiques*, t. II. The two conics constitute a curve of degree and class 4, any element of which determines two elements of the other kind, for a line of the one conic determines two points of the other,

and reciprocally. But there are proper quartics of degree and class 4, for the equations (Salmon, *Curves*, p. 65)

$$2\delta+3\kappa=8=2\tau+3\iota$$

are satisfied not only when $\delta=4$, $\kappa=0$, the case of two conics, but also when

$$\delta=1=\tau, \quad \kappa=2=\iota.$$

In this case the possibility of there being polygons at once inscribed and circumscribed, that is, Poncelet polygons of the curve, has been discussed in two papers by R. A. Roberts (*Proc. Lond. Math. Soc.*, Vol. XVI., p. 53, and Vol. XXIII., p. 202). My object is to make some additions to the list given by Roberts. Roberts's method, which reduces the present problem directly to that for two conics, permits us to apply directly tables given by Halphen; but it is to be remarked that this analytical solution is not complete till the questions of reality have been considered. I have preferred to use a method where the algebra keeps in touch with the geometry, and where there is no question as to what factors of the result apply to the problem in hand. As in other cases Roberts's list appears complete, I take here only the one where the cusps may, by a real transformation, be placed at the circular points; the curve is now the limaçon.

The limaçon is described by the vertex $x$ of a parallelogram $Ox_1xx_2$, of which the vertex $O$ is fixed, while the vertices $x_1$ and $x_2$ describe circles about $O$ with angular velocities whose ratio is two. Thus it is the map on the $x$ plane of the circle $|t|=1$, where

$$x=2at+\beta t^2;$$

and, since we are concerned with shape only, not with size, we put $\beta=1$. Thus the equations of the curve in circular coordinates are

$$x=2at+t^2, \quad y=2a/t+1/t^2. \tag{1}$$

Since a change of sign in $a$ merely turns the curve through $\pi$, we take $a>0$. A line $ax+by+c=0$ meets the curve in four points whose parameters are given by

$$a\,(2at+t^2)+b\,(2a/t+1/t^2)+c=0,$$

so that four points $t_1$, $t_2$, $t_3$, $t_4$ of the curve are collinear when

$$\Sigma t=-2a=\Sigma\, 1/t. \tag{2}$$

Some special points and lines of the curve are collected for reference.

(i.) *The node*, $t_i$.—Here two values of $t$ give the same value of $x$, say, $t_1$, $t_2$. Then, from (1),

$$t_1+t_2=-2a, \quad t_1t_2=-x.$$

The first equation shows that the points are conjugate points of the $t$-plane, and, since they are to be on the unit circle,

$$t_1t_2=1.$$

Hence $$x=-1, \quad t_i^2+2at_i+1=0.$$

The node is a point, or rather two points, of the curve only when the points $t_i$ are on the unit circle, that is, when $a \leqq 1$.

(ii.) *The outer vertex.*—Here $t=1$, $x=1+2a$.

(iii.) *The inner vertex.*—Here $t=-1$, $x=1-2a$.

(iv.) *The double tangent.*—From equations (2),

$$t+t'=-a=1/t+1/t'.$$

Hence the points of contact are given by

$$t_\tau^2+1+at_\tau=0,$$

and the line itself, in circular coordinates, by

$$\begin{aligned} x+y &= 2a\,(t_\tau+1/t_\tau)+t_\tau^2+1/t_\tau^2 \\ &= -2a^2+a^2-2 \\ &= -(2+a^2). \end{aligned}$$

(v.) *The inflexions.*—In equations (2) let three values of $t$ be $t_\iota$.

Then $$3t_\iota+t=-2a=3/t_\iota+1/t,$$

so that $$3a\,(t_\iota^2+1)+2\,(a^2+2)\,t_\iota=0.$$

The points are on the curve only when

$$(a^2+2)^2<9a^2,$$

*i.e.*, when $$1 \leqq a \leqq 2.$$

The line of inflexions is

$$\begin{aligned} x+y &= 2a\,(t_\iota+1/t_\iota)+t_\iota^2+1/t_\iota^2 \\ &= 2\,(8-13a^2-4a^4)/9a^2. \end{aligned}$$

The equation of any tangent is

$$x\,(at+1)+yt^3\,(a+t)=2t^3\,(2a^2+1)+3at\,(1+t^2);$$

and this line meets the real axis where

$$x\{\alpha(t+1/t)+t^2+1/t^2\} = 2(2\alpha^2+1)+3\alpha(t+1/t).$$

Hence the inflexional tangents meet the axis at

$$x_\iota\{-3\alpha^2(\alpha^2+2)+2(\alpha^2+2)^2-9\alpha^2\} = 9\alpha^2(\alpha^2-1),$$

or

$$x_\iota = -9\alpha^2/(8+\alpha^2).$$

Among the various practical constructions of the limaçon the following deserves emphasis from its practical convenience. In a sheet of hard cardboard cut two perpendicular slits, bisecting each other. Let each slit be guided by a fixed pin; then a pencil held in a hole drilled through any point of the sheet will describe a limaçon as the sheet moves. This is, in fact, the elliptic compasses; we have two planes $A$ and $B$, and two points of $A$ move on two lines of $B$. With reference to $B$ any point of $A$ describes an ellipse, but with reference to $A$ any point of $B$ describes a limaçon.

The reciprocal of a limaçon, being a curve with the same Plücker numbers, is also a limaçon if we so choose the base conic that the flex-tangents are the reciprocals of the cusps; this requires that the point $x_\iota$ just found is the centre of the conic. *The two reciprocal limaçons have the same shape*, that is, $\alpha$ is the same for both. For consider on the axis the points

| | |
|---|---|
| $0$ | where the cusp-tangents meet, |
| $\infty$ | on the line of cusps, |
| $-1$ | the node, |
| $-(1+\alpha^2/2)$ | on the double tangent; |

and their reciprocal lines, which are vertical and meet the axis at the points

| | |
|---|---|
| $(8-13\alpha^2-4\alpha^4)/9\alpha^2$ | on the lines of flexes, |
| $-9\alpha^2/(8+\alpha^2)$ | where the flex tangents meet, |
| $-(1+\alpha^2/2)$ | on the double tangent, |
| $-1$ | the node. |

Corresponding double-ratios of these two fours are equal, and, in fact, it is verified at once that the equation

$$(8+\alpha^2)\,xx'+9\alpha^2(x+x') = 8-13\alpha^2-4\alpha^4$$

sends the one four into the other, and also sends one vertex into the other.*

## 2. *The Correspondence of Cotangential Points.*

The curve depends on the single positive constant $a$; the critical values of $a$ are $a = 1$ (the cardioidal case) and $a = 2$ (the undulational case). When $a < 1$, the flexes are imaginary, but the curve has two loops; with this we begin. When $1 < a < 2$ the curve is an indented oval (§ 4), and when $a > 2$ the curve is an unindented oval, which obviously does not admit a real polygon.

From equations (2), we see that the tangent at $t_3$ meets the curve at $t_1$, $t_2$, when

$$t_1+t_2+2a = -2t_3,$$

$$1/t_1+1/t_2+2a = -2/t_3,$$

*i.e.*, when $$(t_1+t_2+2a)(1/t_1+1/t_2+2a) = 4. \qquad (3)$$

The equation defines a symmetric two-to-two correspondence. The values of $t_2$ will coincide when

$$\left[a\,(t_1^2+1)+(2a^2-1)\,t_1\right]^2 = (2at_1+1)(2at_1+t_1^2),$$

that is $$(t_1^2+1+2at_1)\left[a\,(t_1^2+1)+2\,(a^2-2)\,t_1\right] = 0;$$

the former factor gives the nodal points, the latter the points where the flex tangents meet the curve again.

We now introduce elliptic functions; the notation and especially the necessary $\sigma$-formulæ are explained in the treatise on the *Theory of Functions* by Harkness and myself (Ch. VII.).

The substitution $$\frac{1-t}{1+t} = iz/\kappa, \qquad (4)$$

where $\kappa$ is a real constant, and therefore, for

$$|\,t\,| = 1,$$

$z$ is real, brings the quartic last written into the even form

$$Qz = \left(z^2-\kappa^2\frac{1+\alpha}{1-\alpha}\right)\left(z^2-\kappa^2\frac{1+\gamma}{1-\gamma}\right),$$

---

* It is therefore possible to so choose a base conic that the limaçon is its own reciprocal; the axes of such a conic are given by

$$a^2 = 4\,(1-a^2)(4-a^2)^2/(8+a^2)^2, \quad b^2 = \pm 4\,(1-a^2)\sqrt{1-a^2\,.\,4-a^2}/(8+a^2).$$

where $$\gamma = a - 2/a,$$

so that $$\frac{1+\gamma}{1-\gamma} = -\frac{(1-a)(2+a)}{(1+a)(2-a)}.$$

Therefore (3) can be written

$$\frac{dz_1}{\sqrt{Qz_1}} \pm \frac{dz_2}{\sqrt{Qz_2}} = 0, \tag{5}$$

whence, if we put $$z = \sigma_2 u/\sigma u = \sqrt{pu - e_2}, \tag{6}$$

with $$e_1 - e_2 = \kappa^2 \frac{1+a}{1-a},$$

$$e_3 - e_2 = -\kappa^2 \frac{(1-a)(2+a)}{(1+a)(2-a)},$$

$$e_1 + e_2 + e_3 = 0;$$

then $$u_2 \pm u_1 = \text{a constant } c,$$

where $u_1$ and $u_2$ correspond to cotangential points.

We choose, in the case $0 < a < 1$,

$$\kappa^2 = (1-a^2)(2-a),$$

and have $$e_1 - e_2 = (1+a)^2(2-a) = 2+3a-a^3,$$

$$e_2 - e_3 = (1-a)^2(2+a) = 2-3a+a^3,$$

$$e_3 - e_1 = -4,$$

so that $e_1$, $e_2$, $e_3$ are known real numbers.

We have then
$$\left.\begin{aligned} x &= 2at + t^2, \\ \frac{1-t}{1+t} &= i\tau/\kappa = i\sigma_2 u/\kappa\sigma u, \end{aligned}\right\} \tag{7}$$

equations which map the limaçon into (1) a unit circle, (2) the real axis in the $\tau$-plane, (3) a broken line in the $u$-plane. To determine this broken line, let $2\omega_1$ be the real, $2\omega_3$ the imaginary, period, and note that when

$$t = -1, \quad \tau = \infty, \quad u = 0,$$

$$t = 1, \quad \tau = 0, \quad u = \omega_1 + \omega_3 = \omega_2, \quad \text{say};$$

$$t^2 + 2at + 1 = 0, \quad \tau^2 = e_1 - e_2, \quad pu = e_1, \quad u = \omega_1 \text{ or } \omega_2 + \omega_3.$$

Also observe that the mapping is conform for the points in question, except for the node, near which, since, if $u = \omega_1 + u'$,

$$\sigma_2 u/\sigma u = \sqrt{e_1 - e_2}\,\sigma_3 u'/\sigma_1 u' = P_2 u',*$$

we have $$x+1 = P_1(t-t_\delta) = P_2(u-\omega_1),$$

so that angles at the node are double of angles at the corresponding $u$-points.

Thus, since, for real values of $\tau$, $pu$ is real and $> e_2$, we may take the $u$-path as in Fig. 1.

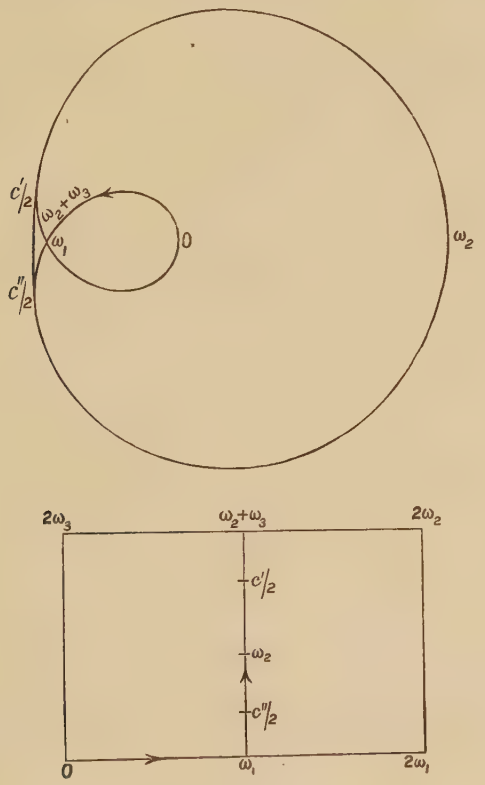


FIG. 1.

Again, if $u_1$ and $u_2$ correspond to cotangential points, we have, as already said

$$u_2 \pm u_1 = c, \qquad (8)$$

and the sign is to be determined.

* By $P_n$ is meant a Taylor series beginning with the $n^{\text{th}}$ power.

Evidently we are concerned only with points on the outer loop, *i.e.*, with points on the vertical part of the $u$-path, so that $c$ is imaginary. We must distinguish *inner* tangents which touch the inner loop, for which the points $u_1, u_2$ include the point of contact, from outer tangents which touch the outer loop, for which the point of contact lies beyond $u_1$ or $u_2$. For the former, $u_2$ and $u_1$ increase together, and we have

$$u_2 - u_1 = c.$$

For the latter, $u_1$ and $u_2$ move oppositely, and we have

$$u_2 + u_1 = c',$$

where the relation of $c'$ and $c$ is found, by considering the transition case of a nodal tangent, to be for the upper part of the figure

$$c' + c = 2(\omega_2 + \omega_3),$$

and for the lower half $\quad c'' - c = 2\omega_1.$

Accordingly, starting from $u_1$, we construct $u_2$ for an inner tangent in the $u$-plane by going a vertical distance $c$, along the vertical part of the $u$-path ; but for an outer tangent this brings us to the end of the vertical part, and we merely turn back and complete the distance $c$. Thus, in any case, regarding the vertical part as an endless chain, we measure along it an absolute distance $| c |$ in a definite direction, to pass from $u_1$ to $u_2$. And, if, going on in this way, we ever get back to $u_1$, we have a Poncelet polygon. Thus, since the length of the chain is $| 4\omega_3 |$, we must have

$$nc = 4m\omega_3, \tag{9}$$

where $n$ and $m$ are integers.

### 3. *Poncelet Polygons when* $\alpha < 1$.

In the preceding equation let $m$ be 1, so that one trip round the chain brings $u$ to the starting point. We have then, when $n = 2h$,

$$\sigma_3 hc/2 = 0;$$

when $n$ is odd, $\qquad \sigma nc/2 = 0.$

To determine $c$ we take a special case, *e.g.*, the double tangent. From (8),

$$\kappa^2 \left(\frac{1-t}{1+t}\right)^2 = -\sigma_2^2 u/\sigma^2 u = -(pu - e_2),$$

and for the double tangent we may take

$$u = c''/2 = c/2 + \omega_1,$$

$$t + 1/t = -a.$$

Hence $$\kappa^2\left(1-\frac{4}{2-a}\right) = -\left\{p\left(\frac{c}{2}+\omega_1\right)-e_2\right\},$$

or $$(1-a^2)(2+a) = e_1-e_2+p\left(\frac{c}{2}+\omega_1\right)-e_1,$$

or $$-2a\,(1+a) = \frac{e_1-e_2\,.\,e_1-e_3}{p\,c/2-e_1},$$

or $$p\,c/2-e_1 = -2\,(1+a)(2-a)/a,$$

so that $$p\,c/2-e_2 = (1+a)(2-a)(1+a-2/a),$$

$$= -(1-a^2)(4-a^2)/a,$$

$$p\,c/2-e_3 = -2\,(1-a)(2+a)/a.$$

Therefore $$\sigma_2^2 c/2\,\sigma_3 c/2 : \sigma_3^2 c/2\,\sigma_1^2 c/2 : \sigma_1^2 c/2\,\sigma_2^2 c/2$$

$$= (1-a)(2+a) : 2 : (1+a)(2-a),$$

and (H. and M., p. 316), if we write $\sigma_\lambda$ for $\sigma_\lambda c$,

$$\sigma_1 : \sigma_2 : \sigma_3 = 1+a : 1-a^2 : 1-a,$$

$$\sigma_2\sigma_3 : \sigma_3\sigma_1 : \sigma_1\sigma_2 = 1-a : 1 : 1+a. \qquad (10)$$

Hence, if $\tau_\lambda$ stand for $\sigma_\lambda 2c$, we have (*loc. cit.*)

$$\tau_1 : \tau_2 : \tau_3 = 1+4a : 1+2a^2 : 1-4a.$$

If, now, the limaçon admits a $2h$-gon, we have

$$\sigma_3 hc/2 = 0.$$

If $h = 3$,

$$\sigma_3 3c/2 = 0,$$

$$\sigma_1\sigma_2 = \sigma_3(\sigma_1+\sigma_2),$$

$$1+a = 2-a,$$

$$a = 1/2.$$

If $h = 4$,

$$\sigma_3 2c = 0,$$

*i.e.*,

$$\tau_3 = 0,$$

$$a = 1/4.$$

If $h = 5$, $\qquad \sigma_3 5c/2 = 0.$

Here we are delayed a minute by not having the appropriate $\sigma$-formula in readiness.

We have, from (60), p. 315, putting $u = 2u$, $v = u/2$,

$$\sigma_\lambda 5u/2\,\sigma 3u/2 = \sigma_\lambda 2u\,\sigma 2u\,\sigma_\mu\mu/2\,\sigma_\nu u/2 - \sigma_\lambda u/2\,\sigma_\mu u/2\,\sigma_\mu 2u\,\sigma_\nu 2u\,;$$

or, writing $\sigma_\lambda$ for $\sigma_\lambda u$, so that

$$\sigma 2u = 2\sigma\sigma_1\sigma_2\sigma_3,$$

$$\frac{\sigma_\lambda 5u/2\,\sigma 3u/2}{\sigma_\lambda u/2\,\sigma u/2} = 4\sigma_1\sigma_2\sigma_3\sigma_\lambda 2u\sigma_\mu^2 u/2\,\sigma_\nu^2 u/2 - \sigma_\mu 2u\,\sigma_\nu 2u,$$

$$= 2\sigma_1\sigma_2\sigma_3\,(\sigma_\mu + \sigma_\nu)\,\sigma_\lambda 2u - \sigma_\mu 2u\,\sigma_\nu 2u\,;$$

whence, substituting for $\sigma_\lambda 2u$ its value in terms of $\sigma$ (p. 316), and removing the factor

$$\frac{\sigma 3u/2}{\sigma u/2} \quad \text{or} \quad \sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2,$$

we have $$\frac{\sigma_\lambda 5u/2}{\sigma_\lambda u/2} = 2\sigma_\mu\sigma_\nu\,(-\sigma_\mu^2\sigma^2 + \sigma_\nu^2\sigma_\lambda + \sigma_\lambda^2\sigma_\mu^2)$$

$$-(-\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2)(\sigma_2\sigma_3 - \sigma_3\sigma_1 + \sigma_1\sigma_2)(\sigma_2\sigma_3 + \sigma_3\sigma_1 - \sigma_1\sigma_2),$$

a result which may be written

$$\frac{\sigma_\lambda 5u/2}{\sigma_\lambda u/2} = 2\sigma_\lambda 2u\,\sigma_\mu u\,\sigma_\nu u - \frac{\sigma 3u\,\sigma u/2}{\sigma 3u/2\,\sigma u}.$$

Hence in our case the equation $\sigma_3 5c/2 = 0$ entails, from (10),

$$2(1+a)(1-4a) - (1+2a)(1-2a) = 0,$$

or $$1 - 6a - 4a^3 = 0,$$

or $$a = \frac{-3+\sqrt{13}}{4}.*$$

For greater values of $h$, $a$ is a number of a higher irrationality. For example, if $h = 6$, we have

$$\sigma_3 3c = 0,$$

$$\tau_1\tau_2 = \tau_3\,(\tau_1 + \tau_2),$$

$$16a^3 + 8a^2 + 8a = 1.$$

---

* We observe incidentally that now $/5$ is a number of the form $a + b\sqrt{13}$; we have, in fact, when

$$e_1 = 2 + \sqrt{13}/24, \quad e_2 = -\sqrt{13}/12, \quad e_3 = -2 + \sqrt{13}/24,$$

$$p\,2\omega_3/5 = -9/4 - 29\sqrt{13}/24.$$

Again, for odd polygons, we have

$$\sigma n c/2 = 0.$$

From $\sigma 5v\sigma^3 v = \sigma 4v\sigma^2 2v - \sigma v\sigma^3 3v,$

let $v = c/2\,;$

then $\sigma 2v = \sigma,$

$$\sigma 3v = \sigma v\,(\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2),$$

$$\sigma 4v = 2\sigma\sigma_1\sigma_2\sigma_3\,;$$

and, since $\sigma_2+\sigma_3 = 2\sigma_2^2 v\sigma_3^2 v,$

$$(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)(\sigma_1+\sigma_2) = 8\,(\sigma_1 v\sigma_2 v\sigma_3 v)^4 = \sigma^4/2\sigma^4 v,$$

or $\sigma^4 v = \tfrac{1}{2}\sigma^4/(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)(\sigma_1+\sigma_2).$

Hence $$\sigma 5\,c/2\,\sigma^3 c/2 = 2\sigma^4\sigma_1\sigma_2\sigma_3 - \tfrac{1}{2}\,\frac{\sigma^4\,(\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2)^3}{(\sigma_2+\sigma_3)(\sigma_3+\sigma_1)(\sigma_1+\sigma_2)}$$

$$= \sigma^4\sigma_1\sigma_2\sigma_3\,(2-P^3/2Q),$$

where $P = \sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2,$

$$Q = \sigma_1(\sigma_2+\sigma_3)\,\sigma_2(\sigma_3+\sigma_1)\,\sigma_3(\sigma_1+\sigma_2).$$

In our case $P = 3, \quad Q = 2\,(4-\alpha^2)\,;$

and therefore $\sigma 5c/2 = 0,$

when $8\,(4-\alpha^2) = 27, \quad$ or $\quad \alpha = \sqrt{10}/4,$

the condition for a pentagon.

Similarly $$\sigma 7v\sigma^3 v = \sigma 5v\sigma^3 3v - \sigma 2v\sigma^3 4v$$

$$= \sigma^3 v\sigma 5v\,(\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2) - 8\sigma^4\sigma_1^3\sigma_2^3\sigma_3^3$$

$$= \sigma^4\sigma_1\sigma_2\sigma_3\,\{2P^3 - P\,6/2\,Q - 8\sigma_1^2\sigma_2^2\sigma_3^2\}.$$

This is zero when

$$2\,.\,3^3 - 3^6/4\,(4-\alpha^2) - 8\,(1-\alpha^2) = 0,$$

*i.e.*, when $$32\alpha^4+56\alpha^2-7 = 0,$$

$$\alpha = \tfrac{1}{4}\sqrt{-14+6\sqrt{7}}.$$

### 4. *Poncelet Polygons when* $1 < \alpha < 2.$

Turning now to the case $1<\alpha<2$, we observe that $\kappa^2, = (1-\alpha^2)(2-\alpha)$, is now negative, and $\tau$ in equation (4) is a pure imaginary. We have the same equation as before, only now $pu$ is real and $< e_2$.

The points where the flex-tangents meet the curve now take the place of the double point; they are

$$u = \omega_3, \quad u = \omega_1 + \omega_2.$$

The inner tangents touch at points on the indentation, between the inflexions; and the outer tangents at points between the inflexions and the points of contact of the double tangent. The mapping is indicated in Fig. 2.

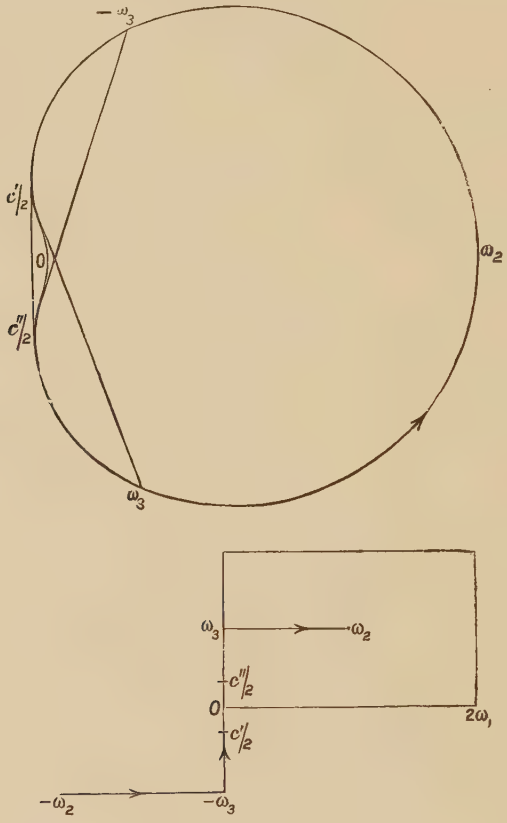


FIG. 2.

For inner tangents $u_2 - u_1 = c$,

and for outer tangents $u_2 + u_1 = c'$ or $c''$,

where $c' - c = 2(\omega_1 + \omega_2)$,

$c'' + c = 2\omega_3$.

The lower point of contact of the double tangent is $c''/2$ or $c/2 + \omega_3$.

Hence, repeating the process of the last article, we have

$$(1-a^2)(2+a) = p\left(\frac{c}{2}+\omega_3\right)-e_2,$$

whence

$$pc/2-e_3 = \quad 2(1-a),$$

$$pc/2-e_1 = -2(1+a),$$

$$pc/2-e_2 = \quad a(1-a^2),$$

$$\sigma_2^2 c/2\,\sigma_3^2 c/2 : \sigma_3^2 c/2\,\sigma_1^2 c/2 : \sigma_1^2 c/2\,\sigma_2^2 c/2 = a^3-a : 2 : a^2+a,$$

$$\sigma_1 : \sigma_2 : \sigma_3 = 1+a : a^2-1 : 1-a,$$

and

$$\sigma_2\sigma_3 : \sigma_3\sigma_1 : \sigma_1\sigma_2 = 1-a : -1 : 1+a\,; \qquad (11)$$

and these results may be verified by observing that, if $c_+$ be the present constant $c$, and $c_-$ that of the previous article when $a$ was $<1$, while $c''$ is analytically the same in both, we had

$$c'' = c_+ + 2\omega_3 = c_- + 2\omega_1,$$

so that

$$\sigma_\lambda c_+ = \sigma_\lambda(c_- + 2\omega_1 - 2\omega_3),$$

or (H. and M., p. 302, f. 34)

$$\sigma_1 c_+ : \sigma_2 c_+ : \sigma_3 c_+ = -\sigma_1 c_- : \sigma_2 c_- : -\sigma_3 c_-.$$

The ratios of $r_\lambda$ are as before.

The condition of closure is, as before,

$$nc = 4m\omega_3.$$

Running through the preceding $\sigma$-formulæ with the new values (11) instead of (10), we find only two new cases: *first*, when

$$\sigma_3(5c/2) = 0\,;$$

in this case

$$2(1+a)(1-4a)-3(1+2a)(1-2a) = 0,$$

or

$$4a^2-6a-1 = 0,$$

or

$$a = (3+\sqrt{13})/4,$$

the condition for a decagon.*

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* It will be observed that, whereas $c<\omega_3$ when $a<1$, $c>\omega_3$ when $a>1$. Thus in the present decagon a figure would show that

$$10c = 12\omega_3\,;$$

while for the *decagon of the preceding article*

$$10c = 4\omega_3.$$

*Second*, we have, when $\sigma 7c = 0$,

$$8a^2+1+32a^2(1-a^2) = 0,$$

whence $$a^2 = (5 \pm 3\sqrt{3})/8.$$

We have then the following *table of the simpler Poncelet polygons* :—

| Number of Sides... | 5 | 6 | 7 | 8 | 10 |
|---|---|---|---|---|---|
| Value of $a$ ..... | $\sqrt{10}/4$ | 1/2 | $\sqrt{-14+6\sqrt{7}}/4$<br>$\sqrt{10+6\sqrt{3}}/4$ | 1/4 | $(\sqrt{13}+3)/4$<br>$(\sqrt{13}-3)/4$ |

Four of these, namely those with an even number of sides, are not given by Roberts.

### 5. *The Hexagonal Case.*

When $a = 1/2$, the curve is the trisectrix $r = \cos\theta/3$. The intersection of the cusp-tangents, $x = 0$, is now the inner vertex; reciprocally the join of the inflexions is the tangent at the outer vertex.

Referring to Fig. 3, let $t_1, t_2, \ldots, t_6$ be the points of the hexagon, and let $t_{12}$ be the point of contact of the line $t_1, t_2$. We have, from (3),

$$(t_1+t_2+1)(1/t_1+1/t_2+1) = 4.$$

The roots of this quadratic in $t_1$ are $t_1$ and $t_3$; hence

$$t_1 t_3 = t_2,$$

$$t_1+t_3 = -\frac{t_2^2-t_2+1}{t_2+1};$$

or, since $$t_2 = 1/t_5,$$

$$t_1 t_3 t_5 = 1,$$

and $$t_1+t_3 = -\frac{1-t_5+t_5^2}{t_5+t_5^2}.$$

Hence $$t_1+t_3 = -\frac{t_1 t_3-1+t_5}{1+t_5},$$

or, for odd suffixes, $$\Sigma t_1+\Sigma t_1 t_3 = 1,$$

or, for all suffixes, $\sum_{1}^{6} t_\lambda = 1.$

Now, from (2), $2t_{12} = -1 - t_1 - t_2.$

Therefore $t_{12} + t_{34} + t_{56} = -2.$

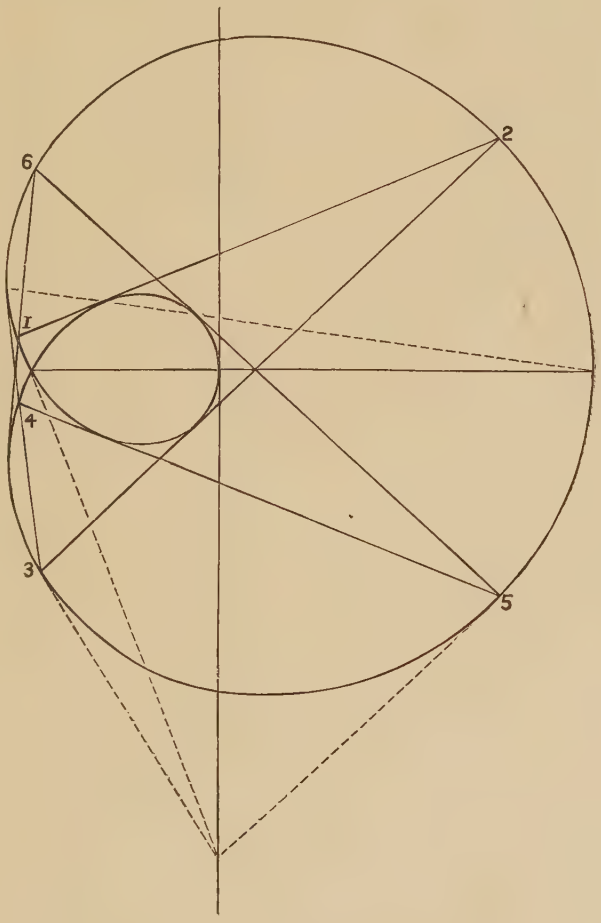


FIG. 3.

Therefore the points $t_{12}$, $t_{34}$, $t_{56}$, 1 lie on a line; that is, *the points of contact of alternate sides lie on a line through the outer vertex.*

Reciprocally, *the tangents at alternate angular points meet on the tangent at the inner vertex.*

## *On the Calculus of Equivalent Statements. (Seventh Paper.)*

*By* Hugh MacColl, B.A. Received November 10th, 1897. Communicated by Mr. Tucker, November 11th, 1897.

Def. 1.—Let $\phi$ denote any implication whose elementary constituents may change their values, being sometimes of the class $\epsilon$, sometimes of the class $\eta$, and sometimes of the class $\theta$. Let $\phi^\alpha$ assert that $\phi$ is *true* for *all* values of its constituents; let $\phi^\beta$ denote $\phi'^\alpha$ and assert that $\phi$ is *false* for *all* values of its constituents; and let $\phi^\gamma$ denote $\phi^\alpha\phi^\beta$, and assert that $\phi$ is neither true for all values of its constituents, nor false for all values of its constituents. Thus $\phi^\alpha$ would apply to the statement $AB : B+C$; $\phi^\beta$ would apply to the statement $A+A' : BB'$; and $\phi^\gamma$ would apply to $A : BC$.

The symbols $\alpha$, $\beta$, $\gamma$, thus interpreted, will be analogous to, but not quite identical with, $\epsilon$, $\eta$, $\theta$. For example, $\phi^\epsilon$ asserts that $\phi$ is *always true*, but it does not expressly state *why;* whereas the symbol $\phi^\alpha$ asserts that $\phi$ is always true *because it would be an inconsistency of language, or a contradiction to our definitions, to say that it is false.* In my Fifth Paper I have drawn no distinction between $\phi^\epsilon$ and $\phi^\alpha$, using the former symbol in both senses. With regard to many problems and investigations this may safely and conveniently be done; but cases might arise in which it would be necessary to differentiate between the two significations. When such cases present themselves we shall always have the formulæ $A^\alpha : A^\epsilon$, $A^\beta : A^\eta$, $A^\gamma : A^\gamma$; but *not* always the converse statements $A^\epsilon : A^\alpha$, $A^\eta : A^\beta$, $A^\gamma : A^\theta$. For example, $A^\beta : A^\eta$ may be read: "If the statement $A$ is *self-contradictory*, it asserts an *impossibility*"; whereas $A^\eta : A^\beta$ may be read: "If $A$ asserts an *impossibility*, it is *self-contradictory*, which is not a necessary or legitimate inference. The symbol $A^\eta$, but not the symbol $A^\beta$, would apply to such a statement as "He swam across the Atlantic in five minutes"; the symbols $A^\beta$ and $A^\eta$ would each apply to such a statement as "The whole is less than its part." Briefly, $A^\beta$ asserts that $A$ is a *formal* impossibility; while $A^\eta$ simply asserts that $A$ is an impossibility, asserting nothing as to the *kind* of impossibility.

In connexion with these symbolic interpretations I may here reply to some criticism on my Fifth Paper. It has been objected that it is incorrect to speak of a statement as "sometimes true and sometimes

false"; that a statement once made belongs either to the class of truths or to the class of falsehoods once for all, and can never again pass from that class to its opposite. As this criticism evidently proceeds from a misapprehension of my meaning, it seems necessary to explain that meaning a little more fully. My point of view is this, that, just as the implication $A : B$ may be true or false according to the nature of the statements denoted by $A$ and $B$, and is, as a matter of fact, true in the case of some statements and false in the case of others, so certain statements, such as "Mrs. Brown is not at home," are sometimes used to convey, and do convey, a true information, while at other times they are used (*in the very same form of words*) to convey, and do convey, a false information. This class of statements I call *variables*. Let $A$ denote a statement of this class, and (to fix our ideas) suppose it to have been made on five different occasions: *truly* on the *first*, *third*, and *fifth* occasions; *falsely* on the *second* and *fourth*. Let the symbols $A_1$, $A_2$, $A_3$, $A_4$, $A_5$ denote these five separate statements, each expressed *in the same form of words*. The *general* symbol $A$ (*not* any of the *particular* symbols $A_1$, $A_2$, $A_3$, $A_4$, $A_5$) denotes this form of words; and, therefore, I hold that, though none of the statements $A_1, A_2$, &c., can be spoken of separately as *sometimes true and sometimes false*, yet their general representative $A$, which may denote any of them, whether true or false, can be thus spoken of. When $A$ represents $A_1$, *then $A$ is true because $A_1$ is true;* when $A$ represents $A_2$, *then $A$ is false because $A_2$ is false;* and similarly for the other cases.

In this it seems to me that I am only imitating a common and useful convention among mathematicians, who speak of the points $(x_1, y_1)$, $(x_2, y_2)$, &c., as *fixed*, their fixed positions being determined by the *constants* $x_1$, $y_1$, or $x_2$, $y_2$, &c.; while they speak of the point $(x, y)$ as a *current* or *moving* point whose position *varies* with the *varying* values of its coordinates $x$ and $y$.

Def. 2.—The symbol $A^{u \cdot v}$ (especially when $A$ is a complicated expression) may be used as a convenient abbreviation for $A^u A^v$. When thus used it must not be confounded with the symbol $A^{uv}$, which (as already defined) means $(A^u)^v$, nor with $A^{u,v}$, which means $A^u + A^v$. Thus, if $\kappa, \lambda, \mu$ respectively denote (1) statements *known and admitted to be true, or provable from the data;* (2) statements *known and admitted to be false, or provable false from the data;* (3) statements *belonging neither to the class $\kappa$ nor to the class $\lambda$*, and therefore *doubtful*, we get $A^{\lambda \cdot \epsilon} = A^\lambda A^\epsilon = \eta$; but $A^{\lambda\epsilon} = (A^\lambda)^\epsilon$, which merely asserts that *$A$ is always known to be false.*

Def. 3.—Let $P$, $Q$ denote two *individuals* of the same class; and, to fix our ideas, suppose them to be *points* limited to some line, area, or volume. Then, if $A$, $B$, $C$, &c., denote *statements*, the symbols $P_A$, $P_{\phi(A,B)}$, &c., will assert that the statements $A$, $\phi(A, B)$, &c., are true *when applied to the point* $P$. The symbol $P^u_A$ means $(P_A)^u$, and asserts (in accordance with previous definitions) that the statement $P_A$ belongs to the class of statements denoted by $u$. Thus $P'_A$ or $P^{\iota}_A$ asserts that $A$ is false when applied to the point $P$, no assertion being made as to whether $P$ is fixed or variable; $P^{\epsilon}_A$ asserts that $A$ is true for all permissible positions of the varying point $P$; $P^{\eta}_A$ asserts that $A$ is false for all permissible positions of the point $P$; and $P^{\theta}_A$ asserts that $A$ is true for some permissible positions of the point $P$, but not for all.

From these definitions we get several self-evident formulæ, such as

$$P_{AB'} = P_A P'_B, \quad P_{A+B'} = P_A + P'_B, \quad P_{A:B} = P_A : P_B,$$

and, generally, $$P_{\phi(A,B)} = \phi(P_A, P_B).$$

When we have to speak of the mutual relations of two or more points $P$, $Q$, &c., such statements as $P_A + Q_B$, $P_A : Q'_B$, &c., explain their own meaning, and require no fresh definition. When there is no risk of ambiguity, $P\alpha$ may replace $P_\alpha$.

To give some geometrical illustrations, let the symbols $\alpha^u$, $\alpha^v$, $\alpha^w$ respectively assert (as before) that $\alpha$ is *positive*, that $\alpha$ is *negative*, that $\alpha$ is *imaginary;* and let $\alpha^0$ (or some other appropriate symbol in the few cases in which there might be danger of ambiguity) assert that $\alpha$ is *zero*. Then, assuming our *universe of points* to be restricted to the same plane, adopting the usual conventions as to rectangular axes, and putting $\phi$ or $\phi(x, y)$ for $x^2+y^2-r^2$, the symbol

$$P\phi^u Q\phi^v R\phi^0 S\phi^w$$

necessarily asserts that $P$ is *outside* the circle $\phi^0$ (*i.e.*, the circle $x^2+y^2-r^2=0$), that $Q$ is *inside*, that $R$ is *on the circumference*, and that $S$ does not belong to our *universe of points; i.e.*, that, if the point $S$ exists at all, it must be in some other plane. With the same symbolic conventions, we should have the easily verified formula

$$(PQ-2r)^u : (P\phi^v : Q\phi^u) : P\phi^{v_\iota} + Q\phi^u,$$

which may be read: "If $PQ$ is greater than $2r$, then, if $P$ is inside the circle $\phi^0$, $Q$ must be outside this circle; a conditional assertion

which implies the alternative that either $P$ is *not* inside or else that $Q$ is outside." We should also have

$$(PQ-2r)^0 : (P\phi^0 : Q\phi^u + Q\phi^0),$$

which may be read: "If $PQ$ is equal to the diameter, then, if $P$ is on the circumference, $Q$ must be either *outside* the circle $\phi^0$ or else on its circumference."

As another example, let $P$ be a point taken at random in the accompanying square, the equation to the enclosed circle being

$$x^2+y^2-r^2=0,$$

or, briefly, $\phi^0$, in which $\phi$ denotes

$$x^2+y^2-r^2.$$

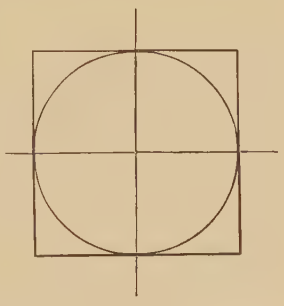

Fig. 1.

Then we have $P^s\phi^u P^s\phi^v$, in which $P^s\phi^u$ and $P^s\phi^v$ are abbreviations for $(P\phi^u)^s$ and $(P\phi^v)^s$. The factor $(P\phi^u)^s$ asserts that the statement $\phi^u$ in reference to the point $P$ may be true or false; and the factor $(P\phi^v)^s$ asserts that the statement $\phi^v$ in reference to $P$ may also be true or false.

In the mathematical applications of symbolic logic, and especially in probability, the number of ways in which we can economize time, space, and brain labour (without sacrificing one iota of clearness and accuracy), by varying the meanings we give to our symbols of relation (indices, subscripts, &c.), is almost unlimited. In my last two papers the symbol $A^s$ asserts that the chance of the statement $A$ being true (assuming the condition $\epsilon$) is more than zero and less than unity. But there is no reason in the world why the symbol $\theta$ should not, when possible and convenient, be replaced by the *exact value* of this chance. Thus $A^{\frac{2}{3}}$ would be equivalent to the equation

$$\frac{A}{\epsilon} = \frac{2}{3},$$

and would assert that the chance of $A$ being true (the condition $\epsilon$ being assumed throughout) was $\frac{2}{3}$. It being understood that $A$ is a *statement* (and in certain investigations it might be convenient to reserve capitals for statements), this meaning could not possibly be confounded with the customary meaning of the same symbol when $A$ is a number or ratio, for the simple reason that to talk of the

square of the cube root of a *statement*, or of the cube root of its square, would be to talk nonsense. To show the working of this convention, I will take the following definitions :—

Def. 4.—The symbol $A^x$ is short for $\frac{A}{\epsilon} = x$, and asserts that the chance of $A$ being true (on the understood assumption $\epsilon$) is $x$.

Def. 5.—The symbol $\frac{B}{A^x}$ (in conformity with the notation of my last paper) denotes the *chance* that the statement $B$ is true on the assumption that the statement $A^x$ is true. Thus we get

$$\left(\frac{B}{A^x} = y\right) = A^x : B^y = \left(\frac{B^y}{A^x} = 1\right);$$

each of the three equivalences asserting that when the chance of $A$ being true is $x$ the chance of $B$ being true is $y$.

From these definitions we readily get several formulæ, such as

$$(1) \quad A^a B^b (AB)^c : (A+B)^{a+b-c},$$

$$(2) \quad A^a B^b (AB)^c : \left(\frac{A}{B} = \frac{c}{b}\right),$$

$$(3) \quad A^a B^b (AB)^c : \left(\frac{B}{A} = \frac{c}{a}\right),$$

$$(4) \quad A^a B^b : \left(\frac{A}{B} = \frac{a}{b}\ \frac{B}{A}\right)\left(\frac{B}{A} = \frac{b}{a}\ \frac{A}{B}\right),$$

$$(5) \quad A^a B^b : \left\{\frac{A}{B} - \frac{A}{B'} = \frac{1}{1-b}\left(\frac{A}{B} - a\right)\right\}.$$

In the last formula, when $A$ and $B$ are independent, we have

$$\frac{A}{B} = \frac{A}{\epsilon} = a,$$

so that we get

$$\frac{A}{B} - \frac{A}{B'} = 0,$$

as we ought to have.

Def. 6.—The symbol $a'$, when $a$ denotes a *chance*, means $1-a$; so

that when $a, b, c$, &c., are respectively used to denote $\frac{A}{\epsilon}$, $\frac{B}{\epsilon}$, $\frac{C}{\epsilon}$, &c., then $a'$, $b'$, $c'$, &c., will denote $\frac{A'}{\epsilon}$, $\frac{B'}{\epsilon}$, $\frac{C'}{\epsilon}$, &c.

Using the notation of Def. 6, we prove Formula 5 as follows :—

$$\frac{A}{B} - \frac{A}{B'} = \frac{A}{B} - \frac{a}{b'}\frac{B'}{A} \text{ (by Formula 4)}$$

$$= \frac{A}{B} - \frac{a}{b'}\left(1 - \frac{B}{A}\right)$$

$$= \frac{A}{B} - \frac{a}{b'}\left(1 - \frac{b}{a}\frac{A}{B}\right) \text{ (by Formula 4)}$$

$$= \frac{A}{B} - \frac{a}{b'} + \frac{b}{b'}\frac{A}{B}$$

$$= \left(1 + \frac{b}{b'}\right)\frac{A}{B} - \frac{a}{b'}$$

$$= \frac{1}{b'}\left(\frac{A}{B} - a\right),$$

since $$b + b' = 1.$$

To prove Formula 4, we have

$$\frac{B}{A} = \frac{AB}{\epsilon} \div \frac{A}{\epsilon} = \frac{B}{\epsilon}\frac{A}{B} \div \frac{A}{\epsilon} = b\frac{A}{B} \div a = \frac{b}{a}\frac{A}{B}.$$

In connexion with the preceding formulæ, 4 and 5, I may take a problem which Dr. Keynes, in his *Formal Logic* (Third Edition, p. 374), gives as an example of "numerical logic"; but it may fairly be called a problem in probability :—

"Show that if the proportion of $B$'s out of the class $A$ is greater than out of the class *not-A*, then the proportion of $A$'s out of the class $B$ will be greater than that out of the class *not-B*."

This is equivalent to proving

$$\left(\frac{B}{A} - \frac{B}{A'}\right)^u : \left(\frac{A}{B} - \frac{A}{B'}\right)^u,$$

in which the index $u$ asserts (as in previous conventions) that the

expression affected by it is real and positive. We have (by Formula 5)

$$\left(\frac{B}{A}-\frac{B}{A'}\right)^{u} : \left(\frac{A}{B}-\frac{A}{B'}\right)^{u} = \left\{\frac{1}{a'}\left(\frac{B}{A}-b\right)\right\}^{u} : \left\{\frac{1}{b'}\left(\frac{A}{B}-a\right)\right\}^{u}$$

$$= \left(\frac{B}{A}-b\right)^{u} : \left(\frac{A}{B}-a\right)^{u}$$

$$= \left(\frac{b}{a}\,\frac{A}{B}-b\right)^{u} : \left(\frac{A}{B}-a\right)^{u} \text{(by Formula 4)}$$

$$= \left(\frac{A}{B}-a\right)^{u} : \left(\frac{A}{B}-a\right)^{u} = \epsilon;$$

for $$\left(\frac{b}{a}\,\frac{A}{B}-b\right)^{u} = \left\{\frac{a}{b}\left(\frac{b}{a}\,\frac{A}{B}-b\right)\right\}^{u} = \left(\frac{A}{B}-a\right)^{u}.$$

"The chance that, if $A$ is true, $B$ is true" is a somewhat ambiguous expression. It might mean $\frac{A:B}{\epsilon}$, or $\frac{A'+B}{\epsilon}$, or $\frac{B}{A}$. I showed in my last paper that the second of these is generally greater and never less than the first. I will now show that the second is also generally greater and never less than the third. We have

$$\frac{A'+B}{\epsilon}-\frac{B}{A} = 1-\frac{AB'}{\epsilon}-\frac{B}{A} = 1-\frac{A}{\epsilon}\,\frac{B'}{A}-\frac{B}{A}$$

$$= 1-\frac{A}{\epsilon}\left(1-\frac{B}{A}\right)-\frac{B}{A} = \left(1-\frac{A}{\epsilon}\right)\left(1-\frac{B}{A}\right).$$

Now, since $\frac{A}{\epsilon}$ and $\frac{B}{A}$ are *chances*, neither of these two bracket factors is negative; so that the chance $\frac{A'+B}{\epsilon}$ can never be less than the chance $\frac{B}{A}$. But the two chances may be equal, firstly, when $A$ is a *certainty*, and, secondly, when $A$ *implies* $B$.

I have in my Fifth Paper used the symbol $W\phi$ to denote the weakest datum or antecedent (expressed in terms of $\epsilon$ or $\eta$ or $\theta$ as indices or predicates) from which we can infer $\phi$, and $S\phi$ to denote the strongest conclusion we can draw from $\phi$ as our only datum. I now propose, as an almost necessary complement to these symbols, to employ another symbol $V\phi$ to denote the antecedents from which (without further data) we can infer neither $\phi$ nor $\phi'$. For example, if $\phi$ denote

$\phi\,(A, B, C)$, the symbol $V\phi$ will denote the sum (or the simplified equivalent of the sum) of the terms from which we cannot (in the absence of further data) infer either $\phi$ or $\phi'$; the total number of terms of which these $V$-terms form a part being the 27 given by the product $A^{\epsilon,\eta,\theta}\,B^{\epsilon,\eta,\theta}\,C^{\epsilon,\eta,\theta}$; and the remaining terms being those belonging to $W\phi$ (from each of which we can infer $\phi$) and those belonging to $W\phi'$ (from each of which we can infer $\phi'$). It is evident that $V\phi = V\phi'$, so that we have

$$\phi = W\phi + \phi\,.V\phi,$$

$$\phi' = W\phi' + \phi'.V\phi,$$

$$\epsilon = W\phi + W\phi' + V\phi.$$

For example, let it be required to find $W\phi$, $W\phi'$, $S\phi$, $S\phi'$, and $V\phi$, when we have

$$\phi_{,} = A_{0B} : A_{B'+C}.$$

By simple inspection, we get

$$\phi\,!\;\; A^{\eta} + B^{\eta} + C^{\epsilon};$$

and therefore

$$\phi' : A^{\eta\iota} B^{\eta\iota} C^{\epsilon\iota},$$

two considerations which will enable us to shorten the operations of finding $W\phi$, $W\phi'$, &c. These may be found by any of the methods in my Fifth Paper, and, when reduced to their simplest forms, are

$$W\phi = A^{\eta} + B^{\theta\iota} + C^{\epsilon},$$

$$W\phi' = A^{\epsilon} B^{\theta} C^{\eta},$$

$$S\phi = W'\phi' = (W\phi')' = A^{\epsilon\iota} + B^{\theta\iota} + C^{\eta\iota},$$

$$S\phi' = W'\phi = (W\phi)' = A^{\eta\iota} B^{\theta} C^{\epsilon\iota},$$

$$V\phi = V\phi' = (A^{\theta} C^{\eta} + A^{\eta\iota} C^{\theta})\, B^{\theta}.$$

Though we can infer neither $\phi$ nor $\phi'$ from any of the three terms in $V\phi$, we can, nevertheless, draw other conclusions, as follows:—

$$A^{\theta} C^{\eta} B^{\theta} \phi = A^{\theta} B^{\theta} C^{\eta}\,(A_{0B} : A_{B'+C})$$

$$= A^{\theta} B^{\theta} C^{\eta}\,(A_{B} + A_{B'+C});$$

for, $A$, $B$, $C$ being understood to be *singulars*, which implies that each is always of the same class, be it $\epsilon$ or $\eta$ or $\theta$, it follows that $A_{0B}$ and

$A_{B'+C}$ are *constants*. Hence

$$A^{\beta}B^{\gamma}C^{\eta}\phi = A^{\beta}B^{\gamma}C^{\eta}(A_B + A_{B'+\eta})$$
$$= A^{\beta}B^{\gamma}C^{\eta}(A_B + A_{B'}).$$

Also
$$A^{\alpha}B^{\gamma}C^{\theta}\phi = A^{\alpha}B^{\gamma}C^{\theta}(\epsilon_B + \epsilon_{B'+C})$$
$$= A^{\alpha}B^{\gamma}C^{\theta}B_C;$$

for
$$B^{\theta}\epsilon_B = \eta, \quad \text{and} \quad \epsilon_{B'+C} = B_C.$$

But from the third of the three terms which constitute $V\phi$ we can draw no conclusion; we only get

$$A^{\beta}B^{\gamma}C^{\theta}\phi = A^{\beta}B^{\gamma}C^{\theta}(A_B + A_{B'+C}) = A^{\beta}B^{\gamma}C^{\theta}\phi,$$

which is a truism.

All these results may be illustrated diagrammatically. Take, for example,

$$W\phi' = A^{\alpha}B^{\theta}C^{\eta},$$

which asserts that $\phi$ is false when $A$ is a *certainty*, $B$ a *variable*, and $C$ an *impossibility*. We draw a boundary $A$, including the *whole universe of discourse* $E$; then a boundary $B$, including *a part, but not the whole*, of the universe of discourse $E$; and, finally, a boundary $C$, *wholly excluded* from the universe of discourse $E$. This may be done in various ways of which the accompanying figure (Fig. 2) is one.

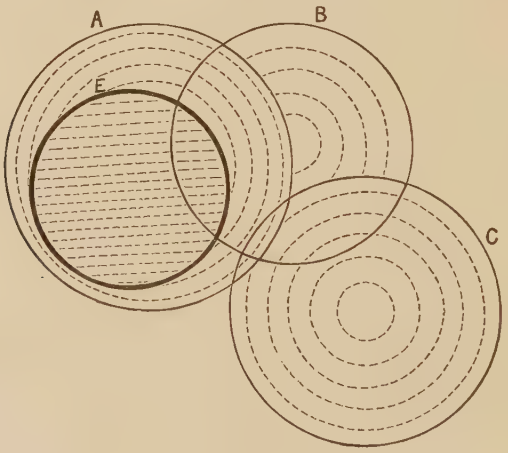


Fig. 2.

The statement $\phi$ denotes $A_B + A_{E'+C}$, which is equivalent to $A_B + (AB)_C$,

and asserts an alternative, namely, that any point taken at random in the collection $E$ must either be in $B$ if it is in $A$, or else must be in $C$ if it is in both $A$ and $B$. Both terms of the alternative are false, so that $\phi$ is false. Or, putting $\phi$ into the (here) equivalent form $A_{oB} : (AB)_C$, and employing the phraseology of the ordinary logic, the implication may be read: "If some $A$ is not $B$, then all $AB$ is $C$"—a false conditional, since its antecedent is true and consequent false.

Next, take $$W\phi = A^{\eta} + B^{\theta\iota} + C^{\epsilon}.$$

The middle term is equivalent to $B^{\epsilon} + B^{\eta}$, so that the equation asserts that each of the four terms $A^{\eta}$, $B^{\epsilon}$, $B^{\eta}$, $C^{\epsilon}$ implies $\phi$. To illustrate the case $B^{\eta}$, we may draw the boundaries $A$ and $C$ as we please, provided the points in the collection $B$ (if such there be) are *wholly excluded from our universe of discourse, the collection* $E$.

Let the four boundaries be drawn as in the accompanying figure (Fig. 3). The antecedent $A_{oB}$ of the implication $A_{oB} : (AB)_C$ asserts

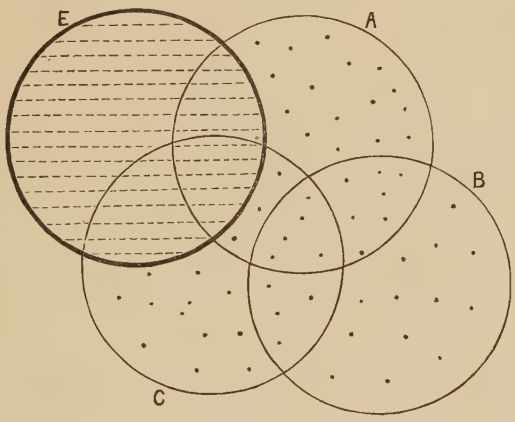


Fig. 3.

that a point taken at random in $E$ may be in the collection $A$ without being in the collection $B$, which is true; while the consequent $(AB)_C$ asserts that a point taken at random in $E$ cannot be in both $A$ and $B$ and excluded from $C$, which is also true since such a point cannot be in $B$. Hence $\phi$ in this case holds good.

In my Fifth Paper I referred to Dr. Schroder's error, that in Formula 13 of my Third Paper, namely, the formula

$$A_x + B_x : (AB)_x,$$

the sign of implication (:) should be the sign of equality (=). The accompanying figure (Fig. 4) will make the error evident. For, speaking of a point taken at random in the universe of discourse $E$,

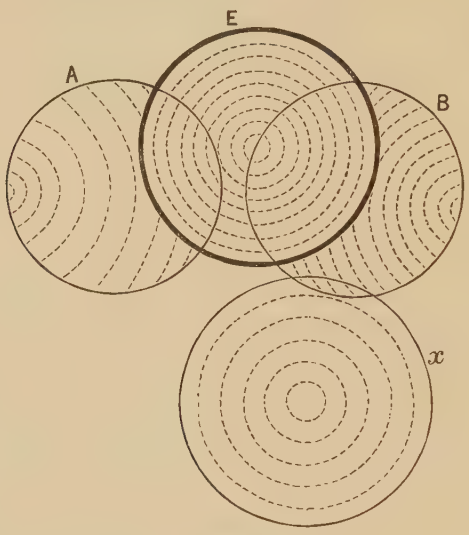


Fig. 4.

to say that if it falls in $A$ it will fall in $x$, or that if it falls in $B$ it will fall in $x$, is to assert an *impossibility;* while to say that its falling in both $A$ and $B$ (*i.e.*, in the non-existent area common to both), but not in $x$, is an impossibility (which is the consequent assertion $AB : x$ or its equivalent $ABx' : \eta$) is to affirm a *certainty*, since

$$AB = \eta, \quad \text{and} \quad \eta : x = \epsilon.$$

Thus Formula 13 in the case here illustrated reduces to the form $\eta : \epsilon$, which always $= \epsilon$; whereas, if the sign of equality replaced the sign of implication in the formula, it would become $\eta = \epsilon$, which always $= \eta$.

Of course, if my symbol $\alpha : \beta$ were always synonymous with my symbol $\alpha' + \beta$, then, since $A_x + B_x$ would in that case be synonymous with $(A' + x) + (B' + x)$, while $(AB)_x$ would be synonymous with $(AB)' + x$, that is, with $A' + B' + x$, the antecedent and consequent of my implication $A_x + B_x : (AB)_x$ would be equivalent, and the symbol (=) might be substituted for the symbol (:); but I have always insisted in my previous papers published years ago in the *Proceedings*, *Mind*, and elsewhere, that, though $\alpha : \beta$ always implies $\alpha' + \beta$, it is *not*

equivalent to it. To account for the divergence of my view from that of most other logicians in regard to this point I may here repeat, what I long ago stated in my Third Paper, that my logical calculus owes its birth to a question in probability (No. 3440) proposed by Mr. Stephen Watson twenty-six years ago in the *Educational Times*. The method having thus sprung from probability, it is not surprising that in its subsequent developments it should throughout bear the marks of its origin. While the majority of other logicians, starting from the traditional logic, naturally thought of their *universes*, their *classes* or *properties*, &c., I, from the mere accident of circumstances, entering upon the subject by a different road, necessarily regarded it from a different point of view, and thought of *occurrences* with their causes and consequences; of *statements* with their implications and equivalences; of *chances* always restricted between the limits of unity and zero. Hence the fact that, though in my words I may, like others, have sometimes confounded the *true* with the *certain*, the *false* with the *impossible*, I always kept clear of these ambiguities in my symbolic formulæ.

[*A Correction.*—By actual application of the experimental method described in Note 7, p. 569, of my Sixth Paper, I have been led to doubt the correctness of the chance, ·18906, therein given, and revision of my calculation has shown this doubt to have been well founded. The true chance is ·25441. I had made the mistake of taking $\log 2$ as meaning $\log_{10} 2$, whereas it, of course, means $\log_e 2$. The mistake is repeated on p. 573. This numerical error does not, of course, invalidate the algebraic formulæ or general principle; and the matter only deserves notice as an example of the advantage of testing theory by experiment in probability, as in other branches of science. The actual result was 35 quadratics with real roots in a total of 125 quadratics; a result which, by the formula $M \Big/ \left(M + \frac{1}{n+1}\right)$, gives a probability of about 2 to 1 *against* the correctness of ·18906, and about 8 to 1 *in favour of* the correctness of ·25541.]

*Note on Bessel Functions.* *By* H. M. Macdonald.

Received November 10th, 1897. Read November 11th, 1897.

The solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

can be represented by convergent series in ascending powers of $x$ for all values of $n$, and by semiconvergent series in descending powers of $x$ for values of $n$, which are such that the real part of $n$ is greater than $-\frac{1}{2}$. If $y_1$ and $y_2$ are two solutions of the first kind, $y_1'$ and $y_2'$ of the second, then they are connected by relations of the form

$$y_1'=ay_1+by_2,\quad y_2'=cy_1+dy_2.$$

The constants in the above relations are usually obtained by calculating the numerical value of the two sides of the equation for certain values of $x$ (Stokes, *Camb. Phil. Trans.*, Vol. IX., X.; Weber, *Crelle*, Vol. LXXV.; *Math. Ann.*, Vol. XXXVII.). The object of the following note is to show how the one form of solution can be directly obtained from the other.

1. *Solution of the Differential Equation.*

The solutions of the differential equation have been given by Sonine (*Math. Ann.*, Vol. XVI.) in the form

$$\frac{1}{2\pi\iota}\int e^{\frac{1}{2}x(t-1/t)}\frac{dt}{t^{n+1}},$$

the paths of integration being such that they begin and end at places where $e^{\frac{1}{2}x(t-1/t)}$ vanishes, $n$ being arbitrary. These solutions are there obtained from the difference equation; they can be obtained from the differential equation in the following manner.

Assume

$$y=\int_{s_0}^{s_1}e^{xs}S\,ds,$$

where $S$ is a function of $s$ not involving $x$. Then, substituting in the differential equation, after having multiplied by $x^2$,

$$\int_{s_0}^{s_1} e^{xs}\,(x^2s^3+xs+x^2-n^2)\,S\,ds = 0;$$

integrating by parts

$$e^{xs}\left[x\,(s^2+1)\,S-\frac{d}{ds}\left\{(s^2+1)\,S\right\}+sS\right]\Big|_{s_0}^{s_1}$$
$$+\int_{s_0}^{s_1} e^{xs}\left[\frac{d^2}{ds^2}\left\{(s^2+1)\,S\right\}-\frac{d}{ds}\,(sS)-n^2S\right]ds = 0.$$

Hence
$$e^{xs}\left[x\,(s^2+1)\,S-\frac{d}{ds}\left\{(s^2+1)\,S\right\}+sS\right]\Big|_{s_0}^{s_1} = 0, \qquad (1)$$

$$\frac{d^2}{ds^2}\left\{(s^2+1)\,S\right\}-\frac{d}{ds}\,(sS)-n^2S = 0. \qquad (2)$$

Putting $s = \sinh\eta$, (2) becomes

$$\frac{1}{\cosh\eta}\,\frac{d}{d\eta}\left\{\frac{1}{\cosh\eta}\,\frac{d}{d\eta}\,(S\cosh^2\eta)\right\}-\frac{1}{\cosh\eta}\,\frac{d}{d\eta}\,(S\sinh\eta)-n^2S = 0,$$

that is,
$$\frac{d^2}{d\eta^2}\,(S\cosh\eta)-n^2S\cosh\eta = 0.$$

A solution of this is
$$S\cosh\eta = Ae^{-n\eta},$$

whence
$$y = A\int_{\eta_0}^{\eta_1} e^{x\sinh\eta}\,e^{-n\eta}\,d\eta$$

is a solution of the original differential equation, $\eta_0$ and $\eta_1$ being given by

$$e^{x\sinh\eta}\left[xS\cosh^2\eta-\frac{d}{d\eta}\,(S\cosh\eta)\right]\Big|_{\eta_0}^{\eta_1} = 0,$$

that is, by
$$e^{x\sinh\eta}\,(x\cosh\eta+n)\,e^{-n\eta}\Big|_{\eta_0}^{\eta_1} = 0.$$

Writing $e^{\eta} = t$, $y$ takes the form

$$A\int_{t_0}^{t_1} e^{\frac{1}{2}x\,(t-1/t)}\,\frac{dt}{t^{n+1}},$$

where $t_0$, $t_1$ satisfy
$$e^{\frac{1}{2}x\,(t-1/t)}\left[\frac{x}{2}\,(t+t^{-1})+n\right]t^{-n}\Big|_{t_0}^{t_1} = 0.$$

The following values of $t_0$ and $t_1$ satisfy this condition:—

$$t_0 = \infty e^{\alpha\iota}, \qquad t_1 = \infty e^{\beta\iota},$$
$$t_0 = 0e^{(\pi-\alpha)\iota}, \quad t_1 = 0e^{(\pi-\beta)\iota},$$
$$t_0 = 0e^{(\pi-\alpha)\iota}, \quad t_1 = \infty e^{\beta\iota},$$

where the real parts of $xe^{\alpha\iota}$, $xe^{\beta\iota}$ are less than zero. The solutions corresponding to these sets of values for the limits may be denoted by $y_1$, $y_2$, $y_3$; they are the same as those given by Sonine. The solution $y_1$ is immediately seen to be equivalent to the series which is usually taken to define $J_n(x)$, the constant $A$ being put equal to $\frac{1}{2\pi\iota}$. The solution $y_2$ is equivalent to $e^{-n\pi\iota}J_{-n}(x)$ and $y_3$ to

$$\frac{J_{-n}(x) - e^{-n\pi\iota}J_n(x)}{2\iota \sin n\pi}.$$

These relations are established in the paper quoted.

## 2. *The Semiconvergent Forms of the Solutions.*

When a function can be represented, for certain values of the variables involved, by two series one of which is convergent and the other semiconvergent, it seems natural to expect that it can be represented by a double series of the form $\Sigma_p \Sigma_q u_{pq}$, where summation with respect to $q$ first gives one of the series, and with respect to $p$ first the other. In the present case it is required to find a double integral which shall be equal to the integral above given, change in the order of integration giving the semiconvergent form.

Taking the solution

$$y_3 = \frac{1}{2\pi\iota}\int_{0\,e^{(\pi-\alpha)\iota}}^{\infty e^{\beta\iota}} e^{\frac{1}{2}x(t-1/t)}\frac{dt}{t^{n+1}},$$

and putting $xt = 2\tau$,

$$y_3 = \frac{1}{2\pi\iota}\left(\frac{x}{2}\right)^n \int_{0\,xe^{(\pi-\alpha)\iota}}^{\infty xe^{\beta\iota}} e^{\tau-(x^2/4\tau)}\frac{d\tau}{\tau^{n+1}}.$$

If the real part of $n$ is greater than $-1$, the upper limit may be taken to be $c+\infty\iota$, where $c$ is any real positive quantity, for the part of the integral from $\infty xe^{\beta\iota}$ to $c+\infty\iota$ vanishes. Further, if the real part of $x$ is taken to be greater than zero, the lower limit can be chosen so that the real part of $\tau$ is positive all along the path of

integration. Now

$$\Pi(n-\tfrac{1}{2}) = \tau^{n+\frac{1}{2}} \int_0^\infty e^{-\tau s} s^{n-\frac{1}{2}} ds,$$

when the real part of $\tau$ is positive and the real part of $n$ is greater than $-\frac{1}{2}$; therefore

$$y_3 = \frac{1}{2\pi\iota} \frac{x^n}{2^n \Pi(n-\frac{1}{2})} \int_0^{c+\infty\iota} \int_0^\infty e^{\tau - \tau s - (x^2/4\tau)} s^{n-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau\, ds;$$

where the real part of $n$ is greater than $-\frac{1}{2}$, and the real part of $x$ is greater than zero. This may be written

$$y_3 = \frac{1}{2\pi\iota} \frac{x^n}{2^n \Pi(n-\frac{1}{2})} \left[ \int_0^{c+\infty\iota} \int_0^1 e^{\tau - \tau s - (x^2/4\tau)} s^{n-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau\, ds \right.$$
$$\left. + \int_0^{c+\infty\iota} \int_1^\infty e^{\tau - \tau s - (x^2/4\tau)} s^{n-\frac{1}{2}} \tau^{-\frac{1}{2}} d\tau\, ds \right].$$

Writing in the first integral

$$\tau(1-s) = t,$$

and in the second

$$\tau(s-1) = t,$$

it becomes

$$y_3 = \frac{1}{2\pi\iota} \frac{x^n}{2^n \Pi(n-\frac{1}{2})} \left[ \int_0^{c+\infty\iota} \int_0^1 e^{t - [x^2(1-s)/4t]} s^{n-\frac{1}{2}} (1-s)^{-\frac{1}{2}} t^{-\frac{1}{2}} dt\, ds \right.$$
$$\left. + \int_0^{c+\infty\iota} \int_1^\infty e^{-t - [x^2(s-1)/4t]} s^{n-\frac{1}{2}} (s-1)^{-\frac{1}{2}} t^{-\frac{1}{2}} dt\, ds \right].$$

Now, by the above,

$$\int_0^{c+\infty\iota} e^{t - [x^2(1-s)/4t]} \frac{dt}{t^{\frac{1}{2}}} = \iota\sqrt{\pi}\, e^{\iota x\sqrt{(1-s)}},$$

and

$$\int_0^{c+\infty\iota} e^{-t - [x^2(s-1)/4t]} \frac{dt}{t^{\frac{1}{2}}} = \sqrt{\pi}\, e^{-x\sqrt{(s-1)}};$$

hence, changing the order of integration in the expression for $y_3$,

$$y_3 = \frac{1}{2\sqrt{\pi}} \frac{x^n}{2^n \Pi(n-\frac{1}{2})} \left[ \int_0^1 e^{\iota x\sqrt{(1-s)}} s^{n-\frac{1}{2}} (1-s)^{-\frac{1}{2}} ds \right.$$
$$\left. + \frac{1}{\iota} \int_1^\infty e^{-x\sqrt{(s-1)}} s^{n-\frac{1}{2}} (s-1)^{-\frac{1}{2}} ds \right].$$

Writing $D$ for $\frac{d}{dx}$, and observing that the expressions under the

integral sign are uniform between the limits of integration,

$$y_3 = \frac{x^n}{2^{n+1}\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\left[\int_0^1 e^{\iota x\sqrt{(1-s)}}(1-s)^{-\frac{1}{2}}\,ds + \frac{1}{\iota}\int_1^\infty e^{-x\sqrt{(s-1)}}(s-1)^{-\frac{1}{2}}\,ds\right],$$

that is,
$$y_3 = \frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x}. \qquad (3)$$

Substituting for $y_3$ the equivalent expression

$$\frac{J_{-n}(x)-e^{-n\pi\iota}J_n(x)}{2\iota\sin n\pi} = \frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x}. \qquad (3)$$

Similarly, it may be shown that

$$\frac{J_{-n}(x)-e^{n\pi\iota}J_n(x)}{2\iota\sin n\pi} = \frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{e^{-\iota x}}{\iota x}. \qquad (4)$$

These expressions can be extended to the case where the real part of $x$ is negative after the manner of Weber, *Math. Ann.*, Vol. XXXVII. The symbolical forms for the Bessel functions (3) and (4) have been given by Hargreave, *Phil. Trans.*, 1848, for $J_n(x)$ and $Y_n(x)$. From the above, subtracting (4) from (3),

$$J_n(x) = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{\sin x}{x};$$

adding,
$$\frac{J_{-n}(x)-J_n(x)\cos n\pi}{\sin n\pi} = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{\cos x}{x};$$

when $n$ is an integer, this latter becomes

$$\frac{1}{\pi}Y_n = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{\cos x}{x},$$

where $Y_n$ is Hankel's second solution of the equation.

The integral forms can be obtained from the symbolical as follows:—

$$\frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x} = \frac{x^n}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})}(1+D^2)^{n-\frac{1}{2}}\frac{1}{\iota}\int_0^\infty e^{-x(t-\iota)}\,dt,$$

that is, $$y_3 = \frac{x^n}{2^n \sqrt{\pi}\, \Pi(n-\frac{1}{2})} \cdot \frac{1}{\iota} \int_0^\infty e^{-x(t-\iota)} (-2\iota t + t^2)^{n-\frac{1}{2}} dt;$$

hence $$y_3 = \frac{x^n\, 2^{n-\frac{1}{2}}}{2^n \sqrt{\pi}\, \Pi(n-\frac{1}{2})} e^{\iota x - \frac{1}{2}(2n+1)\pi\iota} \int_0^\infty e^{-xt} t^{n-\frac{1}{2}} \left(1 - \frac{t}{2\iota}\right)^{n-\frac{1}{2}} dt.$$

Putting $xt = s$,

$$y_3 = \frac{e^{\iota x - \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 - \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds,$$

that is,

$$\frac{J_{-n}(x) - e^{-n\pi\iota} J_n(x)}{2\iota \sin n\pi} = \frac{e^{\iota x - \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 - \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds,$$

$$\frac{J_{-n}(x) - e^{n\pi\iota} J_n(x)}{2\iota \sin n\pi} = -\frac{e^{-\iota x + \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 + \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds.$$

These are equivalent to the relations given by Weber, *Math. Ann.*, XXXVII., and can be extended to the case where the real part of $x$ is negative as before.

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*The Integral* $\int P_n^2 dx$, *and Allied Forms in Legendre's Functions, between Arbitrary Limits.* *By* R. HARGREAVES, M.A. Received October 28th, 1897. Read November 11th, 1897; and received, in revised form, December 15th, 1897.

For different positive integers the indefinite integral $\int P_m P_n\, dx$ has been expressed in simple form, and when $m = n$ the value of the definite integral between the special limits 0 and 1, or $-1$ and $+1$, is due to Legendre. The fundamental theorem of the present paper expresses the difference $(2n+3)\, P_{n+1}^2 - (2n+1)\, P_n^2$ as the differential coefficient of a simple expression involving $P_n$ and $P_{n+1}$. From this follows the difference of two consecutive integrals between arbitrary limits, and a direct summation gives the value of the single integral. As the argument turns on the use of sequence equations, it is at once applicable to the forms $P_n Q_n$ and $Q_n^2$, and moreover the index $n$ may

be fractional. With the integrals mentioned above are associated the group of integrals

$$\int (1-x^2)\left(\frac{dP_n}{dx}\right)^2 dx, \quad \int (1-x^2)\frac{dP_n}{dx}\frac{dQ_n}{dx}\,dx,$$

and

$$\int (1-x^2)\left(\frac{dQ_n}{dx}\right)^2 dx,$$

to which corresponding results apply.

1. Starting from the relations

$$(1-x^2)\,P'_{n+1} = (n+1)(P_n - xP_{n+1}), \qquad (a)$$

$$(1-x^2)\,P'_n = (n+1)(xP_n - P_{n+1}), \qquad (b)$$

square and subtract; then

$$(1-x^2)^2\,(P'^2_n - P'^2_{n+1}) = (n+1)^2\left\{(xP_n - P_{n+1})^2 - (P_n - xP_{n+1})^2\right\}$$

$$= (n+1)^2\,(1-x^2)\,(P^2_{n+1} - P^2_n),$$

or

$$(1-x^2)\,(P'^2_n - P'^2_{n+1}) = (n+1)^2\,(P^2_{n+1} - P^2_n). \qquad (1)$$

Hence

$$(n+1)\left\{(2n+3)\,P^2_{n+1} - (2n+1)P^2_n\right\}$$

$$= (n+1)(n+2)\,P^2_{n+1} - n(n+1)\,P^2_n + (n+1)^3\,(P^2_{n+1} - P^2_n)$$

$$= (n+1)(n+2)\,P^2_{n+1} - n(n+1)\,P^2_n + (1-x^2)(P'^2_n - P'^2_{n+1})$$

$$= \frac{d}{dx}\left\{(1-x^2)(P_nP'_n - P_{n+1}P'_{n+1})\right\},$$

in virtue of

$$\frac{d}{dx}(1-x^2)\,P'_n = -n(n+1)\,P_n,$$

and a similar equation for $P_{n+1}$. Thus

$$\left.\begin{aligned}(2n+3)\,P^2_{n+1} - (2n+1)\,P^2_n &= \frac{d}{dx}\left\{\frac{1-x^2}{n+1}(P_nP'_n - P_{n+1}P'_{n+1})\right\} \\ &= \frac{d}{dx}\left\{x\,(P^2_n + P^2_{n+1}) - 2P_nP_{n+1}\right\},\end{aligned}\right\} \qquad (2)$$

the second form following from the first by use of $(a)$ and $(b)$. In integral form these are :—

$$\left.\begin{aligned}\int_0^x (2n+3)\,P^2_{n+1}dx - \int_0^x (2n+1)\,P^2_n dx &= \frac{1-x^2}{n+1}(P_nP'_n - P_{n+1}P'_{n+1}) \\ &= x\,(P^2_{n+1} + P^2_n) - 2P_nP_{n+1},\end{aligned}\right\} (3)$$

no constant being needed, as each right-hand member vanishes with $x$. As the right-hand member also vanishes with $x = 1$, we have as a particular case

$$\int_0^1 (2n+3)\, P_{n+1}^2\, dx = \int_0^1 (2n+1)\, P_n^2\, dx = \&c. = \int_0^1 P_0^2 dx = 1,$$

or

$$\int_0^1 P_n^2\, dx = \frac{1}{2n+1},$$

this being Legendre's result.

If the relation (3) is written for consecutive values of $n$ down to $n = 0$, summation gives

$$(2n+1)\int_0^x P_n^2\, dx$$

$$= x - \frac{1-x^2}{n} P_n P_n' - (1-x^2)\left\{ \frac{P_{n-1}P_{n-1}'}{n\,(n-1)} + \frac{P_{n-2}P_{n-2}'}{(n-1)(n-2)} + \ldots \frac{P_1 P_1'}{2.1} \right\}$$

$$= x P_n^2 - \frac{2\,(n-1)}{2n-1} P_n P_{n-1}$$

$$+ 2\left\{ \frac{P_{n-1}P_{n-2}}{(2n-1)(2n-3)} + \frac{P_{n-2}P_{n-3}}{(2n-3)(2n-5)} + \ldots \frac{P_1 P_0}{3.1} \right\}, \quad (4)$$

the second form following from the first by use of

$$(2n+1)(1-x^2)\, P_n' = n\,(n+1)(P_{n-1} - P_{n+1}). \qquad (c)$$

Also in the second form the bracket does not appear if $n = 1$.

The connexion with the integral $\int (1-x^2)\, P_n'^2\, dx$ is got by multiplying $(1-x^2)\, P_n'' - 2x P_n' + n\,(n+1)\, P_n = 0$ by $P_n$, viz.,

$$\left.\begin{aligned} &\frac{d}{dx}(1-x^2)\, P_n P_n' + n\,(n+1)\, P_n^2 = (1-x^2)\, P_n'^2, \\ \text{or in integral form} \\ &(1-x^2) P_n P_n' + \int_0^x n\,(n+1)\, P_n^2 dx = \int_0^x (1-x^2)\, P_n'^2 dx, \end{aligned}\right\} \quad (5)$$

through which a form analogous to (4) may be easily found for

$$\int_0^x (1-x^2)\, P_n'^2\, dx.$$

It may be remarked that when the limits are any of the quantities $0$, $\pm 1$, or any root of $P_n = 0$ or $P_n' = 0$,

$$\int (1-x^2)\, P_n'^2\, dx = n\,(n+1) \int P_n^2 dx.$$

2. The formulæ are immediately applicable to $Q_n$ if we note, firstly, the difference in the opening terms, and, secondly, the necessity of keeping distinct the two cases $x>1$ and $x<1$, for the first of which $Q_0 = \frac{1}{2}\log\frac{x+1}{x-1}$, for the second $\frac{1}{2}\log\frac{1+x}{1-x}$, $Q_n$ being in each case given by $P_nQ_0 - Z_n$.

The series (4) are then modified to

$$\left.\begin{aligned}
&\int_0^x \{(2n+1)\,Q_n^2 - Q_0^2\}\,dx \\
&= Q_0 - \frac{1-x^2}{n}\,Q_nQ_n' - (1-x^2)\left\{\frac{Q_{n-1}Q_{n-1}'}{n(n-1)} + \dots \frac{Q_1Q_1'}{2.1}\right\} \\
&= Q_0 - Q_1Q_0 + xQ_n^2 - \frac{2(n-1)}{2n-1}\,Q_nQ_{n-1} \\
&\qquad\qquad + 2\left\{\frac{Q_{n-1}Q_{n-2}}{(2n-1)(2n-3)} + \dots \frac{Q_1Q_0}{3.1}\right\},
\end{aligned}\right\} \quad (6)$$

and the integral $\int_x^\infty$ with the other form of $Q_0$ requires the sign of the right-hand members of (6) reversed. To apply (3) to the case of $x=1$, we have to evaluate $x(Q_{n+1}^2 + Q_n^2) - 2Q_nQ_{n+1}$. As the zero of a vanishing rational factor outweighs the infinite element in $Q_0$ or $Q_0^2$, the limit is that of $(Z_{n+1}-Z_n)^2$ for $x=1$. But in this case the sequence equation for $Z$ is

$$(n+1)(Z_{n+1}-Z_n) = n(Z_n - Z_{n-1}) = \&c. = 1,$$

because $Z_0 = 0$, $Z_1 = 1$. Hence (3) gives for the two forms of $Q_0$

$$\left.\begin{aligned}
\int_1^\infty (2n+1)\,Q_n^2\,dx - \int_1^\infty (2n-1)\,Q_{n-1}^2\,dx &= -\frac{1}{n^2}, \\
\int_0^1 (2n+1)\,Q_n^2\,dx - \int_0^1 (2n-1)\,Q_{n-1}^2\,dx &= \frac{1}{n^2},
\end{aligned}\right\} \quad (7)$$

from which follow

$$\left.\begin{aligned}
&\int_1^\infty (2n+1)\,Q_n^2\,dx = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots \text{ ad inf.}, \\
\text{and } &\int_0^1 (2n+1)\,Q_n^2\,dx = \frac{\pi^2}{4} - \left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots \text{ ad inf.}\right\},
\end{aligned}\right\} \quad (8)$$

with a limit for $n$ infinite, zero in the former case, $\frac{\pi^2}{4}$ in the latter. The integrals

$$\int_1^\infty Q_0^2 dx = \frac{\pi^2}{6} \quad \text{and} \quad \int_0^1 Q_0^2 dx = \frac{\pi^2}{12}$$

admit of direct evaluation, and are required to obtain the form (8).

In (8) attention may be called to the fact that the sum of the two forms is $\frac{\pi^2}{4}$, and independent of $n$; this appearing as analogue to the result of Legendre, that

$$\int_0^1 (2n+1)\, P_n^2 dx = 1.$$

3. A slight modification is wanted for the integral $\int P_n Q_n dx$. Since $(a)$ and $(b)$ apply equally to $P_n$ or $Q_n$, we have in lieu of (1)

$$(1-x^2)(P_n' Q_n' - P_{n+1}' Q_{n+1}') = (n+1)^2 (P_{n+1} Q_{n+1} - P_n Q_n); \qquad (9)$$

and therefore in lieu of (4)

$$\int_1^x (2n+3)\, P_{n+1} Q_{n+1} dx - \int_1^x (2n+1)\, P_n Q_n dx$$

$$= \frac{x^2-1}{n+1} (P_{n+1} Q_{n+1}' - P_n Q_n') = \frac{x^2-1}{n+1} (P_{n+1}' Q_{n+1} - P_n' Q_n)$$

$$= x\,(P_{n+1} Q_{n+1} + P_n Q_n) - (P_n Q_{n+1} + P_{n+1} Q_n), \qquad (10)$$

the equivalence of the first two forms depending on

$$P_n Q_n' - P_n' Q_n = \frac{1}{1-x^2}.$$

For $x<1$ the integrals are written $\int_x^1$, and the sign of the right-hand members of (10) changed. With $x = 0$,

$$P_n Q_{n+1} + P_{n+1} Q_n = -(P_n Z_{n+1} + P_{n+1} Z_n) = \frac{(-1)^{n+1}}{n+1},$$

using the sequence equation, and noting the opening values $P_0 = 1$, $P_1 = 0$, $Z_0 = 0$, $Z_1 = 1$. Hence

$$\int_0^1 (2n+1)\, P_n Q_n dx - \int_0^1 (2n-1)\, P_{n-1} Q_{n-1} dx = \frac{(-1)^n}{n}; \qquad (11)$$

and therefore, as $\int_0^1 Q_0 dx = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ ad inf.,

$$\int_0^1 (2n+1) P_n Q_n dx$$
$$= (-1)^{n+1}\left\{\frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \ldots \text{ ad inf.}\right\}.$$

The corresponding integral $\int_1^\infty$ is infinite with $\int_1^\infty Q_0 dx$, though, as in (11), the difference of consecutive values is finite, viz.,

$$\frac{1}{2n+1} + \frac{1}{2n-1} - \frac{1}{n}.$$

The summation of (10) leads to

$$\int_1^x \{(2n+1) P_n Q_n - Q_0\} dx$$
$$= \frac{x^2-1}{n} P_n' Q_n + (x^2-1)\left\{\frac{P_{n-1}' Q_{n-1}}{n(n-1)} + \ldots \frac{P_1' Q_1}{2.1}\right\}$$
$$= -P_0 Q_1 + x P_n Q_n - \frac{2(n-1)}{2n-1} P_{n-1} Q_n + 2\left\{\frac{P_{n-2} Q_{n-1}}{(2n-1)(2n-3)} + \ldots \frac{P_0 Q_1}{3.1}\right\}$$
$$+ \frac{1}{n(2n-1)} + \frac{1}{(n-1)(2n-3)} + \ldots \frac{1}{2.3}, \qquad (12)$$

and two closely related forms, the right-hand member needing reversal of sign to represent $\int_x^1$ for the case of $x<1$. Corresponding to (5) is

$$(1-x^2) P_n' Q_n' - n(n+1) P_n Q_n = -\frac{d}{dx}(1-x^2) P_n Q_n' = -\frac{d}{dx}(1-x^2) P_n' Q_n,$$

and the particular values of $(1-x^2) P_n' Q_n$ for $x = 1, 0, \infty$ may be given, viz., 0; $-1$ for $n$ odd, 0 for $n$ even; and $-\frac{n}{2n+1}$ respectively.

4. I propose to take as example of fractional order the case where the order is $n-\frac{1}{2}$, $n$ being 0 or a positive integer. The coefficient of the last integral $2(-\frac{1}{2})+1$ vanishing, all terms above this are exactly integrable. These functions have been treated at some length under the name "toroidal functions," by Dr. Hicks, who modifies Heine's notation, writing

$$P_n = \int_0^\pi (x+\sqrt{x^2-1}\cos\theta)^{n-\frac{1}{2}} d\theta, \quad Q_n = \int_0^\infty \frac{d\theta}{(x+\sqrt{x^2-1}\cosh\theta)^{n+\frac{1}{2}}},$$

the more general notation describing these as $P_{n-\frac{1}{2}}$ and $Q_{n-\frac{1}{2}}$. Adopting this notation, the sequence equation is

$$(2n+1)\,P_{n+1}+(2n-1)\,P_{n-1}=4nxP_n,$$

and the opening terms

$$P_0=2k^{\frac{1}{2}}K', \quad P_1=2k^{-\frac{1}{2}}E',$$

$$Q_0=2k^{\frac{1}{2}}K, \quad Q_1=2k^{-\frac{1}{2}}(K-E).$$

The modulus $$k=x-\sqrt{x^2-1},$$

$K$ and $E$ are complete elliptic integrals to modulus $k$, $K'$ and $E'$ to the complementary modulus $k'$, and $x$ has any real value greater than 1. For $x=1$, $P_n=\pi$, and equation (3) is replaced by

$$\left.\begin{aligned}\int_1^x (n+1)\,P_{n+1}^2\,dx-\int_1^x nP_n^2\,dx &= \frac{x^2-1}{2n+1}(P_{n+1}P'_{n+1}-P_nP'_n)\\ &= \frac{x}{2}(P_n^2+P_{n+1}^2)-P_nP_{n+1},\\ \int_1^x P_1^2\,dx &= (x^2-1)(P_1P'_1-P_0P'_0)\\ &= \frac{x}{2}(P_0^2+P_1^2)-P_0P_1,\end{aligned}\right\} \tag{13}$$

and, by summation,

$$\left.\begin{aligned}&\int_1^x (n+1)\,P_{n+1}^2\,dx\\ =\frac{x^2-1}{2n+1}&P_{n+1}P'_{n+1}+2(x^2-1)\left\{\frac{P_nP'_n}{(2n+1)(2n-1)}+\dots\frac{P_1P'_1}{3.1}\right\}-(x^2-1)\,P_0P'_0\\ =-\frac{2n-1}{4n}&P_nP_{n+1}+\frac{x}{2}(P_{n+1}^2+P_0^2)-P_0P_1+\tfrac{1}{4}\left\{\frac{P_nP_{n-1}}{n\,(n-1)}+\dots\frac{P_1P_0}{1}\right\},\end{aligned}\right\} \tag{14}$$

the former true to $n=0$, the latter to $n=1$. For the $Q_n$ functions it is convenient to write the integrals $\int_x^\infty$; the sign of the right-hand member is reversed, and the last equation needs a constant, viz.,

$$\int_x^\infty Q_1^2\,dx=Q_0Q_1-\frac{x}{2}(Q_0^2+Q_1^2)+\frac{\pi^2}{4}, \tag{15}$$

because the limit of $xQ_0^2$ for $x$ infinite is $\frac{\pi^2}{2}$. The integrals between limits 1 and $\infty$ are finite for $Q$, infinite for $P$. In the former case

for $x=1$, $Q_0=2K$, $Q_1=2(K-E)=2K-2$, and $K$ is infinite. The part of $Q_n$ due to $K$ is the same for all values of $n$, and the whole value $2K-2q_n$, where $q_n$ obeys the sequence equation

$$(2n+1)\,q_{n+1}+(2n-1)\,q_{n-1}=4nq_n,$$

and starts with $q_0=0$, $q_1=1$; thus

$$(2n+1)(q_{n+1}-q_n)=(2n-1)(q_n-q_{n-1})=1.$$

Thus $Q_nQ_{n+1}-\frac{x}{2}(Q_n^2+Q_{n+1}^2)=-\frac{1}{2}(Q_{n+1}-Q_n)^2$

$$=-2(q_{n+1}-q_n)^2=-\frac{2}{(2n+1)^2},$$

so that $$\int_1^\infty (n+1)\,Q_{n+1}^2dx-\int_1^\infty nQ_n^2dx=-\frac{2}{(2n+1)^2},$$

and $$\int_1^\infty Q_1^2dx=\frac{\pi^2}{4}-2,$$

and by summation

$$\int_1^\infty nQ_n^2dx=2\left\{\frac{1}{(2n+1)^2}+\frac{1}{(2n+3)^2}+\ldots \text{ ad inf.}\right\},\qquad (16)$$

analogous to (8) for functions of integral order.

For the form $P_nQ_n$, we have (true to $n=0$)

$$\int_1^x (n+1)\,P_{n+1}Q_{n+1}dx-\int_1^x nP_nQ_ndx$$

$$=\frac{x^2-1}{2n+1}(P_{n+1}Q'_{n+1}-P_nQ'_n)$$

$$=\frac{x}{2}(P^nQ_n+P_{n+1}Q_{n+1})-\tfrac{1}{2}(P_nQ_{n+1}+P_{n+1}Q_n).\qquad (17)$$

For $x=\infty$, the right-hand member

$$=\frac{\pi}{4}\left(\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right)$$

in the general case, and is infinite in the last case.

The character of the integrals evaluated here may be made more distinct by writing them in elliptic integrals. The $P_1^2$ integral is then

$$\int_k^1\frac{2k^2E'^2dk}{k^3}=\left(k+\frac{1}{k}\right)\left[\frac{E'^2}{k}+kK'^2\right]-4E'K',$$

or, in the complementary form,

$$\int_0^k \frac{2k^3E^2dk}{k'^4} = \left(k' + \frac{1}{k'}\right)\left[\frac{E^2}{k'} + k'K^2\right] - 4EK.$$

The $Q_1^2$ integral is

$$\int_0^k \frac{2k'^2}{k^3}(K-E)^2\,dk = 3K^2 - 4KE - k^2K^2 - \frac{1+k^2}{k^2}(K-E)^2 + \frac{\pi^2}{4},$$

and, when $k = 1$ for the upper limit, the integral

$$= \frac{\pi^2}{4} - 2.$$

The $P_1Q_1$ integral is

$$\int_k^1 \frac{2k'^2(K-E)\,E'\,dk}{k^3} = 2EK' + \frac{k'^2}{k^2}KE' - k'^2KK' - \left(1 + \frac{1}{k^2}\right)EE',$$

or, in complementary form,

$$\int_0^k \frac{2k^3E\,(K'-E')\,dk}{k'^4} = 2E'K + \frac{k^2}{k'^2}EK' - k^2KK' - \left(1 + \frac{1}{k'^2}\right)EE'.$$

The general case gives under the integral sign a quadratic function of $E$ and $K$ or $E'$ and $K'$, with coefficients, rational and integral functions of $k$.

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*Thursday, December 9th,* 1897.

Professor ELLIOTT, F.R.S., President, in the Chair.

Fourteen members present.

The following persons were elected members:—Miss Ada Isabel Maddison, B.Sc. London, Bryn Mawr College, Pennsylvania, U.S.A., and Mr. William Fleetwood Sheppard, M.A., LL.M., formerly Fellow of Trinity College, Cambridge.

Mr. A. Berry and Mr. E. T. Whittaker were admitted into the Society.

The President announced that, the Rev. T. R. Terry being unable to act as Auditor, he had, through the Secretary, requested Mr. E. G. Gallop to act in that capacity. The sanction of the meeting was granted for the President's action, and then the Auditor's report was

presented. The adoption of the Treasurer's report, coupled with a vote of thanks to Mr. Gallop, was then moved by Professor Hudson, seconded by Mr. J. J. Walker, and carried unanimously.

Miss Hardcastle communicated "A Theorem concerning the Special Systems of Point Groups on a Particular Type of Base Curve."

Mr. Love gave an abstract of a paper by Professor W. Burnside, "The Construction of the Straight Line joining Two Given Points."

Impromptu communications were made by Mr. Berry ("On some Orthogonal Covariants"), Mr. Whittaker ("On Multiform and Uniform Functions," on the lines of a memoir by M. Poincaré), and Mr. Macaulay ("A Geometrical Theorem"). Messrs. Larmor, Mathews, and Cunningham, and the President, joined in discussions upon the communications.

The following presents were made to the Library:—

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 6; 1897.

"Nyt Tidsskrift for Matematik," A. Aargang 8, Nos. 4, 5; B. Aargang 8, No. 3; Copenhagen, 1897.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXVI., Heft 2; Jahrgang 1895; Berlin, 1897.

"Archives Néerlandaises des Sciences exactes et naturelles," Série 2, Tome I., Liv. 2, 3; La Haye, 1897.

Graf, Dr. J. H.—"Der Mathematiker Jakob Steiner von Utzenstorf," 8vo; Bern, 1897.

Stokes, Sir G. G.—"On the Nature of the Röntgen Rays," pamphlet, 8vo; Manchester, 1897. (Offprint, Vol. XLI., Pt. IV., of Proceedings of the Manchester Literary and Philosophical Society.)

"Proceedings of the American Philosophical Society," Vol. XXXVI., No. 155; Philadelphia, May, 1897.

"Proceedings of the Physical Society," Vol. XV., Pt. 11, No. 86; London, 1897.

"Kansas University Quarterly," Vol. VI., No. 4, October, 1897.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. III., Fasc. 8–10; Napoli, 1897.

"Bulletin des Sciences Mathématiques," Tome XXI., Oct., 1897; Paris.

"Bulletin of the American Mathematical Society," Vol. IV., No. 2; November, 1897.

"The John Crerar Library, First and Second Annual Report for 1895–6," 8vo; Chicago, 1897.

Boehm, R.—"I. Das Calebassencurare, II. Das Topfcurare, III. Uber einige Curarerinden," 8vo; Leipzig, 1897.

"Transactions of the Cambridge Philosophical Society," Vol. XVI., Pt. 2; November, 1897.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VI., Fasc. 9, 10; Roma, 1897.

"Educational Times," December, 1897.

"Indian Engineering," Vol. XXII., Nos. 17–20, Oct. 23–Nov. 13, 1897.

"Bulletins de l'Académie Royale de Belgique," 1895, Tome XXX.; 1896, Tomes XXXI., XXXII.; 1897, Tome XXXIII.; Bruxelles.

"Annuaire de l'Académie Royale des Sciences, des Lettres et des Beaux Arts, 1896–7, et Réglements"; Bruxelles.

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## *The Construction of the Straight Line joining Two Given Points.*

*By* W. BURNSIDE. Received November 29th, 1897. Read December 9th, 1897.

Euclid's postulate: "Let it be granted that a straight line may be drawn from any one point to any other point" implies the use of a ruler or straight-edge of any required finite length. The postulate is clearly not intended to apply to the case in which the distance between the two points is infinite, *i.e.*, in which no finite number of repetitions of a finite operation will lead from one point to the other. In fact, Prop. XXXI, Book I, gives a compass and ruler construction for the line when one of the points can be reached while the other cannot. The other exceptional case, when neither point can be reached, *i.e.*, when the two given points are the points at infinity on two non-parallel lines, is not dealt with by Euclid. It is, however, not difficult to show by elementary geometrical considerations that in this case no point of the joining line can be reached by a finite number of finite operations.

In elliptic space any one point can be reached from any other by a finite number of finite operations. The line joining two given points can therefore be always constructed with the ruler alone.

In hyperbolic space as in Euclidean, there are, so long as we deal with elementary geometry, three cases of the construction to consider. These are (i) that in which the two points to be joined are points which can be reached from any assigned point by a finite number of finite operations, or say "finite points"; (ii) that in which one of the points is a finite point, and the other a point at infinity on a given straight line; (iii) that in which both the points are points at infinity.

If, however, we deal with projective geometry, we must assume that *every* two straight lines in a plane determine a point. When

the two straight lines are non-intersectors the point can neither be a finite point nor a point at infinity. Such a point is termed an "ideal point"; and there can be no projective geometry in hyperbolic space without the consideration of ideal points (lines and planes). The problem of constructing the straight line joining two given points involves therefore three further cases; namely, (iv) that in which one of the points is a finite point and the other an ideal point; (v) that in which one is a point a infinity and the other an ideal point; (vi) that in which both points are ideal points.

For these last three cases, however, the problem may be put in an alternative form. A pencil of lines in a plane which pass through a common ideal point is the same thing as a set of lines which have a common perpendicular. Hence the problem of constructing the line that joins a given point to the point determined by two non-intersectors in a plane is the same as that of finding the common perpendicular of the two non-intersectors and letting fall on it a perpendicular from the given point.

In the first only of the six cases can the construction be carried out by the ruler alone. Some, at least, of the constructions given below for the other five cases are either known or are equivalent to known constructions. The constructions are, however, here verified by purely geometrical and very simple reasoning; whereas, so far as I am aware, those given hitherto depend either directly or indirectly on considerations derived from analysis.

The geometrical results of which use is made were published in a paper "On the Kinematics of non-Euclidean Space," in Vol. xxvi of the Society's *Proceedings*. It will save the reader the trouble of reference if the results required are summarized here.

Let $AEB$, $CFD$ be any two lines, and $EF$ their common perpendicular. Then successive rotations through two right angles about $AB$ and $CD$ are equivalent to a translation of amplitude $2EF$, of which the line $EF$ is the axis, together with a rotation round $EF$ through twice the angle between the planes $AEF$ and $EFC$. In particular, successive rotations through two right angles about two intersecting lines are equivalent to a rotation about a line through their point of intersection perpendicular to their plane; successive rotations through two right angles about two non-intersectors in a plane are equivalent to a translation along their common perpendicular, and successive rotations through two right angles about two lines which meet at infinity give a displacement *sui generis* for which no finite point or straight line remains unaltered.

A rotation through two right angles about a line $AB$ will be represented by the symbol $\pi AB$, and in dealing with a plane figure a rotation through two right angles about a line through $A$ perpendicular to the plane of the figure will be represented by $\pi A$. The successive components of a compound displacement are to be read from left to right.

## I.

*To join a given finite point to a given point at infinity.*

Let $A$ be the given finite point, and $I$, on the line $BI$, the point at infinity. Draw $AB$ at right angles to $BI$, and $AE$ at right angles to $AB$. Take an arbitrary point $C$ on $BI$ and with $A$ as centre and $BC$ as radius describe a circle. From $C$ draw $CDE$ at right angles to $AE$, meeting the circle in $D$. Then $AD$ is the line required.

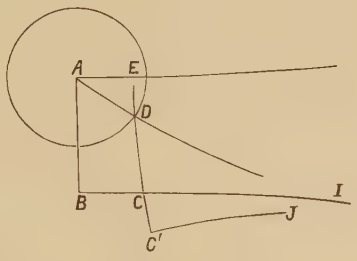


Fig. 1.

Produce $EC$ to $C'$, so that $CC'$ is equal to $ED$, and through $C'$ draw $C'J$ at right angles to $CC'$. Then,

$$\begin{aligned}\text{translation } 2DA \,.\, \text{translation } 2BC &= \pi D \,.\, \pi A \,.\, \pi B \,.\, \pi C,\\ &= \pi D \,.\, \pi AE \,.\, \pi BC \,.\, \pi C,\\ &= \pi C'J \,.\, \pi C \,.\, \pi BC \,.\, \pi C,\\ &= \pi C'J \,.\, \pi BC.\end{aligned}$$

Now, if $AD$ and $BC$ intersect in a finite point, a translation $2DA$ followed by a translation $2BC$ is equivalent to a translation along a line which meets $BC$ at an acute angle. On the other hand, $\pi C'J \,.\, \pi BC$ is equivalent to a rotation round a point of $BC$, a translation along a line at right angles to $BC$, or a displacement which keeps no finite point or line fixed, according as $C'J$ and $BC$ meet in a finite point, an ideal point, or a point at infinity. Hence $AD$ and $BC$ cannot intersect in a finite point.

If $AD$ and $BC$ intersect in an ideal point, let $PQ$ be their common perpendicular. On $ADP$, $BCQ$ produced, take $p$, $q$ such that

$$Pp = Qq = BC,$$

and through $p$ and $q$ draw $pp'$ and $qq'$ at right angles to $AD$ and $BC$. Then a translation $2DA$ followed by a translation $2BC$ is equivalent to $\pi pp'$ followed by $\pi qq'$. If $pp'$ and $qq'$ intersect, this is equivalent to a rotation round some point lying between $AD$ and $BC$; and, if they do not, it is equivalent to a translation along a line which is not at right angles to $BC$. In neither case can it be equivalent to $\pi C'J.\pi BC$. Hence $AD$ and $BC$ do not meet in an ideal point. Finally, then, $AD$ and $BC$ must meet in a point at infinity, or, in other words, $AD$ passes through $I$.

## II.

*To join a given finite point to a given ideal point.*

Since the joining line is at right angles to the common perpendicular of the two non-intersectors, which determine the ideal point, this is the same as :—*To find the common perpendicular to two given non-intersectors.*

(The preceding construction gives a test to determine whether or no two given lines are non-intersectors.)

*Lemma.*—Let $AA'$ and $BB'$ be at right angles to $AB$, and let $CC'$ bisect $AB$ at right angles. From $C'$ draw $C'A'$ at right angles to $AA'$ and $C'B'$ at right angles to $BB'$.

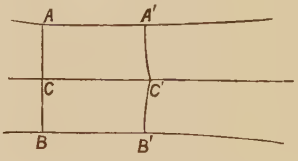


Fig. 2.

Then, translation $2A'A$ . translation $2BB'$

$$= \pi A'C' . \pi AB . \pi AB . \pi B'C'$$

$$= \pi A'C' . \pi B'C'.$$

Hence translations of equal amplitude and opposite sense along two non-intersectors can always be taken so small that they are equivalent to a rotation. They are, in fact, equivalent to a rotation unless the

translations are so great that $A'C'$, $B'C'$, constructed as in Fig. 2, are either non-intersectors or meet at infinity.

Let $AB$, $CD$ be the two given non-intersectors whose common perpendicular is required. On $AB$ take two points $A$ and $B$ sufficiently near together. Draw $AC$ and $BD$ at right angles to $CD$, and produce them to $A'$ and $B'$, so that

$$CA' = AC \quad \text{and} \quad DB' = BD.$$

Then, since $\pi CD$ changes $AB$ into $A'B'$, the common perpendicular of $AB$ and $CD$ is also at right angles to $A'B'$. Produce $CA$ and $DB$ to $a$ and $b$, so that

$$Aa = CA \quad \text{and} \quad Bb = DB;$$

and through $a$ and $b$ draw $aa'$ and $bb'$ at right angles to $Aa$ and $Bb$.

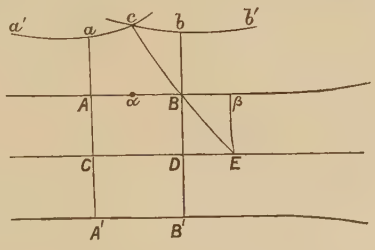


Fig. 3.

If $AB$ is sufficiently small, $\pi B \,.\, \pi A \,.\, \pi A' \,.\, \pi B'$ is a rotation (Lemma). Hence also $\pi A \,.\, \pi A' \,.\, \pi B' \,.\, \pi B$ is a rotation. Now

$$\begin{aligned} \pi A \,.\, \pi A' \,.\, \pi B' \,.\, \pi B &= \text{translation } 2aC \,.\, \text{translation } 2Db \\ &= \pi aa' \,.\, \pi CD \,.\, \pi CD \,.\, \pi bb' \\ &= \pi aa' \,.\, \pi bb'. \end{aligned}$$

Hence $aa'$ and $bb'$ must intersect. Let $c$ be the point of intersection. Join $cB$ and produce it. Then, since $Bb$ is equal to $BD$, while the angles at $D$ and $b$ are right angles and the vertically opposite angles at $B$ are equal, $cB$ produced must meet $CD$ in a point $E$, such that $BE$ is equal to $Bc$.

Now
$$\begin{aligned} \pi B \,.\, \pi A \,.\, \pi A' \,.\, \pi B' &= \pi B \,.\, \pi ac \,.\, \pi bc \,.\, \pi B \\ &= \text{rotation round } E. \end{aligned}$$

Hence finally by the lemma, if $E\beta$ is at right angles to $AB$, and $\alpha$ is taken on $AB$, so that $\beta\alpha$ is equal to and in the same sense as $BA$, then the common perpendicular to $AB$ and $CD$ passes through $\alpha$.

## III.

*To draw the line joining two ideal points.*

If $AB$, $CD$ are the non-intersectors determining one ideal point, and $A'B'$, $C'D'$ those giving the other; $\alpha\beta$, the common perpendicular to $AB$, $CD$, and $\alpha'\beta'$, that to $A'B'$, $C'D'$, can be found by the preceding construction.

If $\alpha\beta$, $\alpha'\beta'$ are non-intersectors, their common perpendicular, which is the required line, can also be found by the preceding construction.

If $\alpha\beta$ and $\alpha'\beta'$ meet at infinity or are intersectors, there are no finite points on the required line.

## IV.

*To draw the line joining a given point at infinity to a given ideal point.*

This is the same thing as :—

*To draw a perpendicular on a given line from a given point at infinity.*

Let $AB$ be the given line and $I$ the given point at infinity. Join any two points $C$ and $D$ of $AB$ to $I$. From $A$ and $B$ draw $Aa$ and $Bb$ at right angles to $CI$, and produce then to $A'$ and $B'$, so that

$$aA' = Aa, \quad bB' = Bb.$$

From $A'$ and $B'$ draw $A'a'$ and $B'b'$ at right angles to $DI$ and produce them to $A_1$ and $B_1$, so that

$$a'A_1 = A'a', \quad b'B_1 = B'b'.$$

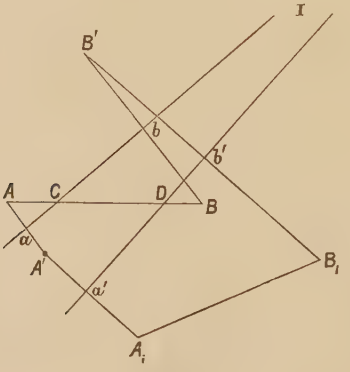


Fig. 4.

Also from $A$ and $B$ construct similarly $A_2$ and $B_2$, except that in this case the perpendiculars are first let fall on $DI$, and afterwards on $CI$. Then $A_1B_1$ is the line that results from giving $AB$ the displacement $\pi CI.\pi DI$; and $A_2B_2$ is the line that results from giving $AB$ the inverse displacement $\pi DI.\pi CI$.

If $C$ and $D$ are taken so that, $J$ being the point at infinity on $CD$ produced, $ICJ$ and $IDJ$ are both acute angles, then $AB$ and $A_1B_1$ will certainly intersect, as also will $AB$ and $A_2B_2$.

For, by the displacement $\pi CI.\pi DI$, $C$ is displaced to a point $C_1$, on the far side of $AB$ from $I$. Also, if $JJ'$ is the perpendicular on $CI$ from $J$, and $J'J_1$ the perpendicular from $J'$ on $DI$, $J'$ and $J_1$ both being points at infinity, then $JJ'$ is on the same side of $AB$ as $I$ and $J'J_1$ is on the same side of $JJ'$ as $I$; so that $J_1$, the displaced position of $J$, is on the same side of $AB$ as $I$. Hence $A_1B_1$ has points on both sides of $AB$, or, in other words, $AB$ and $A_1B_1$ intersect.

Let $E$ and $F$ be the points in which $A_2B_2$ and $A_1B_1$ intersect $AB$, and let $G$ be the middle point of $EF$. Then $GI$ will be the line required.

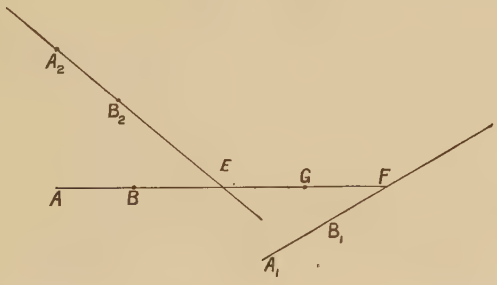


Fig. 5.

For $A_2B_2$, $AB$, $A_1B_1$ are three consecutive sides of a regular polygon described about a limit-circle with its centre at $I$; and $E$, $F$ are two consecutive angular points of the polygon. Hence $G$ is the point of contact of the side $AB$, and $GI$ is therefore at right angles to $AB$.

## V.

*To draw the straight line joining two points at infinity.*

Let $I$ and $J$ be the two points at infinity. Join $I$ and $J$ to any finite point $A$, and draw $AK$ bisecting the angle $IAJ$. Then $\pi AK$ interchanges $I$ and $J$, and therefore leaves the line $IJ$ unchanged; so

that $IJ$ is at right angles to $AK$. The line $IJ$ is therefore given on

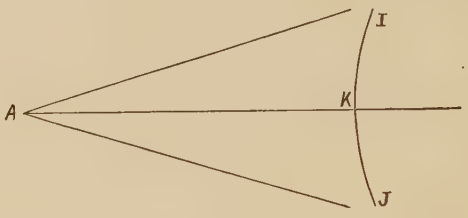


Fig. 6.

drawing the perpendicular from $I$ on $AK$ by the preceding construction.

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*A Theorem concerning the Special Systems of Point-Groups on a Particular Type of Base-Curve.* *By* Miss Frances Hardcastle. Received November 22nd, 1897. Read December 9th, 1897.

The "special" systems of point-groups on a base-curve of deficiency $p$ are known to be connected in pairs, viz. (employing the usual notation, originally introduced by Brill and Noether*), the special system $g^r_R$ always exists simultaneously with a $g^q_Q$ when the relations

$$Q+R=2p-2, \qquad \text{(i.)}$$

$$Q-R=2(q-r) \qquad \text{(ii.)}$$

hold among the positive integers $Q$, $R$, $q$, $r$.

Given $p$, these equations, as they stand, are capable of being satisfied by an infinite number of sets of values of $Q$, $R$, $q$, $r$,† but by means of a further relation, an inequality, namely, $Q \geqslant 2q$ (which also implies that $R \geqslant 2r$), the number of these sets, or, in

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* *Math. Ann.*, Vol. VII., 1873, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie."

† This is, of course, only true if we assume no knowledge of the application of the equations, and hence no connexion at all among the unknown integers.

other words, the number of different possible special systems of point-groups $g^r_Q$, $g^r_R$ on a base-curve of given deficiency is limited. The equations (i.) and (ii.) are due to Brill and Noether, and are generally known as the Riemann-Roch equations. The inequality $Q \geqslant 2q$ was first established in connexion with the theory of Abelian functions, by Clifford,* but it is easily proved independently, directly from the definitions and properties of the special systems ;† it applies to all base-curves of whatever nature their singularities may be; it is, in fact, the sole necessary restriction on all curves on the range of values of $Q$, $R$, $q$, $r$, when these quantities are expressed in terms of $p$, the deficiency of the curve (in particular, it holds, whether the given base-curve be "general" for the given value of $p$, or not). But for particular types of base-curves it appears not impossible to assign a narrower limitation to the range of values of $Q$, $R$, $q$, $r$, in terms of $p$, and, in the following pages, I state, and prove by an elementary method, such a limitation with reference to a particular type of base-curve.

The given base-curve is of order $n$, deficiency $p$, its singularity of highest order is an $(n-3)$-fold point, and its only other singularities are triple and double points, say $T$ triple points and therefore $2n-5-p-3T$ double points. An adjoint curve of order $n-3$, $C_{n-3}$ (in future, a curve of order $n$ will be denoted by $C_n$) has, by definition, an $(n-4)$-fold point at the $(n-3)$-fold point of the base-curve; it can therefore have no other singularity, and, if $T>0$, every adjoint $C_{n-3}$ must degenerate into $T$ straight lines joining the $(n-3)$-fold point to the $T$ triple points—these lines have no more points of intersection with the base-curve—together with a $C_{n-3-T}$ with an $(n-4-T)$-fold point at the $(n-3)$-fold point of the base-curve, which passes once through each of the $T$ triple points, and once through each of the $2n-5-p-3T$ double points—this $C_{n-3-T}$ cuts the base-curve in $2p-2$ points, exclusive of the base-points.

Any two such curves $C_{n-3-T}$ cut each other in

$$(n-3-T)^2-(n-4-T)^2-T-(2n-5-p-3T)=p-2$$

points, exclusive of the intersections at the base-points; these $p-2$ points are not necessarily on the base-curve, but it is clear that, whatever may be the number of intersections of two such curves which do actually lie on the base-curve, it can never, *provided the*

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* "Classification of Loci," *Phil. Trans.*, 1878; *Coll. Works*, p. 329.
† *Cf.* Bertini, *Annali di Mat.*, XXII., 1894.

*curves are non-degenerate*, exceed $p-2$. If $T=0$, any two adjoints $C_{n-3}$—each of which cuts the base-curve in $2p-2$ points, exclusive of the base-points—cut each other in $(n-3)^2-(n-4)^2-(2n-5-p)$ points (exclusive of the base-points), which is again $=p-2$, and we shall therefore, in what follows, deal with a $C_{n-3-T}$, where $T \geqslant 0$ (with an $(n-4-T)$-fold point, passing once through each of the $2n-5-p-3T$ double points, and, if $T \geqslant 1$, once through the $T$ triple points), as the exact equivalent, in association with the given base-curve, of the adjoint $C_{n-3}$ associated, in general, with every base-curve.

We proceed to prove that, *on the given base-curve, the only pairs of complementary special systems of point-groups* $g^r_R$, $g^q_Q$ *which can be found satisfy one of the condition* $Q \geqslant 3q$, $R \geqslant 3r$.

The $g^r_R$ being, by hypothesis, special and complete, is determined on the given base-curve by an $r$-ply infinite system of curves $C_{n-3-T}$, unconditioned, save by passage through the base-points, and by passage through any arbitrarily selected $G^q_Q$ of the residual $g^q_Q$. If $r=1$, we have a singly infinite system of curves $C_{n-3-T}$, and two linearly independent curves of this system cut each other in $Q$ points on the base-curve, as well as at the base-points; but, if non-degenerate, they have been shown not to cut each other in more than $p-2$ points, exclusive of the base-points; *if*, therefore, $Q>p-2$, *the* $g'_R$ *is certainly not determined on the given base-curve by means of proper curves* $C_{n-3-T}$, *nor, a fortiori, can a* $g^r_R$, $r>1$, *be so determined.* Now either $Q$ or $R$ is always $>p-2$ (because $Q+R=2p-2$); if, therefore, the systems $g^r_R g^q_Q$ are determined on the base-curve by *proper curves* $C_{n-3-T}$, one or other of the quantities $r$ and $q$ must be zero; hence *non-degenerate systems of curves* $C_{n-3-T}$ *can only determine, on the given base-curve, systems of point-groups which satisfy the condition we have to prove* (for, if $q=0$, $Q \geqslant 3q$; and, if $r=0$, $R \geqslant 3r$). But this does not yet prove that no systems of point-groups contradicting this condition can be found on the given base-curve; it does not even prove that it is *impossible* to find a $g'_R$ on the given base-curve, although $Q>p-2$, for the argument does not apply if the $G^q_Q$ be so chosen that the system of curves $C_{n-3-T}$ *degenerates* into systems of curves of lower order (the sum of whose orders is, of course, $n-3-T$) which together satisfy the necessary conditions of passage through the base-points. In fact, a single example suffices to show that a $g'_R$ (whether $q \geqslant p-2$) can be determined by such a *degenerate* system of curves $C_{n-3-T}$. For let the $G^q_Q$ be so chosen that the system of curves $C_{n-3-T}$ degenerates into a straight line through the

$(n-3)$-fold point of the base-curve, and into a proper $C_{n-4-T}$ (with an $(n-5-T)$-fold point at the $(n-3)$-fold point, and passing once through each of the $T$ triple points, and of the $2n-5-p-3T$ double points); then we obtain a $g'_3$ $(R=3)$ whose elements are the $G'_3$s formed by the three points in which this straight line cuts the base-curve as it rotates about the $(n-3)$-fold point with one degree of freedom in the plane. Here $R=3r$, and this particular example falls under the rule we have to prove, but it still remains to show that even a degenerate system of curves $C_{n-3-T}$ *cannot* determine on the base-curve a $g^q_Q\, g^r_R$, unless the combination of values of $Q, q, R, r$ satisfies one of the condition $Q \geq 3q$, $R \geq 3r$.

For this purpose, consider the possible ways in which a $C_{n-3-T}$ can degenerate.* It must have an $(n-4-T)$-fold point at the $(n-3)$-fold point of the base-curve, and, since a *proper* $C_{n-3-T-x}$ can have no singularity of higher order than an $(n-4-T-x)$-fold point, a $C_x$ which together with this curve makes up a $C_{n-3-T}$ must have an $x$-fold point at the $(n-3)$-fold point of the base-curve, *i.e.*, it consists of $x$ straight lines through this point. These $x$ straight lines may consist of :—

(1) $t$ lines free to rotate about the $(n-3)$-fold point of the base-curve—each of these lines has then one degree of freedom in the plane, and cuts the base curve in three more points.

(2) $s$ lines joining the $(n-3)$-fold point to $s$ of the double points—each of these lines is *fixed* in position, and cuts the base-curve in one more point.

(3) $T'$ lines which coincide with $T'$ of those lines which join the $(n-3)$-fold point to the $T$ triple points—each of these lines is *fixed* in position and cuts the base-curve in *no* more points.

The remaining curve, which makes up the $C_{n-3-T}$, and, together with these $x$ straight lines, fulfils the necessary conditions of passage through the base-points, is a *proper* $C_{n-3-T-x}$ with an $(n-4-T-x)$-fold point at the $(n-3)$-fold point of the base-curve, which passes once through such of the triple and double points of the base-curve as are not on the $x$ straight lines. But, if this $C_{n-3-T-x}$ is a conic (*i.e.*, if $x=n-5-T$), and finally degenerates into two straight lines, one of

* We are not concerned with the conditions which may cause the system of $C_{n-3-T}$ to degenerate, but we show that, even if it must degenerate, no one of all possible degenerate systems can determine complementary systems of point-groups in which $Q, q, R, r$ contradict the above condition.

these may possibly not fall under any one of the three categories already considered, for it may be—

(*a*) A line not passing through any base-point. It then has two degrees of freedom in the plane, and cuts the base-curve in $n$ points ;

(*b*) A line joining two triple points, or two double points, or a triple point to a double point. It is then fixed in position, and cuts the base-curve in $n-6$, $n-4$, $n-5$ more points respectively ; or

(*c*) A line passing through a triple point or a double point. It then has one degree of freedom in the plane, and cuts the base-curve in $n-3$ or $n-2$ more points respectively.

But, in the line of argument which now follows, it is assumed that the $C_{n-3-T-x}$ is a proper curve of order higher than one—the case in which it is a straight line is considered separately in a foot-note; $t$, $s$, $T'$ have any initially assigned values (compatible with the values of $n$, $p$, $T$), and the proof consists in demonstrating that, if $Q < 3q$ *and* $R < 3r$, we are always led to two contradictory inequalities among these numbers, the known number $p$ and another number $m$, also of initially assigned value.

If possible, then, let $Q = 3q-k$, $R = 3r-l$, where $k$ and $l$ are positive integers connected by the equation $2p-2 = 3(q+r)-k-l$.

We begin with the $g^q_{Q-3q-k}$, and inquire under what conditions an element $G^q_Q$ of this system can obtain $q$ degrees of freedom on the base-curve, *i.e.*, how the $Q$ points can be placed among the intersections of the base-curve with $x = t+s+T'$ straight lines and the proper $C_{n-3-T-x}$ in order that $G_Q$ may have exactly $q$ degrees of freedom. Now, if all the $Q$ points are on straight lines, it is clear that $G_Q$ can have at most $\left[\frac{Q}{3}\right]$ $\left(i.e.,\text{ the greatest integer in } \frac{Q}{3}\right)$ degrees of freedom, even in the most favourable case when the points of $G_Q$ are in triplets on some of the $t$ straight lines which have each one degree of freedom,* and $\left[\frac{Q}{3}\right] = \left[q-\frac{k}{3}\right]$ is less than $q$ ; hence, to obtain $q$

* This is also true in the case we excepted above from the general argument. For it clearly applies if the last straight line (when the whole $C_{n-3-T}$ consists of straight lines) falls under the heading (*b*), for *no* degrees of freedom are then acquired if some of the $Q$ points lie on it. It also applies if this line falls under heading (*c*), for then one degree of freedom is added only if $n-3$ or $n-2$ of the $Q$ points lie on the line, and then the total number of degrees of freedom is $\left[\frac{Q-(n-3)}{3}\right]+1$ at most, and this is not greater than $\left[\frac{Q}{3}\right]$, unless $n<6$ [where an $(n-3)$-fold point, being only a double point, is not a base-curve of the type here

degrees of freedom, it is necessary to assume that some, or all, of the $Q$ points, say $m$ of them, are among the intersections of the base-curve with the proper $C_{n-3-T-x}$.

Similarly, some of the $R$ points of an element $G^r_R$ of the $g^r_R$ must be on the proper $C_{n\ 3-T-x}$, for $\left[\frac{R}{3}\right]$ is less than $r$. Now, the total number of points on the $x$ straight lines is $3t+s$; the remaining $2p-2-(3t+s)$ are therefore on the $C_{n-3-T-x}$, and of these $m$ belong to the $G^q_Q$. Hence, $2p-2-3t-s-m$ (the number of points belonging to the $G^r_R$ which are on the $C_{n-3-T-x}$) *must be a positive integer*, whatever may be the values of $t$, $s$, $T'$, $m$.

The assigned positions of the $Q$ points of the $G^q_Q$ and the $R$ points of the $G^r_R$ may be shown by writing them thus

$$G^q_Q = G^q_{(Q-m)+(m)}, \quad G^r_R = G^r_{(3t+s-Q+m)+(2p-2-3t-s-m)},$$

where the numbers in the first brackets of each subscript are the numbers of points on the $x$ straight lines, and those in the second brackets are the numbers on the $C_{n-3-T-x}$. We have to consider how the $q$ and $r$ degrees of freedom are respectively obtained.

The greatest number of degrees of freedom which the $Q-m$ points on straight lines can have is, by a previous argument, $\left[\frac{Q-m}{3}\right]$, and, since $q = \frac{Q+k}{3}$, this is always *less than* $q$. Hence the $q$ degrees of freedom of the element $G^q_Q$ cannot arise wholly from the degrees of freedom of the $Q-m$ points on straight lines; *the $m$ points on the proper $C_{n-3-T-x}$ must have at least one degree of freedom.* But this necessitates that the number of the remaining points on the $C_{n-3-T-x}$, viz., the $2p-2-3t-s-m$ belonging to the $G^r_R$, should certainly not exceed the number of intersections of two curves $C_{n-3-T-x}$, exclusive of those at the base-point (*cf.* similar reasoning on p. 134). And this

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considered]. Similarly it applies if the line falls under heading (*a*), for then one or two degrees of freedom are added only if $n-1$ or $n$ of the $Q$ points lie on the line, and then the total number of degrees of freedom is

$$\left[\frac{Q-(n-1)}{3}\right]+1 \quad \text{or} \quad \left[\frac{Q-n}{3}\right]+2,$$

at most, which are not greater than $\left[\frac{Q}{3}\right]$, unless $n<4$ or $n<6$.

number is

$$(n-3-T-s-t-T')^2-(n-4-T-s-t-T')^2$$
$$-(2n-5-2p-3T-s)-T+T$$
$$=p-2-2t-s-T'$$

(*cf.* p. 133). Thus the condition that the element $G_Q$ should have as many as $q$ degrees of freedom on the base-curve, where $Q=3q-k$, is that it should be possible to choose $t$, $s$, $T'$, $m$ such that

$$2p-2-3t-s-m \leqslant p-2-2t-s-T'$$

or
$$p \leqslant t+m-T'. \qquad (A)$$

The same line of argument can be applied to the case of the $G^r_R$, and shows that the $3t+s-Q+m$ points of the $x$ straight lines can have at most *one third* of this number of degrees of freedom; but, since $Q=3q-k$ and $k-3q=3r-l-(2p-2)$, this number of degrees of freedom is $\frac{1}{3}[3r-l-(2p-2-3t-s-m)]$. But this number is always *less than* $r$, *provided* $l$ *is a positive integer*, for $2p-2-3t-s-m$ was seen, with the same proviso, to be a positive integer. Hence the $r$ degrees of freedom of the element $G^r_R$ cannot arise wholly from the degrees of freedom of the points on straight lines; the $2p-2-3t-s-m$ points on the proper $C_{n-3-T-x}$ must have at least one degree of freedom. And, as in the similar case of the $G^q_Q$, this necessitates that $m$, the number of remaining points on the $C_{n-3-T-x}$, should not exceed the number of intersections of two curves $C_{n-3-T-x}$, exclusive of those at the base-points, *i.e.*, that

$$m \leqslant p-2t-2-s-T'$$

or
$$p \geqslant m+2t+s+2+T'. \qquad (B)$$

But, by (A), $p=m+t-T'-y$ $(y \geqslant 0)$; therefore (B) becomes

$$0 \geqslant 2+t+s+2T'+y,$$

which is impossible, since $t$, $s$, $T'$, and $y$ are, by hypothesis, positive integers, or zero. If, therefore, $Q<3q$ and, at the same time, $R<3r$, we are led to a logical absurdity on the assumption that the $G^q_Q$ was so chosen that the system of curves $C_{n-3-T}$ *degenerates* into systems of lower order; and we have already seen that non-degenerate

systems can only determine systems for which either $Q \geq 3q$ or $R \geq 3r$. Our initial statement is, therefore, completely proved, viz., on the particular type of base-curve in question, it is necessary that either $Q \geq 3q$ or $R \geq 3r$, in order that a pair of complementary systems $g^r_R\, g^q_Q$ should exist.

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*Thursday, January 13th,* 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Nine members present.

Miss Mildred E. Barwell; Mr. Bertram Hopkinson, B.A., Scholar of Trinity College, Cambridge; Mr. Robert Hamilton Pinkerton, B.A. Balliol College, Oxford, Mathematical Lecturer, University College, Cardiff; and Mr. Algernon Charles Legge Wilkinson, B.A., Scholar of Trinity College, Cambridge, Mathematical Lecturer, University College, Cardiff, were elected members.

The President informed the members present of the death, on December 13th, 1897, of their foreign member Signor Brioschi, and dwelt upon the work the deceased member had done and the loss experienced by the mathematical world. He then read a letter from Professor R. Meldola, F.R.S., drawing attention to the proposed "Sylvester Memorial" Fund, circulars relating to which were laid upon the table.

Mr. Love communicated a paper by Mr. B. Hopkinson "On Discontinuous Fluid Motion," and Mr. S. H. Burbury gave a sketch of his paper "On the General Theory of Stationary Motion in a System of Molecules."

The following papers were briefly communicated by the President:—

Note on a Property of Pfaffians: H. F. Baker.

On the Intersections of Two Conics of a given type, and on the Intersections of Two Cubics: H. M. Taylor.

On the Continuous Group defined by any given Group of Finite Order: Professor W. Burnside.

On those Transformations of Coordinates which lead to new Solutions of Laplace's Equation: Professor Forsyth.

Mr. Mathews exhibited copies of a figure connected with Professor F. Morley's paper read at the November meeting.

The following presents to the Library were received:—

"The Japan Imperial University Calendar," 1896–7.

"Proceedings of the Royal Society," Vol. LXII., No. 381.

Zeuthen, H. G.—"Notes sur l'Histoire des Mathématiques," pamphlet.

Maddison, Isabel.—"On Singular Solutions of Differential Equations of the First Order in Two Variables, and the Geometrical Properties of certain Invariants and Covariants of their complete Primitives," pamphlet.

"Report of the Superintendent of the U.S. Naval Observatory," for the years ending June 30th, 1894, and 1897; Washington, 1895, 1897.

"Year-Book of the Royal Society for 1896–7."

"Biblioteca Boncompagni," Pt. 1, 8vo; Roma, 1898. (Sale from 27th January to 12th February.)

"Bulletin of the American Mathematical Society," Series 2, Vol. IV., No. 3; December, 1897.

"Bulletin des Sciences Mathématiques," Tome XXI.; November, 1897.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. III., Fasc. 11; Napoli.

Darboux, G.—"Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes," Tome I., 8vo; Paris, 1898.

"Rendiconti del Circolo Matematico di Palermo," Tome XI., Fasc. 6.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VI., Fasc. 11; Roma, 1897.

"Sachregister der Abhandlungen und Berichte der Math.-Phys. Classe der K. Sachs. Gesellschaft der Wissenschaften zu Leipzig," roy. 8vo; Leipzig, 1897.

"Acta Mathematica," XX., 3, 4, XXI.; Stockholm, 1897.

"Educational Times," January, 1898.

Gordan, P.—"Resultanten ternärer Formen," "Der Hermitesche Reciprocitätssatz," "Le résultant de trois formes ternaires quadratiques," pamphlets.

"Annales de l'Ecole Polytechnique de Delft," Tome VIII., Livr. 3, 4.

"G. Lejeune Dirichlets Werke," herausgegeben von L. Kronecker fortzesetz von L. Fuchs, Band II., 4to; Berlin, 1897.

"Collected Mathematical Papers of Arthur Cayley," Vol. XIII., 4to; Cambridge, 1897.

"Indian Engineering," Vol. XXII., Nos. 21–25; Nov. 20–Dec. 18, 1897.

## *Note on a property of Pfaffians.* *By* H. F. Baker.

Received December 16th, 1897. Read January 13th, 1898.

When $k$ is even, one of the square roots of the skew symmetrical determinant of $k$ rows and columns whose $(r, s)$-th element is $(rs)$ is the quantity defined, by successive steps, by the equation

$$(12 \ldots k) = (12)(34 \ldots k) - (13)(245 \ldots k) + (14)(235 \ldots k) - \ldots \ldots + (1k)(234 \ldots k-1),$$

wherein, in each bracket of $k-2$ indices, the indices are in their natural order. This quantity, when expanded in terms of the quantities $(rs)$, may be written

$$(12 \ldots k) = (12)(34) \ldots (k-1 \,.\, k) \pm \ldots,$$

where the meaning is that we are to set down all the symbols which can be derived from $(12)(34) \ldots (k-1 \,.\, k)$ by interchange of two of the symbols 2, 3, 4, ..., $k$ which do not occur in the same bracket, a minus sign being introduced for every interchange. We thus obtain, for $k = 2n$, an aggregate of $(2n)!/2^n(n!)$ terms, each being a product of $n$ of the original quantities $(rs)$.

Now let $x$, $y$ denote two indices other than 1, 2, ..., $2n$, $2n+1$, and let $(xyrs)$ be given by $(xy)(rs) - (xr)(ys) + (xs)(yr)$ ; then—

(I.) *If in the expanded form of* $(123 \ldots 2n-1 \,.\, 2n)$ *every quantity* $(rs)$ *be replaced by* $(xyrs)$, *the result is equal to* $(xy)^{n-1}(xy1234 \ldots 2n-1 \,.\, 2n)$.

For instance, for $n = 2$,

$$(xy12)(xy34) - (xy13)(xy24) + (xy14)(xy23) = (xy)(xy1234).$$

(II.) *If in the expanded form of* $(y123 \ldots 2n+1)$, *we replace every quantity* $(rs)$ *whose symbol does not contain the index* $y$ *by the quantity* $(xyrs)$, *and replace every quantity* $(yr)$ *which contains* $y$ *by the quantity* $(xr)$, *in which* $x$ *replaces* $y$, *the result is* $(xy)^n(x123 \ldots 2n+1)$.

For instance, for $n = 1$,

$$(x1)(xy23) - (x2)(xy13) + (x3)(xy12) = (xy)(x123).$$

These results, once stated, are easy to verify, either by induction or directly from the expansion of the Pfaffian, or by the remark that the Pfaffian $(123 \ldots 2n)$ is equal to

$$\frac{1}{n!}\left[\epsilon_1\epsilon_2(12) + \epsilon_1\epsilon_3(13) + \ldots + \epsilon_r\epsilon_s(rs) + \ldots\right]^n,$$

where, in the bracket, there are $n(2n-1)$ terms, and $\epsilon_1, \ldots, \epsilon_{2n}$ are imaginary associative symbols satisfying the equations

$$\epsilon_i^2 = 0, \quad \epsilon_i \epsilon_j = -\epsilon_j \epsilon_i, \quad \epsilon_1 \epsilon_2 \ldots \epsilon_{2n} = 1.$$

Also, the expression

$$\epsilon_1\epsilon_2(12) + \epsilon_1\epsilon_3(13) + \ldots + \epsilon_r\epsilon_s(rs) + \ldots$$

can be written $\frac{1}{2}\left[\epsilon_1(1) + \epsilon_2(2) + \ldots + \epsilon_{2n}(2n)\right]^2$,

where $(1), \ldots, (2n)$ are imaginary symbols such that

$$\epsilon_r(r)\,\epsilon_s(s) = \epsilon_r\epsilon_s(rs) = \epsilon_s\epsilon_r(sr).$$

In the equations (I.), (II.), the left sides may be described as Pfaffians whose elements are Pfaffians of (two or) four indices. It would seem to be worth investigation what analogous expansions exist for the general Pfaffian in terms of Pfaffians of any assigned number of indices.

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*On Discontinuous Fluid Motions involving Sources and Vortices.* *By* B. Hopkinson. Received December 23rd, 1897. Read January 13th, 1898.

The theory of fluid motion in two dimensions where the boundary consists partly of straight walls and partly of free stream lines over which the pressure is constant has been investigated in a general manner by Mr. Love.* His method, however, only applies to cases in which there are no sources or other singularities in the moving fluid, the motion being produced entirely by streams proceeding to or from infinity. This paper contains an extension of the theory to cases in which there are singularities present anywhere in the fluid.

In order the more clearly to show the modifications of Mr. Love's method necessitated by this extension, I will quote the preliminary

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* *Proc. Camb. Phil. Soc.*, VII., 1891.

account of the problem and of the method of solving it which appears at the head of his paper :—

"The motion of the fluid is supposed to take place in the plane of a complex variable $z$, and to be given by means of a velocity-potential $\phi$ and a stream-function $\psi$, so that $\phi + i\psi$, or $w$, is a function of $z$. The object of the theory is to show how in any given problem the functional relation between $w$ and $z$ can be discovered. For this purpose we consider two functions $\zeta$ and $\Omega$ such that

$$\zeta = \frac{dz}{dw} \quad \text{and} \quad \Omega = \log \zeta,$$

and it is well known that $\Omega$ is a complex quantity whose real part is the logarithm of the reciprocal of the velocity at the point $z$, and whose imaginary part is the angle which the direction of this velocity makes with the real axis in the $z$-plane. In the kind of motion to which this method applies, the region of the $z$-plane within which the motion takes place is bounded partly by fixed straight lines and partly by free stream-lines. Along the fixed boundaries the direction of the velocity is given, so that the corresponding parts of the boundary in the $\Omega$-plane are lines parallel to the real axis. Along the free stream-lines the velocity is a given constant, so that the corresponding parts of the boundary in the $\Omega$-plane are parts of a straight line parallel to the imaginary axis. Hence the boundary in the $\Omega$-plane is a polygon which we know how to draw. In like manner, the boundary in the $w$-plane, consisting of parts of straight lines parallel to the real axis ($\psi$ = constant), is a polygon which we know how to draw. If, now, we take an auxiliary complex variable $u$, the polygons in the $\Omega$- and $w$-planes can be conformally represented on the half-plane for which the imaginary part of $u$ is positive. The required transformations are given by the theory of Schwarz and Christoffel, and we are thus in a position to write down two relations,

$$\frac{dz}{du} = f_1(u),$$

$$\frac{d\Omega}{du} = f_2(u),$$

from which $$\frac{dz}{du} = C f_1(u) \exp\left\{\int f_2(u)\, du\right\}.$$"

In this way a relation $z = f(u)$ is obtained which is such that the area of motion is conformally represented on that half of the plane of $u$ for which the imaginary part of $u$ is positive.

In cases where sources, sinks, or vortices are present in the moving fluid, it is still possible to obtain the relation $z = f(u)$ by finding $\Omega$ and $w$ in terms of $u$. Mr. Love's method of arriving at these functions here fails, however, because they become infinite at the singularities, and the half-plane of $u$ can no longer be conformally represented on areas in the planes of $\Omega$ and of $w$. We cannot, therefore, have recourse to Schwarz's expression for $\frac{d\Omega}{du}$ and $\frac{dw}{du}$, but must obtain these functions direct from their boundary and infinity conditions. I proceed to show how to do this.

Suppose that the motion goes on in a simply-connected area in the $z$-plane, and that the boundary consists of straight walls and free stream-lines alternately. There may be streams proceeding to or from infinity, and these streams may be bounded either by parallel straight walls or by free surfaces ultimately straight and parallel. $\psi$ is constant over the boundary, save for discontinuities at those points at infinity whence the streams come or whither they go. The velocity is supposed to be finite at all points of the boundary. At isolated points within the area the velocity becomes infinite. The type of these points is $z = \zeta$, where $\frac{dw}{dz}$ becomes finite in the form

$$\frac{A_1}{z-\zeta} + \frac{A_2}{(z-\zeta)^2} + \ldots + \frac{A_m}{(z-\zeta)^m}.$$

Here $A_1, A_2, \ldots, A_m$ may be any complex quantities. The index $m$ is, of course, an integer, and we may call it the order of the singularity at $\zeta$.

The area of motion can always be conformally represented on that half of the plane of a complex variable $u$ for which the imaginary part of $u$ is positive. The problem is to find the relation between $z$ and $u$ in any given case, so that this representation can be effected.

Now, if $w$ be expressed as a function of $u$, it will represent a fluid motion in the half-plane of $u$. In this motion the real axis of $u$ is a rigid boundary over which $\psi$ is constant save for discontinuities of amounts $\pi m_1$, $\pi m_2$, ... at points $t_1, t_2$, ... (which correspond to the discontinuities in $\psi$ on the boundary in the $z$-plane). $\pi m_1$, $\pi m_2$, ... are the rates of discharge of fluid respectively by the several streams in the $z$-plane to which the points $t_1$, $t_2$, ... correspond. There will be singularities at the points corresponding to the singularities in the $z$-plane; and the order of each singularity in the $u$-plane will be equal to the order of the corresponding singularity in the $z$-plane. Thus, at the point

$u = \tau$ corresponding to $z = \zeta \dfrac{dw}{du}$ becomes infinite, like

$$\frac{B_1}{u-\tau} + \frac{B_2}{(u-\tau)^2} + \ldots + \frac{B_m}{(u-\tau)^m},$$

and we note that $B_1 = A_1$.

The relation between $w$ and $u$ must, therefore, be

$$\frac{dw}{du} = \frac{m_1}{u-t_1} + \frac{m_2}{u-t_2} + \ldots + \Sigma\left[\frac{B_1}{u-\tau} + \frac{B_2}{(u-\tau)^2} + \ldots + \frac{B_m}{(u-\tau)^m}\right]$$
$$+ \Sigma\left[\frac{B_1'}{u-\tau'} + \frac{B_2'}{(u-\tau')^2} + \ldots + \frac{B_m'}{(u-\tau')^m}\right],$$

where $B_1', B_2', \ldots, \tau'$ are the complex quantities respectively conjugate to $B_1, B_2, \ldots, \tau$, and the summations extend to all singularities, such as $u = \tau$.

This equation reduces to the form

$$\frac{dw}{du} = \frac{F(u)}{\phi(u)},$$

where $F(u)$ and $\phi(u)$ are rational integral functions of $u$ with real coefficients, and $F(u)$ is of less degree than $\phi(u)$. The real roots of $\phi(u)$ are $t_1, t_2, \ldots$, and no two of them are equal. The complex roots occur in conjugate pairs, and those of them whose imaginary parts are positive are at the points $\tau$ which correspond to the singularities. Each complex root is repeated a number of times equal to the order of the singularity to which it corresponds.

The method of procedure which I adopt is as follows :—Beginning with the motion in the half-plane of $u$ given by the general formula

$$\frac{dw}{du} = \frac{F(u)}{\phi(u)},$$

I shall seek the most general relation $z = f(u)$ which is such that—

(1) The half-plane of $u$ is conformally represented on a simply-connected area in the $z$-plane.

(2) In the motion in that area given by $w$ (expressed as a function of $z$) the velocity of $q$ is constant over some parts of the boundary, while the direction of motion of the fluid is constant over the other parts.

It will then be possible, by proper choice of the functions $F(u)$

and $\phi(u)$ and of the constants which arise in the general form of $z = f(u)$, to arrive at any possible case of motion of the kind described in the $z$-plane. The proper choice of these functions and constants to suit any particular case forms the second part of the problem.

The first step is to determine the infinity and boundary conditions to be satisfied by $\Omega$. Suppose that there are $n$ free pieces in the boundary in the $z$-plane. Corresponding to these there will be $n$ segments of the real axis of $u$—say from $u_1$ to $u_2$, $u_3$ to $u_4$, ..., $u_{2n-1}$ to $u_{2n}$. The remaining $n$ segments, occurring alternately with these, correspond to the straight parts of the boundary. $\Omega$ is a function of $u$, the real part of which does not vary over any one of the first-named segments. On the other segments the imaginary part does not vary, except discontinuously at isolated points which correspond to the points on the straight walls at which the fluid is at rest.

*Infinities of* $\Omega$.—Consider, first, points within the region of motion and not on the boundary. Since the representation of the region of motion on the half-plane of $u$ is conformal, $\log \frac{dz}{du}$ does not become infinite at any such point. Now

$$\Omega = \log \frac{dz}{du} - \log \frac{dw}{du},$$

whence it follows that $\Omega$ becomes infinite at the same points, and in the same manner as $-\log \frac{dw}{du}$. That is, $\Omega$ becomes infinite at every complex root of $F(u)$ and of $\phi(u)$, the imaginary part of which is positive, and at no other points. Let $\sigma$ be such a root of $F(u)$, and let $\sigma'$ be its conjugate root, and let it be repeated $k$ times, so that $F(u)$ contains the factor $(u-\sigma)^k (u-\sigma')^k$. Then $\Omega$ becomes infinite when $u = \sigma$, like $-k \log (u-\sigma)$. Similarly, at the point $\tau$, where $\tau$, as before, is a complex root of $\phi(u)$ within the half-plane, and repeated $m$ times, $\Omega$ becomes infinite, like $m \log (u-\tau)$.

Turning now to points on the boundary, we remember that one of the postulates as to the nature of the motion was that the velocity should not become infinite at any point on the boundary. Hence $\Omega$ can only become infinite on the boundary, if at all, at those points at which the fluid is at rest. These points of rest can clearly only occur on the straight walls, and not on the free stream lines. Now we have

$$\frac{dw}{dz} = \frac{dw}{du} \cdot \frac{du}{dz}.$$

But, if we confine ourselves to points at which both $u$ and $z$ are finite $\frac{dz}{du}$ cannot be zero or infinite on a straight wall. Hence the fluid is at rest, and $\Omega$ is infinite at such roots of $F(u)$ as fall on segments of the real axis which correspond to straight parts of the boundary. If $s$ be such a root of $F(u)$, and if it be repeated $p$ times, then $\Omega$ becomes infinite when $u = s$, like $-p \log (u-s)$. $\Omega$ does not become infinite at any other points on the real axis of $u$, except in certain cases when $u = \infty$.

The motion is supposed to go on in a finite area, except for the streams proceeding to or from infinity. Hence the fluid is not at rest, and $\Omega$ is not infinite for infinite values of $z$.

If the point in the $z$-plane corresponding to $u = \infty$ be either at an infinite distance from the origin, or on a free stream-line, $\Omega$ cannot be infinite there. If, however, it be on a straight wall, $\Omega$ may be infinite. But we have

$$\frac{d\Omega}{du} = -\frac{d}{du}\log\left(\frac{dw}{du}\right) + \frac{d}{du}\log\left(\frac{dz}{du}\right).$$

Now, at the point on the straight wall corresponding to $u = \infty$, $\frac{dz}{du}$ vanishes, like $\frac{\text{const.}}{u^2}$. Also $\frac{dw}{du}$ vanishes when $u = \infty$, like $\frac{\text{const.}}{u}$. Hence $\frac{d\Omega}{du}$ vanishes when $u$ is infinite.

We may now sum up the conditions to be satisfied by the function $\Omega$ :—

(1) Its real part is constant over each of the $n$ segments of the real axis ($u_1$ to $u_2$, $u_3$ to $u_4$, ..., $u_{2n-1}$ to $u_{2n}$) which correspond to the free parts of the boundary in the $z$-plane.

(2) Its imaginary part is constant over each of the remaining $n$ segments, save for certain discontinuous changes.

(3) It becomes infinite at every complex root of $F(u)$ or of $\phi(u)$ which falls within the half-plane. At a root $\sigma$ of $F(u)$, repeated $k$ times, $\Omega$ is infinite, like $-k \log (u-\sigma)$ ; and at a root $\tau$ of $\phi(u)$, repeated $m$ times, it is infinite, like $+m \log (u-\tau)$.

(4) It becomes infinite at those real roots of $F(u)$, of which the type is $s$, which are such that $u_{2r+1} > s > u_{2r}$. The infinity is of the type $-p \log (u-s)$, where the root is repeated $p$ times.

(5) For infinite values of $u$, $\Omega$ may be infinite, but $\frac{d\Omega}{du}$ vanishes.

From these boundary and infinity conditions we have now to determine $\Omega$.

Consider the equation

$$\frac{d\Omega}{du} = \frac{\chi}{\sqrt{(u-u_1)(u-u_2)\ldots(u-u_{2n})}}, \qquad (1)$$

where $\chi$ is a function of $u$ which is real for real values of $u$. It is clear that, as $u$ increases along the real axis from $-\infty$ to $+\infty$, the growth of $\Omega$ is alternately entirely in its real and in its imaginary part. On those segments of the real axis for which

$$u_{2r+1} > u > u_{2r},$$

and which correspond to the straight parts of the boundary in the fluid motion, the real part of $\Omega$ alone varies, and its imaginary part is constant. On the other, and alternate, segments the real part of $\Omega$ is constant. Thus, $\Omega$, if determined from this equation, will satisfy boundary conditions of the right character; and by means of the function $\chi$ we can introduce infinities and discontinuities into $\Omega$ as we please, the only restriction on that function being that it must be real for real values of $u$.

Suppose that we take

$$\chi = C_0 + C_1 u + C_2 u^2 + \ldots + C_{n-1} u^{n-1} - \Sigma \frac{pa}{u-s} - \Sigma k \left( \frac{\beta}{u-\sigma} + \frac{\beta'}{u-\sigma'} \right) + \Sigma m \left( \frac{\gamma}{u-\tau} - \frac{\gamma'}{u-\tau'} \right), \qquad (2)$$

where $C_0, C_1, \ldots, C_{n-1}$ are real. The symbols $p$, $s$, $\sigma$, $\sigma'$, $k$, $\tau$, $\tau'$, $m$ bear the same meanings as before, and the summations extend to all the infinities of $\Omega$. The quantities $a$ are real; and the quantities $\beta'$, $\gamma$ are respectively the complex quantities conjugate to $\beta$, $\gamma$.

Then $\chi$ clearly satisfies the conditions of being real for real values of $u$. Also, at the point $u = s$, $\frac{d\Omega}{du}$ becomes infinite, like

$$-\frac{pa}{\sqrt{(s-u_1)(s-u_2)\ldots(s-u_{2n})}} \frac{1}{u-s},$$

and $\Omega$ becomes infinite, like

$$-\frac{pa}{\sqrt{(s-u_1)(s-u_2)\ldots(s-u_{2n})}} \log(u-s).$$

Hence $\Omega$ will become infinite at $u = s$ in the right way if

$$\alpha = \sqrt{(s-u_1)(s-u_2) \ldots (s-u_{2n})}.$$

Since the root $s$ is so situated that

$$u_{2r+1} > s > u_{2r},$$

it follows that $\alpha$ is real for every such root, as was postulated above.

Similarly, in order that $\Omega$ may become infinite in the right way when $u = \sigma$, and when $u = \tau$, we must take

$$\beta = \sqrt{(\sigma-u_1)(\sigma-u_2) \ldots (\sigma-u_{2n})}$$

and

$$\gamma = \sqrt{(\tau-u_1)(\tau-u_2) \ldots (\tau-u_{2n})},$$

and we shall then have

$$\beta' = \sqrt{(\sigma'-u_1)(\sigma'-u_2) \ldots (\sigma'-u_{2n})},$$

$$\gamma' = \sqrt{(\tau'-u_1)(\tau'-u_2) \ldots (\tau'-u_{2n})},$$

the sign affixed to the square roots being such that $\beta'$ is the complex quantity conjugate to $\beta$, and $\gamma'$ that conjugate to $\gamma$.

The function $\Omega$ determined in this way only becomes infinite at the points whose types are $s$, $\sigma$, $\tau$; and $\dfrac{d\Omega}{du}$ vanishes for infinite values of $u$. Thus $\Omega$ satisfies the right conditions.

The introduction of the $n$ arbitrary constants $C_0, C_1, \ldots, C_{n-1}$ insures that $\Omega$ shall be the most general possible function that satisfies the conditions. This is most readily seen by considering a simple electrical analogy. Suppose that the half-plane of $u$ is a uniformly conducting sheet, the real axis being its edge. Let the segments of the edge from $u_1$ to $u_2$, $u_3$ to $u_4$, ..., $u_{2n-1}$ to $u_{2n}$ be made perfectly conducting and kept at arbitrary potentials, the remaining segments being left free. Let there be electrodes in the sheet and on the edges thereof, at the points whose types are $\sigma$, $\tau$, $s$. Let the electrode at $s$ discharge an amount $-p\pi$ of electricity per second, that at $\sigma$ an amount $-k\pi$, and that at $\tau$ an amount $+m\pi$. Then the electrical potential is clearly the real part of $\Omega$, where $\Omega$ is the function that we have been seeking in the fluid motion problem. The arbitrary potentials at which the $n$ segments of the edge are maintained are respectively the logarithms of the reciprocals of the velocities over the $n$ free pieces of the boundary of the fluid motion. Now the real part of $\Omega$, as determined from equation (1) above, contains $n+1$ arbitrary constants — *i.e.*, the $n$ constants $C_0, C_1, \ldots, C_{n-1}$, and one

constant of integration. These can be determined in one way, and in one way only, for any given set of values of the potentials of the $n$ segments of the edge, and of the potential at infinity.

This electrical problem has some little interest of its own. It has not, so far as I am aware, been solved hitherto, except for cases where all the electrodes are on the edge of the sheet. In such cases the ordinary equation of Schwarz and Christoffel effects an immediate solution. In the particular form of $\Omega$ found above, which corresponds to the fluid motion problem, the strengths of the electrodes are integers. The solution is, however, clearly not confined to cases where the electrodes are of integral strength.

It is worth noting that the above equation for $\frac{d\Omega}{du}$ is a particular case of a more general equation. The ordinary transformation of Schwarz and Christoffel,

$$\frac{dz}{du} = (u-u_1)^{-\alpha_1/\pi}(u-u_2)^{-\alpha_2/\pi}\ldots,$$

gives $z$ as a function of $u$ which is finite and continuous for all values of $u$, and which is such that the argument of $\frac{dz}{du}$ is constant over the real axis save for discontinuous changes at the points $u_1$, $u_2$, ... . If we take, instead of the equation last written, the modified equation

$$\frac{dz}{du} = \chi\,(u-u_1)^{-\alpha_1/\pi}(u-u_2)^{-\alpha_2/\pi}\ldots,$$

where $\chi$ is a function of $u$ which is real for real values of $u$, we shall get a function $z$ which is such that the argument of $\frac{dz}{du}$ is constant on the real axis save for discontinuous changes at the same points and of the same amounts as before. The function $z$ will, however, no longer be finite and continuous for all values of $z$; but we can introduce into it any infinities or discontinuities that we please by proper choice of $\chi$.

The expression for $\chi$ in equation (2) may be put into a more compact form. The complex roots of $F(u)$ and of $\phi(u)$ occur in conjugate pairs, since the coefficients in these functions are all real. Hence the quantities of which the types are $\sigma'$ and $\tau'$ are roots of $F(u)$ and of $\phi(u)$ respectively. We also have

$$\beta' = \sqrt{(\sigma'-u_1)(\sigma'-u_2)\ldots(\sigma'-u_{2n})},$$
$$\gamma' = \sqrt{(\tau'-u_1)(\tau'-u_2)\ldots(\tau'-u_{2n})}.$$

We may therefore write

$$\chi = -\Sigma \frac{\lambda}{u-\sigma} + \Sigma \frac{\mu}{u-\tau}, \tag{3}$$

where $\sigma$ now stands for any root of $F(u)$, real or complex, and in either half-plane, which does not lie on a segment of the real axis of $u$ corresponding to a free part of the boundary. Thus $\sigma$ now includes all the roots of $F(u)$ previously denoted by the symbols $s$, $\sigma$, and $\sigma'$. $\tau$ stands for any complex root of $\phi(u)$ in either half-plane. We also have

$$\lambda = \sqrt{(\sigma-u_1)(\sigma-u_2)\ \ldots\ (\sigma-u_{2n})},$$

$$\mu = \sqrt{(\tau-u_1)(\tau-u_2)\ \ldots\ (\tau-u_{2n})}.$$

If a root be repeated any number of times, the term corresponding to it in the expression for $\chi$ is repeated the same number of times.

We have now, therefore, determined $\Omega$ in terms of $u$. The relation between $z$ and $u$ which we seek is given by

$$\frac{dz}{du} = e^{\Omega} \frac{dw}{du}.$$

This relation will involve a number of constants, such as the roots of $F(u)$ and $\phi(u)$, and the quantities $C_0, C_1, \ldots, C_{n-1}$, $u_1, u_{2r}, \ldots, u_{2n}$. If we assign particular values to these constants, and if, furthermore, we make one particular point in the $z$-plane correspond to one in the half-plane of $u$, so as to fix $z$ as a one-valued function of $u$ in the half-plane, then the half-plane of $u$ will be conformally represented on a determinate region in the plane of $z$. The motion in this region given by $w$ will be such that over $n$ pieces of the boundary the velocity is constant, while over the remaining $n$ pieces, occurring alternately with these, the direction of motion of the fluid is constant. By properly choosing the constants we can arrive at any possible motion whatever which satisfies these conditions. The singularities in the motion will be the same in number and order as those in the corresponding motion in the half-plane of $u$.

In the examples which follow, I investigate in each case the general expressions for a particular class of motion in which there are singularities of a given number and order, and in which the boundary consists of a given number of straight and free pieces. The number and order of the singularities in the motion in the half-plane of $u$ are then known, but their positions and strengths are not known. Three of the constants which occur in the general relation connecting $z$ and $u$

can be arbitrarily assigned without affecting the generality. For, suppose that we take a new complex variable $v$, given by

$$v = \frac{au+b}{a'u+b'},$$

where $a$, $b$, $a'$, $b'$ are real quantities. Then the region in the $z$-plane can be conformally represented on the half-plane of $v$, and the functional relation between $z$ and $v$ will involve the three ratios of $a$, $b$, $a'$, $b'$; to each of which we can give any value we please.

By proper choice of the remaining constants in the general relation between $z$ and $u$, we can arrive at any particular case of motion which comes within the class which is being investigated.

*Example I.*—The boundary consists of a free stream-line and a straight wall. $\psi$ is constant all round the boundary; and the only singularity is a vortex, the circulation round which is $2m\pi$. The velocity on the free stream-line is unity.

If the area of motion be conformally represented on the half-plane of $u$, the motion in that half-plane represented by $w$ will be that due to a single vortex, the circulation round which is $2m\pi$. The representation can always be effected so that this vortex is on the imaginary axis at the point $u=i\gamma$; and, at the same time, the straight wall may be made to correspond to that segment of the real axis of $u$ which lies between $-1$ and $+1$.

The relation between $w$ and $u$ with which we start is, therefore,

$$\frac{dw}{du} = \frac{mi}{u-\imath\gamma} - \frac{mi}{u+\imath\gamma}$$

$$= -\frac{2m\gamma}{u^2+\gamma^2},$$

and from this it must be possible, by a proper transformation, to arrive at any possible motion which satisfies the conditions enumerated above. We note that $\Omega$ is not infinite when $u$ is infinite, since the point corresponding to $u=\infty$ is on the free stream-line.

We shall take the axis of $x$ parallel to the straight wall. $\Omega$ is then a function of $u$, the imaginary part of which vanishes on the real axis between $-1$ and $+1$, while the real part vanishes for real values of $u$ outside these limits. Also, $\Omega$ becomes infinite when $u=\imath\gamma$, like $\log(u-i\gamma)$, and does not become infinite for any other value of $u$. We have, therefore,

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}}\left[\frac{\sqrt{1+\gamma^2}}{u-\imath\gamma} + \frac{\sqrt{1+\gamma^2}}{u+\imath\gamma}\right].$$

Take a new variable $\theta$, given by $u = \sin\theta$, and let $\cos\theta = +\sqrt{1-u^2}$. The effect of this is to conformally represent the area of motion on an area on the plane of $\theta$ bounded by the lines parallel to the imaginary axis and by a segment of the real axis between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; the latter segment corresponding to the straight wall (see Fig. 1). Let $\gamma = \sinh a$, so that the point $u = i\gamma$ corresponds to the point $\theta = ia$.

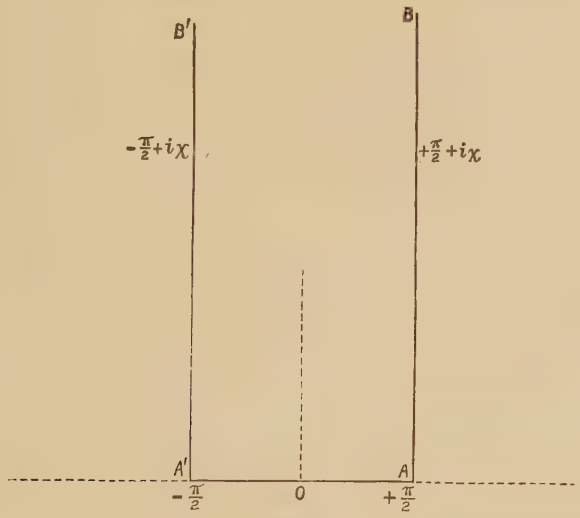


FIG. 1.

The equation for $\Omega$ now integrates in the form

$$\Omega = \log\frac{\cos\theta - \cosh a}{\cos\theta + \cosh a},$$

the constants being so chosen that the imaginary part vanishes if $\theta$ lie on the segment $AA'$ in Fig. 1, while the real part vanishes if it be on $AB$ or $A'B'$.

Thus we find

$$\frac{dz}{du} = e^{\Omega}\frac{dw}{du}$$

$$= -2m\gamma\frac{1}{(\cos\theta+\cosh a)^2}.$$

This is the relation between $z$ and $u$, and it contains one arbitrary constant, *i.e.*, $\gamma$. By choice of $\gamma$ it must be possible to cover every possible case of motion which satisfies the conditions with which we started.

We have $$\frac{dz}{d\theta} = \frac{-2m \sinh a \cos\theta}{(\cos\theta + \cosh a)^2},$$

and, if the origin be taken in the middle of the straight wall, the coordinates of the ends of the wall will be

$$y = 0,$$
$$x = \pm \int_0^{\frac{1}{2}\pi} \frac{2m \sinh a \cos\theta}{(\cos\theta + \cosh a)^2}$$
$$= \pm 2m \left( \frac{1}{\gamma} - \frac{1}{\gamma^2} \tan^{-1}\gamma \right).$$

The expression last written is equal to half the length of the wall, so that we have an equation for determining $\gamma$.

The vortex is at the point corresponding to $\theta = \iota a$, and its coordinates are

$$x = 0,$$
$$y = -\int_0^a \frac{2m\gamma \cosh\phi \, d\phi}{(\cosh\phi + \cosh a)^2}$$
$$= -2m\gamma \left\{ 1 - \frac{1}{\gamma^2} \log(1+\gamma^2) \right\}.$$

Thus the vortex must be on the line of symmetry of the wall, and it must, furthermore, be at a determinate distance from the wall, in order that it may be possible to complete the boundary with a free stream-line.

One half of the free stream-line corresponds to $AB$ in Fig. 1, and the other half to $A'B'$. At points on the first half $\theta$ is equal to $\frac{\pi}{2} + \iota\chi$, and on the other half to $-\frac{\pi}{2} + \iota\chi$, where $\chi$ is real and positive, and ranges from 0 to $a$.

In Fig. 2 I have drawn the case where $m = 1$, and $\gamma = 1$. In this case points on the free stream-line are given by

$$\frac{dz}{d\chi} = -\frac{2\iota \sinh\chi}{(\cosh a - i \sinh\chi)^2}$$

or
$$\frac{dx}{d\chi} = -\frac{2 \sinh\chi \, (3 - \cosh^2\chi)}{(1+\cosh^2\chi)^2},$$

and
$$\frac{dy}{d\chi} = -\frac{4\sqrt{2} \sinh^2\chi}{(1+\cosh^2\chi)^2}.$$

The vortex is at $P$. The figures affixed to the various points are

the values of $u$ corresponding to these points. The length of the wall is $4-\pi$, and the distance of the vortex from it is $4-\log_e 2$.

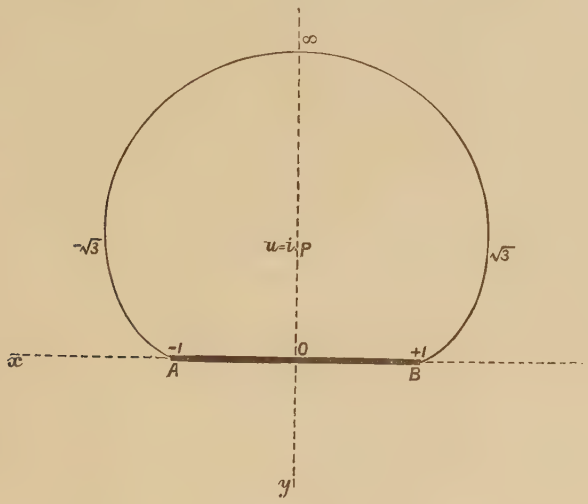


FIG. 2.

*Example II.*—The motion is produced by a vortex and doublet at the same point. The boundary consists wholly of a free stream-line, over which the velocity is unity. The circulation round the vortex is $-2\pi m$, where $m$ is positive.

If the area of motion be conformally represented on the half-plane of $u$, the motion represented by $w$ will be that excited by a vortex and doublet combined. The representation can, in all cases, be effected (by choice of the three constants at our disposal) in such wise that the vortex and doublet are at the point $u = i$, while the axis of the doublet is parallel to the real axis. The form of the boundary in the $z$-plane will manifestly depend solely on the ratio of the strengths of the doublet and vortex; its dimensions only being affected by the absolute magnitudes of these quantities. We therefore assume that the strength of the doublet in the $u$-plane is unity. The strength of the doublet in the $z$-plane will then depend upon the relation between $z$ and $u$.

We have
$$\frac{dw}{du} = \frac{1}{(u-i)^2} + \frac{1}{(u+i)^2} - \iota m \left(\frac{1}{u-i} - \frac{1}{u+i}\right)$$
$$= \frac{2(1+m)(u^2+\beta^2)}{(u^2+1)^2},$$

where
$$\beta^2 = \frac{m-1}{m+1}.$$

The motion in the half-plane of $u$ given by this equation is to be transformed into a motion in the $z$-plane, over the boundary of which the velocity is unity. I propose to investigate the nature of this motion for various values of $m$.

The real part of $\Omega$ is zero for all real values of $u$; and $\Omega$ becomes infinite at those roots and poles of $\frac{dw}{du}$ which are not on the real axis. Two classes of cases, therefore, arise, according as $\beta$ is real or imaginary.

(*a*) $m>1$ and $\beta$ real.—In this case $\Omega$ becomes infinite when $u=\iota\beta$, like $-\log(u-\iota\beta)$; and when $u=i$, like $2\log(u-i)$. We have, therefore,

$$\Omega = \log\frac{u+\iota\beta}{u-\iota\beta} + 2\log\frac{u-i}{u+i},$$

whence

$$\frac{dz}{du} = e^{\Omega}\frac{dw}{du}$$

$$= \frac{2(1+m)(u+\iota\beta)^2}{(u+i)^4}.$$

This gives, on integration,

$$\frac{z}{2(1+m)} = -\frac{1}{u+i} + \frac{i(1-\beta)}{(u+i)^2} + \frac{(1-\beta)^2}{(u+i)^3},$$

the origin being taken at $u=\infty$.

The vortex is at $u=i$, and the fluid is at rest at $u=i\beta$. The equation to the boundary is formed by giving $u$ real values in the equation for $z$. Thus we find for points on the boundary—

$$\frac{x}{2(1+m)} = \frac{u}{3(1+u^2)^3}\left[-3u^4+(1-8\beta+\beta^2)u^2-3\beta^2\right],$$

$$\frac{y}{2(1+m)} = \frac{1}{3(1+u^2)^3}\left[3(2-\beta)u^4+3(1+2\beta-\beta^2)u^2+1+\beta+\beta^2\right].$$

Fig. 3 shows the case where $m=\frac{5}{3}$ and $\beta=\frac{1}{2}$. The vortex and doublet are at $P$, and the point of rest ($u=i\beta$) is at $Q$.

As $m$ diminishes a dimple forms in the boundary. This dimple deepens, and its sides approach until at last they overlap, as shown in Figs. 4 and 5. In these figures the part of the area of motion in the neighbourhood of the point of rest and vortex is shown on an enlarged scale.

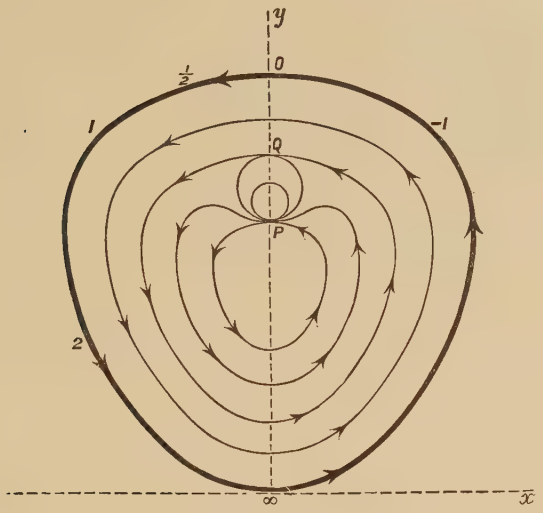


FIG. 3.

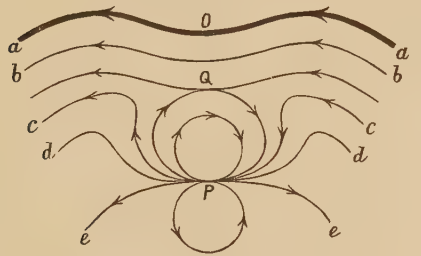


FIG. 4.

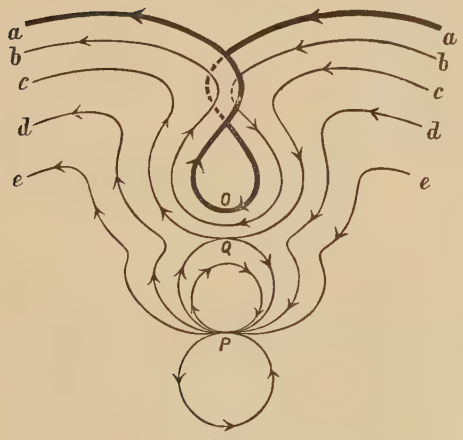


FIG. 5.

Such overlapping will often take place in the general case. By making one particular point in the $z$-plane correspond to a particular point in the $u$-plane, we can always get $z$ as a one-valued function of $u$, but we cannot insure that $u$ shall be a one-valued function of $z$.

A case of motion with overlap is, of course, only of analytical interest; though it is possible to conceive it as going on in a very thin layer.

As $m$ further diminishes the overlap increases, until we reach the limiting case shown in Fig. 6, in which $m = 1$ and $\beta = 0$.

In this case $$\frac{dz}{du} = \frac{2u^2}{(u+i)^4},$$

and $$\Omega = 2 \log \frac{u-i}{u+i}.$$

The point $u = \beta$, which in Figs. 3, 4, and 5 was the point of rest, is now on the boundary, and the fluid is no longer at rest there.

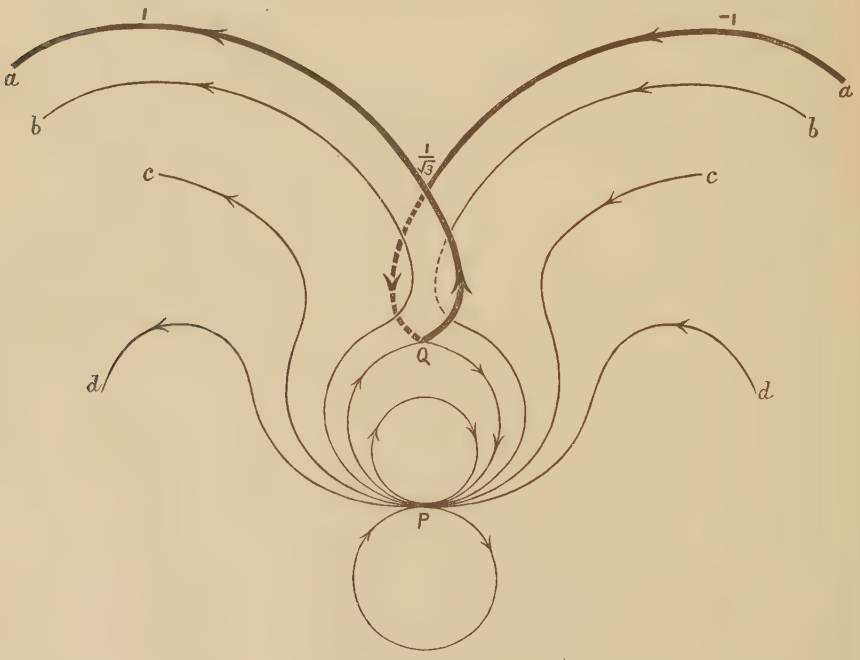


Fig. 6.

In Figs. 3, 4, 5, and 6, the boundary stream-line is drawn thick.

The lighter lines show the general disposition of stream-lines. In Figs. 3 and 6 the boundary is drawn accurately. Figs. 4 and 5 are mere sketches to show how the motion develops as $m$ diminishes. In these figures, as in Fig. 6, the complete drawing would be of unmanageable dimensions; so part only of some of the stream-lines is shown.

In each case $P$ is the vortex and doublet ($u = i$), and $Q$ is the point $u = \iota\beta$, which, in Figs. 3, 4, and 5, is the point of rest.

When there is overlap, the stream-lines of the underlying layer are shown dotted.

(*b*) If $m$ still further diminishes, $\beta$ becomes imaginary.—We have

$$\frac{dw}{du} = \frac{2(1+m)(u^2-\gamma^2)}{(u^2+1)^2},$$

where $\gamma$ is real and equal to $\sqrt{\frac{1-m}{1+m}}$.

$\Omega$ is now infinite at $u = i$, like $2 \log (u-i)$, and at no other point. Its real part vanishes for real values of $u$, as before. We have, therefore,

$$\Omega = 2 \log \frac{u-i}{u+i},$$

and now

$$\frac{dz}{du} = \frac{2(1+m)(u^2-\gamma^2)}{(u+i)^4}.$$

This integrates in the form

$$\frac{z}{2(1+m)} = -\frac{1}{u+i} + \frac{i}{(u+i)^2} + \frac{1+\gamma^2}{3(u+i)^3},$$

the origin being taken at $u = \infty$.

The coordinates of any point on the free stream-line are given by

$$\frac{3x}{2(1+m)} = -\frac{u\{3u^4-(1+\gamma^2)u^2+3\gamma^2\}}{(u^2+1)^3},$$

$$\frac{3y}{2(1+m)} = \frac{6u^4+3(1-\gamma^2)u^2+1+\gamma^2}{(u^2+1)^3}.$$

There is a cusp with the point turned inwards at the points $u = \pm\gamma$. Fig. 7 shows the case where $\gamma^2 = \frac{1}{2}$, $m = \frac{1}{3}$. Fig. 8 is a sketch similar to Figs. 4, 5, and 6, showing an intermediate stage where there is overlap.

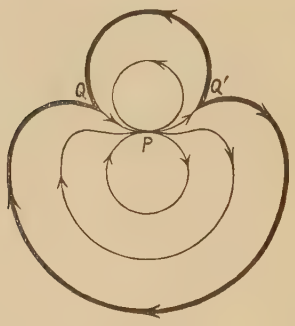


Fig. 7.

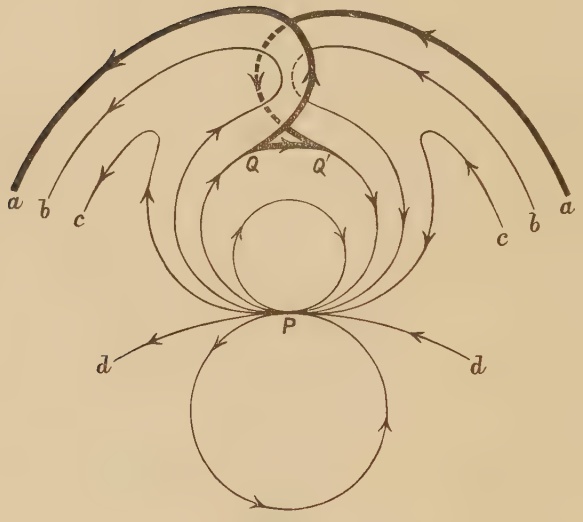


Fig. 8.

It will have been noted that, in this example, the point of rest develops quite continuously with change of $m$ into two cusps on the boundary. The same thing will happen in the same sort of way in the general case of discontinuous fluid motion considered at the beginning of this paper. Suppose that there is an unrepeated complex root $\sigma$ in $\frac{dw}{du}$. Then $\frac{dw}{du}$ contains a factor $(u-\sigma)(u-\sigma')$ in the numerator, and the fluid is at rest at the point corresponding to $u=\sigma$. It will also be remembered that, if

$$\frac{d\Omega}{du}=\frac{\chi}{\sqrt{(u-u_1)(u-u_2)\ \ldots\ (u-u_{2n})}},$$

then $\chi$ contains a term $\frac{\alpha}{u-\sigma}+\frac{\alpha'}{u-\sigma'}$, where

$$\alpha=\sqrt{(\sigma-u_1)(\sigma-u_2)\ldots(\sigma-u_{2n})}.$$

Now let the strengths and positions of the sources or other singularities in the $z$-plane be varied, so that the root $\sigma$ tends to approach a segment of the real axis of $u$ which corresponds to a free part of the boundary. A change in the form of that part of the boundary will then take place, somewhat as shown in Figs. 3, 4, and 5. After a certain amount of this change, $\sigma$ will become real and equal to $\sigma'$. At the same time $\alpha$ becomes a pure imaginary and equal to $-\alpha'$; so that the term in $\chi$ corresponding to $\sigma$ disappears. Thus, at this stage, $\frac{dz}{du}$ vanishes, like const. $\times (u-\sigma)^2$, when $u=\sigma$; for $\Omega$ and $\frac{dz}{dw}$ —hitherto infinite at $u=\sigma$—are now finite at that point. It follows that, when $\sigma$ has just become real, there must be overlap in the neighbourhood of the point corresponding to $u=\sigma$, somewhat like that shown in Fig. 6.

If the change be carried further, the two roots $\sigma$, $\sigma'$—now both real —separate. There is no term in $\chi$ corresponding to either, and so, at each of the points $u=\sigma$, $u=\sigma'$, $\frac{dz}{du}$ vanishes, like $(u-\sigma)$ and $(u-\sigma')$ respectively. Thus, at the point corresponding to each on the bounding curve in the $z$-plane, there is a cusp with its point turned inwards. In this way we get a succession of shapes such as those shown in Figs. 7 and 8.

The overlap will, in general, begin before $\sigma$ has actually become real; and will not disappear until after $\sigma$ and $\sigma'$ have separated some distance.

*Example III.*—A stream escapes from between parallel walls and proceeds to infinity with free surfaces, which are ultimately straight and parallel. The stream is disturbed by a doublet at any point in its path.

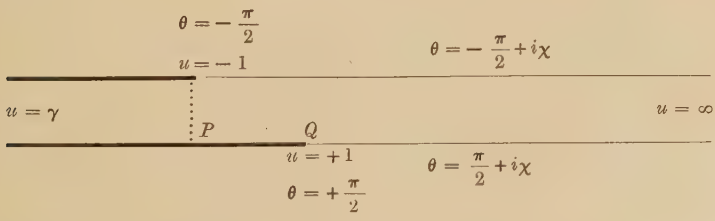


Fig. 9.

Let the area of motion be conformally represented on the half-plane of $u$. The representation can always be effected so that the straight walls correspond to that segment of the real axis of $u$ which lies between $u=-1$ and $u=+1$, while the point at infinity to which the stream proceeds corresponds to $u=\infty$. Let the point at infinity whence the stream comes correspond to $u=\gamma$ (see Fig. 9).

Suppose that the stream discharges an amount $\pi$ of fluid per second. In the motion in the $u$-plane represented by $w$, there is a source at the point $u=\gamma$ discharging at the same rate as the stream. There is no other singularity on the real axis of $u$. There is also a doublet in the $u$-plane corresponding to that in the $z$-plane, say at the point $u=\tau$. We have, therefore,

$$\frac{dw}{du}=\frac{1}{u-\gamma}+\frac{m}{(u-\tau)^2}+\frac{m'}{(u-\tau')^2},$$

which may be written in the form

$$\frac{dw}{du}=\frac{(u-\sigma_1)(u-\sigma_2)(u-\sigma_3)(u-\sigma_4)}{(u-\gamma)(u-\tau)^2(u-\tau')^2}.$$

Now $\Omega$ is a function of $u$ which becomes infinite when $u=\tau$ and when $u=\tau'$, and also at such of the roots $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ as do not lie on the real axis of $u$ outside the limits $\pm1$. Also, assuming that the velocity on the free surfaces is unity, $\Omega$ is a pure imaginary for real values of $u$ outside the limits $\pm1$, and is real if $u$ be within those limits. There are, therefore, five classes of cases, distinguished from one another by the number of cusps appearing on the free part of the boundary, such cusps corresponding in each case to the excepted roots of $\frac{dw}{du}$. The form of the relation between $z$ and $u$ will be different for these several classes; but any one of the classes may be continuously developed into any other in the manner illustrated in the last example.

These classes are as follows:—

I. No excepted roots of $\frac{dw}{du}$.—In this case we have

$$\frac{d\Omega}{du}=\frac{1}{\sqrt{1-u^2}}\left[-\frac{\sqrt{1-\sigma_1^2}}{u-\sigma_1}-\frac{\sqrt{1-\sigma_2^2}}{u-\sigma_2}-\frac{\sqrt{1-\sigma_3^2}}{u-\sigma_3}-\frac{\sqrt{1-\sigma_4^2}}{u-\sigma_4}\right.$$
$$\left.+2\frac{\sqrt{1-\tau^2}}{u-\tau}+2\frac{\sqrt{1-\tau'^2}}{u-\tau'}\right],$$

where it is to be noted that the roots $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ may consist either of two conjugate complex pairs, one complex pair and two real roots, or four real roots. Thus, in this case there may be four points of rest on the boundary, or two on the boundary and one in the fluid, or two in the fluid. The form of $\Omega$ will be the same in all cases of the class.

II. One excepted root. Here

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}}\left[-\frac{\sqrt{1-\sigma_1^2}}{u-\sigma_1}-\frac{\sqrt{1-\sigma_2^2}}{u-\sigma_2}-\frac{\sqrt{1-\sigma_3^2}}{u-\sigma_3}+2\frac{\sqrt{1-\tau^2}}{u-\tau}+2\frac{\sqrt{1-\tau'^2}}{u-\tau'}\right].$$

In this case there is one cusp on the free stream-line, corresponding to the excepted root. There are either two or three points of rest.

III. Two excepted roots, and two cusps on the boundary.—Either one or two points of rest.

IV. Three excepted roots and three cusps on the boundary. One point of rest, which must be on the boundary.

V. Four excepted roots.—In this case there are four cusps on the boundary, and we have

$$\frac{d\Omega}{du} = \frac{1}{\sqrt{1-u^2}}\left[2\frac{\sqrt{1-\tau^2}}{u-\tau}+2\frac{\sqrt{1-\tau'^2}}{u-\tau'}\right].$$

The above equations are readily integrated by the substitution $u = \sin\theta$. If we put $\sigma_1 = \sin\theta_1$, &c., and $\tau = \sin\theta_0$, we shall get for the first class of cases

$$\frac{dz}{d\theta} = \frac{\cos\theta}{\sin\theta-\gamma}\,\frac{\cos^2\frac{1}{2}(\theta+\theta_1)\cos^2\frac{1}{2}(\theta+\theta_2)\cos^2\frac{1}{2}(\theta+\theta_3)\cos^2\frac{1}{2}(\theta+\theta_4)}{\cos^4\frac{1}{2}(\theta+\theta_0)\cos^4\frac{1}{2}(\theta-\theta_0')},$$

and for the second class

$$\frac{dz}{d\theta} = \frac{\cos\theta}{\sin\theta-\gamma}\,\frac{\cos^2\frac{1}{2}(\theta+\theta_1)\cos^2\frac{1}{2}(\theta+\theta_2)\cos^2\frac{1}{2}(\theta+\theta_3)(\sin\theta-\sin\theta_4)}{\cos^4\frac{1}{2}(\theta+\theta_0)\cos^4\frac{1}{2}(\theta+\theta_0')},$$

and so for the other classes, until for the fifth class we have

$$\frac{dz}{d\theta} = \frac{\cos\theta}{\sin\theta-\gamma}\,\frac{(\sin\theta-\sin\theta_1)(\sin\theta-\sin\theta_2)(\sin\theta-\sin\theta_3)(\sin\theta-\sin\theta_4)}{\cos^4\frac{1}{2}(\theta+\theta_0)\cos^4\frac{1}{2}(\theta+\theta_0')}.$$

These five classes will exhaustively cover every possible case.

The quantities $\sin\theta_1$, $\sin\theta_2$, $\sin\theta_3$, and $\sin\theta_4$ are the roots of the equation

$$(u-\tau)^2(u-\tau')^2+(u-\gamma)\left[m(u-\tau')^2+m'(u-\tau)^2\right]=0,$$

and are immediately expressible in terms of $m$, $m'$, $\tau$, $\tau'$, and $\gamma$. In any given case the data are the coordinates and direction of the doublet with reference to the straight walls, the magnitude of the doublet, and the amount $PQ$ by which one wall projects beyond the other. In working out such a case, it is necessary to proceed successively on the five assumptions that the case belongs to each of the classes enumerated above. For each such assumption values will be obtained for $m$, $\tau$, and $\gamma$ in terms of the data. Thus we have—

$$\text{strength of doublet} = \text{mod}\left\{ m \left(\frac{dz}{d\theta}\right)_{\theta=\theta_0} \right\},$$

$$\text{inclination of doublet} = \text{argument of } m \left(\frac{dz}{d\theta}\right)_{\theta=\theta_0},$$

$$\text{doublet is at point corresponding to } \theta = \theta_0,$$

$$PQ = \text{real part of} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \left(\frac{dz}{d\theta}\right) d\theta,$$

which are just sufficient equations for the determination of $m$, $\tau$, and $\gamma$; and, consequently, of $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$.

If the motion is a possible one, it will be found that on one of these assumptions, and one only, the distribution of the roots $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ is appropriate to that assumption. We then know that this is the correct assumption, and can proceed to work out the motion completely.

The effect of the transformation is to conformally represent the area of motion on that portion of the plane of $\theta$ which is shown in Fig. 1. The form of the free stream-lines is given by ascribing to $\theta$ values of the form $\pm\frac{\pi}{2} + \iota\chi$, where $\chi$ is real and positive.

## *On those Transformations of the Coordinates which lead to new Solutions of Laplace's Equation.*

*By* Professor A. R. Forsyth. Received January 11th, 1898. Read January 13th, 1898.

The original aim of the present paper was an investigation of those transformations of the coordinates which, from one general solution* of Laplace's equation $\nabla^2 v = 0$, lead merely by substitution to other solutions. The possible transformations already known consist of

(i.) Change of origin:

(ii.) Orthogonal transformation of the axes:

(iii.) Inversion.†

It appears from the investigation that no other transformations of the required type exist.

Incidentally, two questions have arisen for consideration. One of them is the formation of those families of triply orthogonal surfaces (not necessarily real) which have their radii of curvature everywhere equal to one another and finite: it is proved that the only families are triply orthogonal spheres. The other is the consideration of the corresponding set of simultaneous partial differential equations of the first order: an application of the preceding results has provided an illustration of Lagrange's classification of the integrals of such equations when extended to the case involving three dependent variables.

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* Meaning thereby a solution of some general type, say involving one or more arbitrary functional forms. Thus we should not consider, as included, the transformation $x = Y^2 - Z^2 + \lambda YZ, \quad y = Z^2 - X^2 + \mu ZX, \quad z = X^2 - Y^2 + \nu XY,$

which changes a particular solution

$$ax + by + cz$$

into a particular solution

$$a(Y^2 - Z^2) + b(Z^2 - X^2) + c(X^2 - Y^2) + a'YZ + b'ZX + c'XY.$$

† The method of inversion was, I believe, first announced by Lord Kelvin in a letter to M. Liouville in 1846; see extracts from this letter in Liouville's *Journal de Mathématiques*, 1re Sér., t. xii. (1847), pp. 256–263, and a succeeding Note by M. Liouville himself, *ib.*, pp. 265–290.

1. It is an inference from the general theory* of partial differential equations of the second order in three independent variables, $x$, $y$, $z$, that the argument $u$ of any arbitrary function in the solution of either of the equations

$$\nabla^2 v = 0, \quad \nabla^2 v + \kappa^2 v = 0$$

satisfies the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0.$$

Suppose that $$u = f(x, y, z)$$

is some solution of this equation in $u$, where we may regard $f$ as a known function; then it is known that

$$v = F(f)$$

is a solution of the equation $\nabla^2 v = 0$, $F$ being an arbitrary function. But it is also known that then

$$\frac{1}{r} F(f_1),$$

where $$w = f_1 = f\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

and $$r^2 = x^2 + y^2 + z^2,$$

is also a solution; in other words, $w$ is a solution of the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0;$$

and therefore when any solution of this equation is given, another solution can be constructed from it by substituting $\frac{x}{r^2}$, $\frac{y}{r^2}$, $\frac{z}{r^2}$ for $x$, $y$, $z$ respectively. We are therefore led to the general problem of determining all the substitutions which thus lead to new solutions of

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

and, in consequence, to new solutions of

$$\nabla^2 v = 0.$$

The problem thus suggested can, without loss of generality, be made simpler as follows. Since

$$u = f(x, y, z),$$

* *Phil. Trans.*, 1898, pp. 12, 54.

$f$ being regarded as a known function, we can consider $z$ as a function of $x$, $y$, $u$; denoting derivatives of $z$ in that case by $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{dz}{du}$, we have

$$\left.\begin{aligned} 0 &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{dz}{dx} \\ 0 &= \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{dz}{dy} \end{aligned}\right\}.$$

Substituting in the above equation, and assuming that $f$ involves $z$, or that $u$ is not independent of $z$, it becomes

$$\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1 = 0.$$

So far as concerns this equation, the variable $u$ is parametric; and therefore, if we integrate this new equation, it will be necessary in its integral to make such arbitrary constants, as it contains, arbitrary functions of $u$. As, however, $u$ is itself the argument of an arbitrary function, no generality will be lost by making one of these arbitrary constants actually equal to $u$, and the rest of them arbitrary functions of $u$.

The complete integral of the equation in $z$ is

$$z = ax + by + c,$$

where $a$, $b$, $c$ are arbitrary constants subject to the condition

$$a^2 + b^2 + 1 = 0;$$

hence an argument $u$ can be regarded as determined by

$$z = xp(u) + yq(u) + u,$$

where the arbitrary functions $p$ and $q$ satisfy the relation

$$\{p(u)\}^2 + \{q(u)\}^2 + 1 = 0.$$

From what has been stated, we know that an argument $w$ is derived from $u$ by substituting $\frac{x}{r^2}$, $\frac{y}{r^2}$, $\frac{z}{r^2}$ for $x$, $y$, $z$ respectively; consequently, we have

$$\frac{z}{r^2} = \frac{x}{r^2}p(w) + \frac{y}{r^2}q(w) + w,$$

which, when $w$ is parametric, is a solution of the equation

$$\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1 = 0,$$

and which determines a new quantity $w$ as an argument of arbitrary functions in the solution of the equation

$$\nabla^2 v = 0.$$

Noting the similarity of the equations

$$z = xp(u) + yq(u) + u,$$

$$\frac{z}{r^2} = \frac{x}{r^2} p(w) + \frac{y}{r^2} q(w) + w,$$

the transformation problem can now be stated in the form: Required the most general functions $\xi$, $\eta$, $\zeta$ of $x$, $y$, $z$, such that

$$\zeta = \xi p(v) + \eta q(v) + v,$$

where $p$ and $q$ are arbitrary functions subject to the relation

$$\{p(v)\}^2 + \{q(v)\}^2 + 1 = 0,$$

is a solution of the equation

$$\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1 = 0.$$

2. It is easy to see that the second of the above forms of substitution is the only one which keeps the ratios of $x$, $y$, $z$ unaltered. A substitution which keeps those ratios unaltered can be taken in the form

$$z = xp(\theta) + yq(\theta) + \theta\phi,$$

where $\theta$ is the parametric argument, and $\phi$ is a function of $x$, $y$, $z$ to be determined. We have

$$\frac{dz}{dx} = \frac{p(\theta) + \theta \dfrac{\partial \phi}{\partial x}}{1 - \theta \dfrac{\partial \phi}{\partial z}},$$

$$\frac{dz}{dy} = \frac{q(\theta) + \theta \dfrac{\partial \phi}{\partial y}}{1 - \theta \dfrac{\partial \phi}{\partial z}};$$

and therefore

$$\left\{p(\theta) + \theta \frac{\partial \phi}{\partial x}\right\}^2 + \left\{q(\theta) + \theta \frac{\partial \phi}{\partial y}\right\}^2 + \left\{1 - \theta \frac{\partial \phi}{\partial z}\right\}^2 = 0.$$

Expanding, the terms independent of the derivatives of $\phi$ vanish; whence, dividing out by $\theta$, we have

$$-\left\{2p(\theta)\frac{\partial\phi}{\partial x}+2q(\theta)\frac{\partial\phi}{\partial y}-2\frac{\partial\phi}{\partial z}\right\}$$
$$=\theta\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}$$
$$=\frac{z-xp(\theta)-yq(\theta)}{\phi}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}.$$

In this equation for $\phi$, $p(\theta)$ is an arbitrary function and $q(\theta)$ is a dependent arbitrary function; while $\phi$, as a function of $x, y, z$, is independent of $p$ and $q$. Hence the parts multiplying $p$ alone, the parts multiplying $q$ alone, and the parts independent of $p$ and $q$, on the two sides of the equation, must be respectively equal to one another; and, therefore,

$$\left.\begin{aligned}2\phi\frac{\partial\phi}{\partial x}&=x\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=xJ\\2\phi\frac{\partial\phi}{\partial y}&=y\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=yJ\\2\phi\frac{\partial\phi}{\partial z}&=z\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}=zJ\end{aligned}\right\},$$

where $J$ denotes $\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$.

Squaring and adding, we have

$$4\phi^2J=(x^2+y^2+z^2)J^2.$$

If $J=0$, we can take $\phi=1$; the solution then is

$$z=xp(u)+yq(u)+u.$$

If $J$ is not zero, then $4\phi^2=(x^2+y^2+z^2)J$; so that

$$\frac{1}{x\phi}\frac{\partial\phi}{\partial x}=\frac{1}{y\phi}\frac{\partial\phi}{\partial y}=\frac{1}{z\phi}\frac{\partial\phi}{\partial z}=\frac{2}{x^2+y^2+z^2},$$

or $\phi$ is a constant multiple of $x^2+y^2+z^2$: say it is equal to $x^2+y^2+z^2$. The solution then is

$$\frac{z}{r^2}=\frac{x}{r^2}p(w)+\frac{y}{r^2}q(w)+w,$$

thus establishing the statement made.

3. We begin with the discussion of the relation of the two solutions in § 1 to one another, ignoring (in the first instance) the facts that $p$ and $q$ are arbitrary functions of $u$ and $w$ respectively.

The complete integral of the differential equation

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1=0$$

can be taken in the form

$$z=u+ax+by,$$

where $u$, $a$, $b$ are arbitrary quantities, independent of the variables and subject solely to the relation

$$a^2+b^2+1=0.$$

Any other solution of the equation is, by the general theory, known to be included either in the complete integral or in the general integral, or it is the singular integral.

Such a solution is given by

$$z=\alpha x+\beta y+w\,(x^2+y^2+z^2),$$

where $\alpha$, $\beta$, $w$ are arbitrary quantities, independent of the variables and subject solely to the relation

$$\alpha^2+\beta^2+1=0.$$

In order to determine the class of integral to which this solution belongs, it is sufficient* to equate the two sets of values of $z$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and from these equations (which, owing to the differential equation, are equivalent to two relations) to deduce the values of $u$, $a$, and consequently also of $b$. If the values of $u$ and $a$ are constants, the new solution is a modified form of the complete integral. If they involve the variables, then the solution is an example of the general integral when a functional relation free from $x$, $y$, $z$ exists between $u$ and $a$; and it is an example of the singular integral when no such functional relation exists between $u$ and $a$.

From the equation

$$z=\alpha x+\beta y+w\,(x^2+y^2+z^2),$$

we have

$$\begin{aligned}2wz-1&=\{1-4w^2(x^2+y^2)-4w(\alpha x+\beta y)\}^{\frac{1}{2}}\\&=\{-(\alpha+2wx)^2-(\beta+2wy)^2\}^{\frac{1}{2}},\end{aligned}$$

* See my *Treatise on Differential Equations*, § 181.

because $\alpha^2+\beta^2+1=0$; say we have

$$2wz-1=\Delta.$$

Then
$$\frac{\partial z}{\partial x}(1-2wz)=\alpha+2wx,$$

so that
$$\frac{\partial z}{\partial x}=-\frac{\alpha+2wx}{\Delta};$$

and, similarly,
$$\frac{\partial z}{\partial y}=-\frac{\beta+2wy}{\Delta}.$$

The equation
$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1=0,$$

is manifestly satisfied.

From the given form $z=u+ax+by$

of the complete integral, we have

$$\frac{\partial z}{\partial x}=a,\quad \frac{\partial z}{\partial y}=b,$$

so that
$$a=-\frac{\alpha+2wx}{\Delta},$$

$$b=-\frac{\beta+2wy}{\Delta},$$

and therefore
$$a=i\frac{\alpha+2wx}{\{(\alpha+2wx)^2+(\beta+2wy)^2\}},$$

$$b=i\frac{\beta+2wy}{\{(\alpha+2wx)^2+(\beta+2wy)^2\}};$$

these values being consistent with the relation

$$a^2+b^2+1=0.$$

The value of $a$ is a function of the variables.

Now
$$2wz-1=\Delta;$$

hence for the value of $u$, we have

$$\begin{aligned} u &= z - ax - by \\ &= \frac{1}{2w} + \frac{\Delta}{2w} + \frac{1}{\Delta}(ax + 2wx^2 + \beta y + 2wy^2) \\ &= \frac{1}{2w} + \frac{1}{2w\Delta}\{\Delta^2 + 2wax + 4w^2x^2 + 2w\beta y + 4w^2y^2\} \\ &= \frac{1}{2w} - \frac{1}{2w\Delta}\{a^2 + 2wax + \beta^2 + 2w\beta y\} \\ &= \frac{1}{2w} + \frac{aa + b\beta}{2w}, \end{aligned}$$

say

$$2wu = 1 + aa + b\beta.$$

Hence $u$ also is variable. As the last relation can be written in the form

$$2wu = 1 + aa + \beta i\,(a^2+1)^{\frac{1}{2}},$$

so that $u$ is a function of $a$, it follows that the solution

$$z = ax + \beta y + w\,(x^2 + y^2 + z^2)$$

is an example of the general integral derived from

$$z = u + ax + by.$$

4. This case of the general integral in question can be constructed as follows. Take $u$ as a function of $a$, defined by

$$2wu = 1 + aa + \beta b,$$

so that

$$z = u + ax + by$$

becomes

$$2wz - 1 = a\,(a + 2wx) + b\,(\beta + 2wy)\,;$$

and with this, according to the rule for forming the general integral, combine

$$0 = a + 2wx + \frac{db}{da}(\beta + 2wy).$$

We have

$$a^2 + b^2 = -1,$$

so that

$$a + b\frac{db}{da} = 0\,;$$

and therefore

$$\frac{a}{a + 2wx} = \frac{b}{\beta + 2wy} = \frac{i}{\{(a + 2wa)^2 + (\beta + 2wy)^2\}^{\frac{1}{2}}},$$

the sign of the radical in the latter not being regarded as determinate. Thence

$$2wz-1 = a(\alpha+2wx)+b(\beta+2wy)$$
$$= i\{(\alpha+2wx)^2+(\beta+2wy)^2\}^{\frac{1}{2}},$$

whence $$(2wz-1)^2+(\alpha+2wx)^2+(\beta+2wy)^2 = 0,$$

or, remembering that $\alpha^2+\beta^2+1 = 0$, and dividing out by $4w$, we finally obtain

$$z = \alpha x+\beta y+w(x^2+y^2+z^2),$$

the integral in question.

5. An investigation is thus suggested as to whether other integrals of the differential equation exist which are examples of the general integral deduced from

$$z = u+ax+by,$$

and at the same time—for reasons that have been explained in § 1 —involve linearly two constants such as $a$ and $b$, $\alpha$ and $\beta$, in the respective cases. For this purpose, we assume two quantities $\rho$, $\sigma$, independent of the variables, such that

$$\rho^2+\sigma^2+1 = 0,$$

and another quantity $\nu$, also independent of the variables; and we then require (if possibly attainable) three independent functions $\xi$, $\eta$, $\zeta$ of $x$, $y$, $z$, such that

$$\zeta = \rho\xi+\sigma\eta+\nu$$

furnishes an integral of the differential equation

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1 = 0.$$

Denoting the derivatives of $\xi$ with regard to $x$, $y$, $z$ by $\xi_x$, $\xi_y$, $\xi_z$, and similarly for those of $\eta$ and of $\zeta$, we have

$$\frac{\partial z}{\partial x} = -\frac{\rho\xi_x+\sigma\eta_x-\zeta_x}{\rho\xi_z+\sigma\eta_z-\zeta_z},$$

$$\frac{\partial z}{\partial y} = -\frac{\rho\xi_y+\sigma\eta_y-\zeta_y}{\rho\xi_z+\sigma\eta_z-\zeta_z},$$

consequently, as the differential equation is to be satisfied, it follows that

$$(\rho\xi_x+\sigma\eta_x-\zeta_x)^2+(\rho\xi_y+\sigma\eta_y-\zeta_y)^2+(\rho\xi_z+\sigma\eta_z-\zeta_z)^2 = 0.$$

Now, as $v$ does not occur in this relation, it is not satisfied in virtue of

$$\zeta = \rho\xi + \sigma\eta + v;$$

and therefore the only condition, associated with the relation and assisting its verification, is

$$\rho^2 + \sigma^2 + 1 = 0;$$

in association with this condition, it is an identity. Consequently, we have

$$\xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z = 0,$$

$$\xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z = 0,$$

$$\eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z = 0,$$

$$\xi_x^2 + \xi_y^2 + \xi_z^2 = \eta_x^2 + \eta_y^2 + \eta_z^2 = \zeta_x^2 + \zeta_y^2 + \zeta_z^2,$$

$$= \frac{1}{H^2}, \text{ say.}$$

6. The first three of these equations show that

$$\xi = \text{const.}, \quad \eta = \text{const.}, \quad \zeta = \text{const.}$$

are a family of triply orthogonal surfaces; and the last two impose a limitation upon the families to be selected.

It is easy to see that all the conditions are satisfied for the two cases already considered, which are

$$\text{I.} \quad \xi = x, \quad \eta = y, \quad \zeta = z,$$

giving perpendicular planes for the family; and

$$\text{II.} \quad \xi = \frac{x}{x^2+y^2+z^2}, \quad \eta = \frac{y}{x^2+y^2+z^2}, \quad \zeta = \frac{z}{x^2+y^2+z^2},$$

giving orthogonal spheres through the origin for the family, being in fact the inverse with respect to the origin of the first triply-orthogonal family.

The other two relations impose restrictions upon the families of orthogonal surfaces that can be selected. The quantity $H$ is a function of $x$, $y$, $z$; so that, as $\xi$, $\eta$, $\zeta$ in the left-hand sides of the equations of the form $\xi = \xi(x, y, z)$ can be regarded as parameters, and therefore $x$, $y$, $z$ are expressible in terms of them, $H$ can be

regarded as a function of $\xi$, $\eta$, $\zeta$. But, for any three orthogonal surfaces, if

$$\frac{1}{H_\xi^2} = \xi_x^2 + \xi_y^2 + \xi_z^2,$$

$$\frac{1}{H_\eta^2} = \eta_x^2 + \eta_y^2 + \eta_z^2,$$

$$\frac{1}{H_\zeta^2} = \zeta_x^2 + \zeta_y^2 + \zeta_z^2,$$

it is known* that the radii of curvature of the surface $\xi = \xi(x, y, z)$, at the point where it is cut by the other two surfaces with respective parameters $\eta$ and $\zeta$, are given by

$$\frac{1}{\rho_1} = -\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}, \quad \frac{1}{\rho_2} = -\frac{1}{H_\xi H_\zeta}\frac{\partial H_\xi}{\partial \xi}.$$

In the present case $H_\xi = H_\eta = H_\zeta$; hence the radii of curvature of the surface are equal at every point. As this holds for each of the surfaces, it thus appears that the admissible triply orthogonal surfaces are limited to those which have their radii of curvature everywhere equal.

It is, however, to be noted that this is only an interpretation of the equations and is not of their essence. Though the problem has been suggested by a known instance of real transformation, there is no inherent limitation that the equations of transformation should be purely real; for the equations, determining the arguments $u$ and $w$ respectively, involve functions $p$ and $q$, both of which cannot be entirely real. Moreover, when a solution of Laplace's equation is complex and is expressed in the form $v_1 + iv_2$, then $v_1$ and $v_2$ are each of them a solution of the equation. Consequently, it is not necessary to limit the results to real surfaces; and, in fact, all that is necessary is that the quantities $\xi$, $\eta$, $\zeta$—whether real or complex—shall satisfy the five equations in § 5. Complex solutions, equally with real solutions, would lead to arguments $u$ and $w$.

7. Limiting ourselves, however, for the moment to the case in which the surfaces thus obtained are real, we require those surfaces which have their radii of curvature equal and of the same sign.

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* Lamé, *Leçons sur les Coordonnées curvilignes*, p. 80.

Monge* has shown that the only real surfaces possessing this property are spheres. And it happens that appropriately chosen families of spheres can give triply orthogonal systems of surfaces, the families being chosen in the form

$$\left.\begin{aligned} x^2+y^2+z^2-x\alpha &= 0 \\ x^2+y^2+z^2-y\beta &= 0 \\ x^2+y^2+z^2-z\gamma &= 0 \end{aligned}\right\},$$

where $\alpha$, $\beta$, $\gamma$ are the parameters of the families.

We therefore infer that, if any solution of Laplace's equation be given in the form $V(x, y, z)$, the only real transformation of the variables which leads to another solution is that which corresponds to inversion, viz.,

$$x' = \frac{x}{r^2}, \quad y' = \frac{y}{r^2}, \quad z' = \frac{z}{r^2};$$

and this is the case whether $V$ be a real function or a complex function.

Our investigation has been limited to the consideration of the argument of the arbitrary functions that arise in the solution of $\nabla^2 V = 0$; the transformation of the argument has been established. But it does not follow—and, in fact, it is not the case—that a new solution of Laplace's equation is obtained merely by transforming the argument in the arbitrary function; any variable multiple of the new form will still be a magnitude involving an arbitrary function with the appropriate argument; and accordingly it is necessary to determine whether such variable factor should or should not be introduced.

8. We proceed from the general result without, in the first instance, the restriction as to real transformations.

The quantity $u$, given by

$$z = xp(u) + yq(u) + u,$$

has been expressed in the form

$$u = f(x, y, z);$$

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* *Application de l'Analyse à la Géométrie*, 5me éd., p. 210.

consequently the quantity $v$, given by

$$\zeta = \xi p(v) + \eta q(v) + v,$$

is expressible in the form $v = f(\xi, \eta, \zeta)$. Moreover, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

$$\frac{\partial^2 F(u)}{\partial x^2} + \frac{\partial^2 F(u)}{\partial y^2} + \frac{\partial^2 F(u)}{\partial z^2} = 0,$$

where $F(u)$ is any function of $u$; consequently we also have

$$\left(\frac{\partial v}{\partial \xi}\right)^2 + \left(\frac{\partial v}{\partial \eta}\right)^2 + \left(\frac{\partial v}{\partial \zeta}\right)^2 = 0,$$

$$\frac{\partial^2 F(v)}{\partial \xi^2} + \frac{\partial^2 F(v)}{\partial \eta^2} + \frac{\partial^2 F(v)}{\partial \zeta^2} = 0.$$

The quantity $v$, thus obtained, is the argument of an arbitrary function in the solution of the equation

$$\nabla^2 V = 0.$$

Let $F$ denote this arbitrary function, and suppose that the solution is

$$V = UF(v).$$

Substituting this value in $\nabla^2 V = 0$, we have

$$\begin{aligned} 0 = F(v)\nabla^2 U + 2\left\{(U\xi)\frac{\partial F}{\partial \xi} + (U\eta)\frac{\partial F}{\partial \eta} + (U\zeta)\frac{\partial F}{\partial \zeta}\right\} \\ + U\left\{\nabla^2\xi\frac{\partial F}{\partial \xi} + \nabla^2\eta\frac{\partial F}{\partial \eta} + \nabla^2\zeta\frac{\partial F}{\partial \zeta}\right\} \\ + U\left\{\frac{\partial^2 F}{\partial \xi^2}(\xi_x^2+\xi_y^2+\xi_z^2) + \frac{\partial^2 F}{\partial \eta^2}(\eta_x^2+\eta_y^2+\eta_z^2)\right. \\ \left. + \frac{\partial^2 F}{\partial \zeta^2}(\zeta_x^2+\zeta_y^2+\zeta_z^2)\right\}, \end{aligned}$$

where $(U\theta)$ denotes

$$\frac{\partial U}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial \theta}{\partial z}.$$

But we have $\xi_x^2+\xi_y^2+\xi_z^2 = \eta_x^2+\eta_y^2+\eta_z^2 = \zeta_x^2+\zeta_y^2+\zeta_z^2 = \frac{1}{H^2}$

hence the last set of terms multiplied by $U$ is

$$= \frac{U}{H^2}\left(\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 F}{\partial \zeta^2}\right) = 0.$$

Also $U$ is a function, definite in form and consequently independent of the arbitrary function $F$; hence the aggregates of terms involving the derivatives of $F$ must separately vanish, and therefore we have

$$\left.\begin{aligned} \nabla^2 U &= 0 \\ U\nabla^2\xi + 2(U\xi) &= 0 \\ U\nabla^2\eta + 2(U\eta) &= 0 \\ U\nabla^2\zeta + 2(U\zeta) &= 0 \end{aligned}\right\}.$$

9. Various conditions must be satisfied in order that this set of four simultaneous equations may possess a common solution. To discuss these, we revert to the relations among the derivatives of $\xi$, $\eta$, $\zeta$ as given in § 5; from them we deduce equations of the form

$$\eta_y\zeta_z - \eta_z\zeta_y = \frac{1}{H}\xi_x, \quad \eta_z\zeta_x - \eta_x\zeta_z = \frac{1}{H}\xi_y, \quad \eta_x\zeta_y - \eta_y\zeta_x = \frac{1}{H}\xi_z,$$

there being three such sets; and also

$$\begin{vmatrix} \xi_x, & \xi_y, & \xi_z \\ \eta_x, & \eta_y, & \eta_z \\ \zeta_x, & \zeta_y, & \zeta_z \end{vmatrix} = \frac{1}{H^3}.$$

On solving the last three equations in § 8 for the derivatives of $U$ that are of the first order, we find

$$\left.\begin{aligned} \frac{2}{U}\frac{\partial U}{\partial x} &= -H^2(\xi_x\nabla^2\xi + \eta_x\nabla^2\eta + \zeta_x\nabla^2\zeta) \\ \frac{2}{U}\frac{\partial U}{\partial y} &= -H^2(\xi_y\nabla^2\xi + \eta_y\nabla^2\eta + \zeta_y\nabla^2\zeta) \\ \frac{2}{U}\frac{\partial U}{\partial z} &= -H^2(\xi_z\nabla^2\xi + \eta_z\nabla^2\eta + \zeta_z\nabla^2\zeta) \end{aligned}\right\}.$$

Now, to transform the right-hand sides, we have

$$\xi_x = H(\eta_y\zeta_z - \eta_z\zeta_y),$$

so that $$\xi_{xx} = H_x \begin{vmatrix} \eta_y, & \eta_y \\ \zeta_y, & \zeta_z \end{vmatrix} + H(\eta_{xy}\zeta_z - \eta_{xz}\zeta_y + \eta_y\zeta_{xz} - \eta_z\zeta_{xy}),$$

with similar expressions for $\xi_{yy}$, $\xi_{zz}$; thus

$$\nabla^2\xi = H_x \begin{vmatrix} \eta_y, & \eta_z \\ \zeta_y, & \zeta_z \end{vmatrix} + H_y \begin{vmatrix} \eta_z, & \eta_x \\ \zeta_z, & \zeta_x \end{vmatrix} + H_z \begin{vmatrix} \eta_x, & \eta_y \\ \zeta_x, & \zeta_y \end{vmatrix}$$

$$= \begin{vmatrix} H_x, & H_y, & H_z \\ \eta_x, & \eta_y, & \eta_z \\ \zeta_x, & \zeta_y, & \zeta_z \end{vmatrix}.$$

Similarly $$\nabla^2\eta = \begin{vmatrix} H_x, & H_y, & H_z \\ \zeta_x, & \zeta_y, & \zeta_z \\ \xi_x, & \xi_y, & \xi_z \end{vmatrix}, \quad \nabla^2\zeta = \begin{vmatrix} H_x, & H_y, & H_z \\ \xi_x, & \xi_y, & \xi_z \\ \eta_x, & \eta_y, & \eta_z \end{vmatrix}.$$

These values give

$$\xi_x\nabla^2\xi + \eta_x\nabla^2\eta + \zeta_x\nabla^2\zeta$$

$$= H_x \begin{vmatrix} \xi_x, & \xi_y, & \xi_z \\ \eta_x, & \eta_y, & \eta_z \\ \zeta_x, & \zeta_y, & \zeta_z \end{vmatrix} + H_y \begin{vmatrix} \xi_x, & \xi_z, & \xi_z \\ \eta_z, & \eta_z, & \eta_z \\ \zeta_x, & \zeta_z, & \zeta_z \end{vmatrix} + H_z \begin{vmatrix} \xi_x, & \xi_x, & \xi_y \\ \eta_x, & \eta_x, & \eta_y \\ \zeta_x, & \zeta_x, & \zeta_y \end{vmatrix}$$

$$= \frac{1}{H^3} H_x;$$

and therefore $$\frac{2}{U}\frac{\partial U}{\partial x} = -H^2\frac{1}{H^3}H_x = -\frac{H_x}{H}.$$

Similarly $$\frac{2}{U}\frac{\partial U}{\partial y} = -\frac{H_y}{H},$$

$$\frac{2}{U}\frac{\partial U}{\partial z} = -\frac{H_z}{H};$$

so that the last three equations are consistent with one another, and the common value of $U$ determined by them is

$$U = H^{-\frac{1}{2}}.$$

But, further, we have the equation $\nabla^2 U = 0$, and therefore

$$\nabla^2 H^{-\frac{1}{2}} = 0,$$

being the single condition that must be satisfied in order that the

four equations in $U$ may possess a common solution. This single condition being satisfied, the common solution is given by

$$U = H^{-\frac{1}{2}}.$$

The quantity $H$ is defined by the relations

$$\xi_x^2+\xi_y^2+\xi_z^2 = \eta_x^2+\eta_y^2+\eta_z^2 = \zeta_x^2+\zeta_y^2+\zeta_z^2 = \frac{1}{H^2};$$

hence $\nabla^2 H^{-\frac{1}{2}} = 0$ is a partial differential equation of the third order satisfied alike by $\xi$, by $\eta$, by $\zeta$, when the equations of the triply orthogonal system are taken in the form

$$\xi = \xi(x, y, z), \quad \eta = \eta(x, y, z), \quad \zeta = \zeta(x, y, z).$$

10. Passing now to the particular instance in which the orthogonal surfaces are three families of spheres given by

$$\left.\begin{aligned} x^2+y^2+z^2-x\alpha &= 0 \\ x^2+y^2+z^2-y\beta &= 0 \\ x^2+y^2+z^2-z\gamma &= 0 \end{aligned}\right\},$$

we have $\xi$, $\eta$, $\zeta$ functions of $\alpha$, $\beta$, $\gamma$ respectively, say

$$\xi = f(\alpha), \quad \eta = g(\beta), \quad \zeta = h(\gamma).$$

Now $$\alpha_x^2+\alpha_y^2+\alpha_z^2 = \frac{(x^2+y^2+z^2)^2}{x^4},$$

so that $$\xi_x^2+\xi_y^2+\xi_z^2 = \frac{(x^2+y^2+z^2)^2}{x^4}\{f'(\alpha)\}^2;$$

similarly $$\eta_x^2+\eta_y^2+\eta_z^2 = \frac{(x^2+y^2+z^2)^2}{y^4}\{g'(\beta)\}^2,$$

and $$\zeta_x^2+\zeta_y^2+\zeta_z^2 = \frac{(x^2+y^2+z^2)^2}{z^4}\{h'(\gamma)\}^2.$$

These three quantities must be equal to one another; hence

$$\alpha^2 f'(\alpha) = \beta^2 g'(\beta) = \gamma^2 h'(\gamma) = \text{constant} = -1 \text{ say},$$

so that $$\xi = \frac{1}{\alpha}, \quad \eta = \frac{1}{\beta}, \quad \zeta = \frac{1}{\gamma}.$$

With these values, we have

$$H^{-2} = \xi_x^2 + \xi_y^2 + \xi_z^2 = \frac{1}{r^4},$$

or

$$H^2 = r^4;$$

and therefore

$$U = H^{-\frac{1}{2}} = \frac{1}{r},$$

which satisfies the condition $\nabla^2 H^{-\frac{1}{2}} = 0$.

We thus infer the known theorem that, if $F(x, y, z)$ be a solution of Laplace's equation, another solution is given by

$$\frac{1}{r} F\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right).$$

11. Returning now to the full discussion of the five equations determining the functions $\xi$, $\eta$, $\zeta$, viz.,

$$\left.\begin{aligned} \xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z &= 0 \\ \xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z &= 0 \\ \eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \xi_x^2 + \xi_y^2 + \xi_z^2 &= \eta_x + \eta_y^2 + \eta_z^2 \\ &= \zeta_x^2 + \zeta_y + \zeta_z^2 = \frac{1}{H^2} \end{aligned}\right\},$$

the first three determine a triply orthogonal system and the other two determine surfaces among such systems which satisfy the equation expressing the condition that the radii of curvature are equal and of the same sign.

Surfaces possessing the latter property can be obtained by integrating the differential equation; and it appears* that they are determined by an integral equation involving a couple of arbitrary independent functions. Further it is known that, when a family of surfaces constitutes a portion of a triply orthogonal system, the parameter of the family (or, more generally, the equation of the family) satisfies a partial differential equation of the third order. This remark is originally due to Bouquet; and the establishment of the equation of the third order was made the subject of numerous investigations by him and by others, among whom may be mentioned

* "Note on Surfaces whose Radii of Curvature are equal and of the same sign," *Messenger of Mathematics*, Vol. XXVII. (1898), pp. 129–137.

Bonnet, Serret, Darboux, Lévy, Cayley, and Weingarten.* One method of proceeding accordingly would be to subject the known equation of the surfaces of equal curvature, viz.,

$$\left.\begin{aligned} y+iz+x\theta &= F(\theta) \\ y-iz-x\frac{1}{\theta} &= G(\theta) \end{aligned}\right\},$$

to the equation of the third order characteristic of orthogonality; there would thus arise a relation affecting the arbitrary functions $F$ and $G$. This method is, however, somewhat laborious and indirect.

There are two other processes which can be adopted. We can take the three equations which express the orthogonal property as the primary set and associate the other two with them; the integration of the equations proves to be possible. Or we can take the equations which express the equality of curvatures as the primary set; they are integrable, and their integrals are then subjected to the three simultaneous equations of the first order taken as a subsidiary set.

The first of these processes is as follows:† it leads to the complete result. The second of them will be indicated later, and the result already obtained will be used in illustration of the theory of partial differential equations.

12. From the equations satisfied by $\xi$, $\eta$, $\zeta$, we have

$$\left.\begin{aligned} \frac{1}{H}\xi_x &= \eta_y\zeta_z-\eta_z\zeta_y \\ \frac{1}{H}\xi_y &= \eta_z\zeta_x-\eta_x\zeta_z \\ \frac{1}{H}\xi_z &= \eta_x\zeta_y-\eta_y\zeta_x \end{aligned}\right\},$$

and two other sets; and, further,

$$\begin{vmatrix} \xi_x, & \xi_y, & \xi_z \\ \eta_x, & \eta_y, & \eta_z \\ \zeta_x, & \zeta_y, & \zeta_z \end{vmatrix} = \frac{1}{H^3}.$$

* Full references will be found in Darboux's *Théorie générale des Surfaces;* and Cayley's memoir "On Curvature and Orthogonal Surfaces," *Coll. Math. Papers*, Vol. VIII., pp. 292–315, may be consulted with advantage. [To these must be added Darboux's *Leçons sur les Systèmes orthogonaux*, the first volume of which has appeared since this paper was written.]

† The initial part, § 12, is a modification of Lamé's analysis, applied to triply orthogonal systems not limited by the property of having equal curvatures; see his *Leçons sur les Coordonnées curvilignes*, in particular, Lessons v. and vi.

Now we have

$$\left.\begin{aligned} 1 &= \xi_x \frac{\partial x}{\partial \xi} + \xi_y \frac{\partial y}{\partial \xi} + \xi_z \frac{\partial z}{\partial \xi} \\ 0 &= \eta_x \frac{\partial x}{\partial \xi} + \eta_y \frac{\partial y}{\partial \xi} + \eta_z \frac{\partial z}{\partial \xi} \\ 0 &= \zeta_x \frac{\partial x}{\partial \xi} + \zeta_y \frac{\partial y}{\partial \xi} + \zeta_z \frac{\partial z}{\partial \xi} \end{aligned}\right\},$$

$x$, $y$, $z$ being regarded as functions of $\xi$, $\eta$, $\zeta$; hence

$$\frac{1}{H^3} \frac{\partial x}{\partial \xi} = \begin{vmatrix} \eta_y, & \eta_z \\ \zeta_y, & \zeta_z \end{vmatrix} = \frac{1}{H} \xi_x,$$

and therefore

$$\frac{\partial x}{\partial \xi} = H^2 \xi_x.$$

Similarly for the others; so that, if $\alpha$, $\beta$, $\gamma$ denote $\xi$, $\eta$, $\zeta$ in any order, and if $u$ denote any one of the variables $x$, $y$, $z$, then

$$\frac{\partial u}{\partial \alpha} = H^2 \alpha_u.$$

Also we have

$$\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 = H^4 (\xi_x^2 + \xi_y^2 + \xi_z^2) = H^2;$$

and so for derivatives with regard to $\eta$ and $\zeta$.

If we were dealing with real surfaces, the quantities $H\xi_x$, $H\xi_y$, $H\xi_z$ would be the direction-cosines of the normal to $\xi$ at $x$, $y$, $z$. Denoting them accordingly in any case by $l_\xi$, $m_\xi$, $n_\xi$, we have

$$\left.\begin{aligned} l_\xi &= \frac{1}{H} \frac{\partial x}{\partial \xi} = H \frac{\partial \xi}{\partial x} \\ m_\xi &= \frac{1}{H} \frac{\partial y}{\partial \xi} = H \frac{\partial \xi}{\partial y} \\ n_\xi &= \frac{1}{H} \frac{\partial z}{\partial \xi} = H \frac{\partial \xi}{\partial z} \end{aligned}\right\},$$

and so for $l_\eta$, $m_\eta$, $n_\eta$; $l_\zeta$, $m_\zeta$, $n_\zeta$.

Again, from the equation

$$\frac{\partial x}{\partial \xi} = H^2 \xi_x,$$

we have
$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial}{\partial x}(H^2\xi_x)\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}(H^2\xi_x)\frac{\partial y}{\partial \eta} + \frac{\partial}{\partial z}(H^2\xi_x)\frac{\partial z}{\partial \eta}$$
$$= 2H\frac{\partial H}{\partial \eta}\xi_x + (\xi_{xx}\eta_x + \xi_{xy}\eta_y + \xi_{xz}\eta_z)H^4;$$

and, from
$$\frac{\partial x}{\partial \eta} = H^2 \eta_x,$$

we have, similarly,

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = 2H\frac{\partial H}{\partial \xi}\eta_x + (\eta_{xx}\xi_x + \eta_{xy}\xi_y + \eta_{xz}\xi_z)H^4.$$

Hence, adding and remembering that

$$\xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z = 0,$$

we have
$$\frac{\partial^2 x}{\partial \xi \partial \eta} = H\left(\frac{\partial H}{\partial \eta}\xi_x + \frac{\partial H}{\partial \xi}\eta_x\right).$$

Similarly for the other derivatives of $x$, and for derivatives of $y$, $z$; these can all be included in the form

$$\frac{\partial^2 u}{\partial \alpha \partial \beta} = H\left(\frac{\partial H}{\partial \beta}\alpha_u + \frac{\partial H}{\partial \alpha}\beta_u\right).$$

From the same equation

$$\frac{\partial x}{\partial \xi} = H^2 \xi_x,$$

we have
$$\frac{\partial^2 x}{\partial \xi^2} = 2H\frac{\partial H}{\partial \xi}\xi_x + (\xi_{xx}\xi_x + \xi_{xy}\xi_y + \xi_{xz}\xi_z)H^4$$
$$= 2H\frac{\partial H}{\partial \xi}\xi_x - H^3\frac{1}{H^2}\frac{\partial H}{\partial x}$$
$$= H\frac{\partial H}{\partial \xi}\xi_x - H\frac{\partial H}{\partial \eta}\eta_x - H\frac{\partial H}{\partial \zeta}\zeta_x.$$

Similarly for the other derivatives of $x$, and for derivatives of $y$, $z$: these can all be included in the form

$$\frac{\partial^2 u}{\partial \alpha^2} = H\frac{\partial H}{\partial \alpha}\alpha_u - H\frac{\partial H}{\partial \beta}\beta_u - H\frac{\partial H}{\partial \gamma}\gamma_u.$$

Next, we have $$l_\alpha = \frac{1}{H}\frac{\partial x}{\partial \alpha},$$

so that
$$\frac{\partial l_\alpha}{\partial \alpha} = \frac{1}{H}\frac{\partial^2 x}{\partial \alpha^2} - \frac{1}{H^2}\frac{\partial H}{\partial \alpha}\frac{\partial x}{\partial \alpha}$$
$$= -\frac{\partial H}{\partial \beta}\beta_x - \frac{\partial H}{\partial \gamma}\gamma^x$$
$$= -\frac{1}{H}\frac{\partial H}{\partial \beta}l_\beta - \frac{1}{H}\frac{\partial H}{\partial \gamma}l_\gamma;$$

and
$$\frac{\partial l_\alpha}{\partial \beta} = \frac{1}{H}\frac{\partial^2 x}{\partial \alpha \partial \beta} - \frac{1}{H^2}\frac{\partial H}{\partial \beta}\frac{\partial x}{\partial \alpha}$$
$$= \frac{1}{H}\frac{\partial H}{\partial \alpha}l_\beta.$$

From the latter, we have
$$\frac{\partial}{\partial \gamma}\left(\frac{1}{H}\frac{\partial H}{\partial \alpha}l_\beta\right) = \frac{\partial}{\partial \beta}\left(\frac{1}{H}\frac{\partial H}{\partial \alpha}l_\gamma\right),$$

so that
$$l_\beta\frac{\partial}{\partial \gamma}\left(\frac{1}{H}\frac{\partial H}{\partial \alpha}\right) + \frac{1}{H}\frac{\partial H}{\partial \alpha}\frac{1}{H}\frac{\partial H}{\partial \beta}l_\gamma$$
$$= l_\gamma\frac{\partial}{\partial \beta}\left(\frac{1}{H}\frac{\partial H}{\partial \alpha}\right) + \frac{1}{H}\frac{\partial H}{\partial \alpha}\frac{1}{H}\frac{\partial H}{\partial \gamma}l_\beta,$$

and therefore
$$l_\beta\left(\frac{1}{H}\frac{\partial^2 H}{\partial \alpha \partial \gamma} - \frac{2}{H^2}\frac{\partial H}{\partial \alpha}\frac{\partial H}{\partial \gamma}\right) = l_\gamma\left(\frac{1}{H}\frac{\partial^2 H}{\partial \alpha \partial \beta} - \frac{2}{H^2}\frac{\partial H}{\partial \alpha}\frac{\partial H}{\partial \beta}\right).$$

Corresponding equations subsist with $m_\beta, m_\gamma$; and $n_\beta, n_\gamma$; in place of $l_\beta, l_\gamma$; so that, as the fractions $l_\beta/l_\gamma$, $m_\beta/m_\gamma$, $n_\beta/n_\gamma$ are not equal to one another, we infer
$$\frac{\partial^2 H}{\partial \alpha \partial \beta} - \frac{2}{H}\frac{\partial H}{\partial \alpha}\frac{\partial H}{\partial \beta} = 0,$$

$$\frac{\partial^2 H}{\partial \alpha \partial \gamma} - \frac{2}{H}\frac{\partial H}{\partial \alpha}\frac{\partial H}{\partial \gamma} = 0;$$

and, similarly,
$$\frac{\partial^2 H}{\partial \beta \partial \gamma} - \frac{2}{H}\frac{\partial H}{\partial \beta}\frac{\partial H}{\partial \gamma} = 0.$$

Again, $$\frac{\partial}{\partial\alpha}\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}l_\beta\right)=\frac{\partial^2 l_\alpha}{\partial\alpha\partial\beta}$$

$$=-\frac{\partial}{\partial\beta}\left\{\frac{1}{H}\frac{\partial H}{\partial\beta}l_\beta+\frac{1}{H}\frac{\partial H}{\partial\gamma}l_\gamma\right\};$$

and therefore

$$\left\{\frac{\partial}{\partial\alpha}\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{H}\frac{\partial H}{\partial\beta}\right)\right\}l_\beta+\frac{\partial}{\partial\beta}\left(\frac{1}{H}\frac{\partial H}{\partial\gamma}\right)l_\gamma$$

$$+\frac{1}{H}\frac{\partial H}{\partial\alpha}\frac{1}{H}\frac{\partial H}{\partial\beta}l_\alpha+\frac{1}{H}\frac{\partial H}{\partial\gamma}\frac{1}{H}\frac{\partial H}{\partial\gamma}l_\beta$$

$$-\frac{1}{H}\frac{\partial H}{\partial\beta}\left\{\frac{1}{H}\frac{\partial H}{\partial\alpha}l_\alpha+\frac{1}{H}\frac{\partial H}{\partial\gamma}l_\gamma\right\}=0,$$

so that $$\left\{\frac{\partial}{\partial\alpha}\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{H}\frac{\partial H}{\partial\beta}\right)+\left(\frac{1}{H}\frac{\partial H}{\partial\gamma}\right)^2\right\}l_\beta$$

$$=-l_\gamma\left(\frac{1}{H}\frac{\partial^2 H}{\partial\beta\partial\gamma}-\frac{2}{H^2}\frac{\partial H}{\partial\beta}\frac{\partial H}{\partial\gamma}\right)=0.$$

Hence $$\frac{\partial}{\partial\alpha}\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{H}\frac{\partial H}{\partial\beta}\right)+\left(\frac{1}{H}\frac{\partial H}{\partial\gamma}\right)^2=0;$$

and, similarly,

$$\frac{\partial}{\partial\beta}\left(\frac{1}{H}\frac{\partial H}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\frac{1}{H}\frac{\partial H}{\partial\gamma}\right)+\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}\right)^2=0,$$

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{H}\frac{\partial H}{\partial\gamma}\right)+\frac{\partial}{\partial\alpha}\left(\frac{1}{H}\frac{\partial H}{\partial\alpha}\right)+\left(\frac{1}{H}\frac{\partial H}{\partial\beta}\right)^2=0.$$

13. To solve this set of six equations for $H$, corresponding to Lamé's equations in the more general case, take

$$H=\frac{1}{V}.$$

The first set of three become

$$\frac{\partial^2 V}{\partial\alpha\partial\beta}=0,\quad\frac{\partial^2 V}{\partial\beta\partial\gamma}=0,\quad\frac{\partial^2 V}{\partial\gamma\partial\alpha}=0,$$

so that $$V=\mathrm{A}+\mathrm{B}+\Gamma,$$

where A is an arbitrary function of $\alpha$, B is an arbitrary function of $\beta$, and $\Gamma$ is an arbitrary function of $\gamma$. The second set of three become

$$\frac{\partial^2 V}{\partial \alpha^2}+\frac{\partial^2 V}{\partial \beta^2}=\frac{1}{V}\left\{\left(\frac{\partial V}{\partial \alpha}\right)^2+\left(\frac{\partial V}{\partial \beta}\right)^2+\left(\frac{\partial V}{\partial \gamma}\right)^2\right\},$$

$$\frac{\partial^2 V}{\partial \beta^2}+\frac{\partial^2 V}{\partial \gamma^2}=\frac{1}{V}\left\{\left(\frac{\partial V}{\partial \alpha}\right)^2+\left(\frac{\partial V}{\partial \beta}\right)^2+\left(\frac{\partial V}{\partial \gamma}\right)^2\right\},$$

$$\frac{\partial^2 V}{\partial \gamma^2}+\frac{\partial^2 V}{\partial \alpha^2}=\frac{1}{V}\left\{\left(\frac{\partial V}{\partial \alpha}\right)^2+\left(\frac{\partial V}{\partial \beta}\right)^2+\left(\frac{\partial V}{\partial \gamma}\right)^2\right\},$$

so that $$\frac{\partial^2 V}{\partial \alpha^2}=\frac{\partial^2 V}{\partial \beta^2}=\frac{\partial^2 V}{\partial \gamma^2}=\frac{1}{2V}\left\{\left(\frac{\partial V}{\partial \alpha}\right)^2+\left(\frac{\partial V}{\partial \beta}\right)^2+\left(\frac{\partial V}{\partial \gamma}\right)^2\right\}.$$

The first two equalities give

$$\frac{d^2\mathrm{A}}{d\alpha^2}=\frac{d^2\mathrm{B}}{d\beta^2}=\frac{d^2\Gamma}{d\gamma^2},$$

and each of the three expressions is therefore a constant, say $2a$; so that

$$\mathrm{A}=a\alpha^2+2b'\alpha+c',$$

$$\mathrm{B}=a\beta^2+2b''\beta+c'',$$

$$\Gamma=a\gamma^2+2b'''\beta+c'''.$$

Also $$\left(\frac{d\mathrm{A}}{d\alpha}\right)^2+\left(\frac{d\mathrm{B}}{d\beta}\right)^2+\left(\frac{d\Gamma}{d\gamma}\right)^2=4a\,(\mathrm{A}+\mathrm{B}+\Gamma),$$

which, on substitution, leads to

$$b'^2+b''^2+b'''^2=a\,(c'+c''+c''');$$

and therefore

$$a\,(\mathrm{A}+\mathrm{B}+\Gamma)=(a\alpha+b')^2+(a\beta+b'')^2+(a\gamma+b''')^2.$$

Now $\alpha$, $\beta$, $\gamma$ are $\xi$, $\eta$, $\zeta$ in any order; and the association of constants $b'$, $b''$, $b'''$ with $a\alpha$, $a\beta$, $a\gamma$ does not affect the essential character of $\xi$, $\eta$, $\zeta$; hence we can take

$$H=\frac{1}{V}$$

$$=\frac{1}{\mathrm{A}+\mathrm{B}+\Gamma}$$

$$=\frac{1}{a\,(\xi^2+\eta^2+\zeta^2)}.$$

Using this value of $H$ and transforming the equations, so that they can be regarded as giving $x$, $y$, $z$ in terms of $\xi$, $\eta$, $\zeta$, we have

$$\left.\begin{aligned} x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta &= 0 \\ x_\xi x_\zeta + y_\xi y_\zeta + z_\xi z_\zeta &= 0 \\ x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} x_\xi^2 + y_\xi^2 + z_\xi^2 = x_\eta^2 + y_\eta^2 + z_\eta^2 &= x_\zeta^2 + y_\zeta^2 + z_\zeta^2 \\ &= H^2 \end{aligned}\right\}.$$

Thus

$$\frac{1}{H}\begin{vmatrix} x_\xi, & y_\xi, & z_\xi \\ x_\eta, & y_\eta, & z_\eta \\ x_\zeta, & y_\zeta, & z_\zeta \end{vmatrix}$$

satisfy the equations characteristic of the direction-cosines of three perpendicular lines; and therefore we also have the equivalent equations

$$\left.\begin{aligned} x_\xi y_\xi + x_\eta y_\eta + x_\zeta y_\zeta &= 0 \\ x_\xi z_\xi + x_\eta z_\eta + x_\zeta z_\zeta &= 0 \\ y_\xi z_\xi + y_\eta z_\eta + y_\zeta z_\zeta &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} x_\xi^2 + x_\eta^2 + x_\zeta^2 = y_\xi^2 + y_\eta^2 + y_\zeta^2 &= z_\xi^2 + z_\eta^2 + z_\zeta^2 \\ &= \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2} \end{aligned}\right\},$$

on substituting the value of $H$ and writing $a = \kappa^{-2}$, a mere change of constant.

14. Again, it was seen that a condition $\nabla^2 H^{-\frac{1}{2}} = 0$ must be satisfied; as a matter of fact, it is satisfied by values of $\xi$, $\eta$, $\zeta$ deduced from the foregoing equations with the newly obtained value of $H$, so that no further relation is established by the condition. For, substituting

$$\kappa H^{-\frac{1}{2}} = (\xi^2+\eta^2+\zeta^2)_{\frac{1}{2}}$$

in $\nabla^2 H^{-\frac{1}{2}} = 0$, $\kappa$ being a constant, we find

$$\xi\nabla^2\xi + \eta\nabla^2\eta + \zeta\nabla^2\zeta + \frac{2}{H^2} = 0,$$

after reduction. Values of $\nabla^2\xi$, $\nabla^2\eta$, $\nabla^2\zeta$ have been obtained in § 9,

in terms of derivatives of the first order of $\xi$, $\eta$, $\zeta$; they give

$$\xi\nabla^2\xi+\eta\nabla^2\eta+\zeta\nabla^2\zeta=-\begin{vmatrix} 0, & \xi, & \eta, & \zeta \\ H_x, & \xi_x, & \eta_x, & \zeta_x \\ H_y, & \xi_y, & \eta_y, & \zeta_y \\ H_z, & \xi_z, & \eta_z, & \zeta_z \end{vmatrix};$$

multiplying the second, third, and fourth columns respectively by $H_\xi$, $H_\eta$, $H_\zeta$, and subtracting their sum from the first, the determinant is equal to

$$-\begin{vmatrix} \xi H_\xi+\eta H_\eta+\zeta H_\zeta, & \xi, & \eta, & \zeta \\ 0, & \xi_x, & \eta_x, & \zeta_x \\ 0, & \xi_y, & \eta_y, & \zeta_y \\ 0, & \xi_z, & \eta_z, & \zeta_z \end{vmatrix}$$

$$=-\frac{1}{H^3}(\xi H_\xi+\eta H_\eta+\zeta H_\zeta),$$

so that $$\xi\nabla^2\xi+\eta\nabla^2\eta+\zeta\nabla^2\zeta=\frac{1}{H^3}\left(\xi\frac{\partial H}{\partial\xi}+\eta\frac{\partial H}{\partial\eta}+\zeta\frac{\partial H}{\partial\zeta}\right).$$

Hence the condition is

$$\xi\frac{\partial H}{\partial\xi}+\eta\frac{\partial H}{\partial\eta}+\zeta\frac{\partial H}{\partial\zeta}+2H=0,$$

satisfied identically because $H$ is $\frac{1}{a}(\xi^2+\eta^2+\zeta^2)^{-1}$, a homogeneous function of $\xi$, $\eta$, $\zeta$ of dimension $-2$.

15. It now is possible to integrate the equations that express the orthogonal property. We have proved that, if $u$ denote $x$, $y$, or $z$, and if $\alpha$, $\beta$, $\gamma$ be $\xi$, $\eta$, $\zeta$ in any order, then

$$\frac{\partial^2 u}{\partial\alpha\partial\beta}=H\left(\frac{\partial H}{\partial\beta}\alpha_u+\frac{\partial H}{\partial\alpha}\beta_u\right)$$

$$=\frac{1}{H}\frac{\partial H}{\partial\beta}\frac{\partial u}{\partial\alpha}+\frac{1}{H}\frac{\partial H}{\partial\alpha}\frac{\partial u}{\partial\beta}.$$

Now $$H=\frac{\kappa^2}{\xi^2+\eta^2+\zeta^2}=\frac{\kappa^2}{\alpha^2+\beta^2+\gamma^2},$$

so that the preceding equation becomes

$$(\alpha^2+\beta^2+\gamma^2)\frac{\partial^2 u}{\partial\alpha\partial\beta}+2\left(\beta\frac{\partial u}{\partial\alpha}+\alpha\frac{\partial u}{\partial\beta}\right)=0,$$

or, writing $\rho, \sigma, \tau = \alpha^2, \beta^2, \gamma^2$, it is

$$(\rho+\sigma+\tau)\frac{\partial^2 u}{\partial\rho\partial\sigma}+\frac{\partial u}{\partial\rho}+\frac{\partial u}{\partial\sigma}=0,$$

that is,

$$\frac{\partial^2}{\partial\rho\partial\sigma}\{(\rho+\sigma+\tau)u\}=0.$$

Hence, $x, y, z$, each of them, satisfy the equations

$$\frac{1}{\xi\eta}\frac{\partial^2}{\partial\xi\partial\eta}\{(\xi^2+\eta^2+\zeta^2)u\}=0,$$

$$\frac{1}{\eta\zeta}\frac{\partial^2}{\partial\eta\partial\zeta}\{(\xi^2+\eta^2+\zeta^2)u\}=0,$$

$$\frac{1}{\zeta\xi}\frac{\partial^2}{\partial\zeta\partial\xi}\{(\xi^2+\eta^2+\zeta^2)u\}=0.$$

The most general simultaneous integral of these equations is

$$(\xi^2+\eta^2+\zeta^2)u=E+F+G,$$

where $E$ is an arbitrary function of $\xi$, $F$ of $\eta$, and $G$ of $\zeta$; and therefore, taking distinct arbitrary functions for the three variables, we have

$$\left.\begin{aligned} x&=\frac{E_1+F_1+G_1}{\xi^2+\eta^2+\zeta^2}\\ y&=\frac{E_2+F_2+G_2}{\xi^2+\eta^2+\zeta^2}\\ z&=\frac{E_3+F_3+G_3}{\xi^2+\eta^2+\zeta^2}\end{aligned}\right\}.$$

The functions $E$, $F$, $G$ are arbitrary so far as concerns the immediate equations under consideration; but the values of $x, y, z$ must satisfy other equations.

We have

$$x_\xi=\frac{E_1'}{\xi^2+\eta^2+\zeta^2}-2\xi\frac{E_1+F_1+G_1}{(\xi^2+\eta^2+\zeta^2)^2},$$

and so for the other derivatives; hence, substituting in

$$x_\xi^2+x_\eta^2+x_\zeta^2=\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2},$$

we have

$$(\kappa^4-E_1'^2-F_1'^2-G_1'^2)(\xi^2+\eta^2+\zeta^2) = 4(E_1+G_1+H_1)^2 - 4(E_1+G_1+H_1)(\xi E_1'+\eta G_1'+\zeta H_1'),$$

which must be identically satisfied. To secure this result, the terms depending upon $\xi$ alone, those upon $\eta$ alone, and those upon $\zeta$ alone, must balance on the two sides of the equation: conditions which are necessary, though not sufficient. Writing

$$\kappa^4 = \lambda_1^2+\mu_1^2+\nu_1^2,$$

where $\lambda_1$, $\mu_1$, $\nu_1$ are three arbitrary constants subject to this single relation, the three conditions mentioned give

$$\left.\begin{aligned}(\lambda_1^2-E_1'^2)\,\xi^2 &= 4E_1^2-4\xi E_1E_1'\\ (\mu_1^2-F_1'^2)\,\eta^2 &= 4F_1^2-4\eta F_1F_1'\\ (\nu_1^2-G_1'^2)\,\zeta^2 &= 4G_1^2-4\zeta G_1G_1'\end{aligned}\right\},$$

from which we easily obtain

$$\left.\begin{aligned}E_1 &= a\xi^2+\lambda_1\xi\\ F_1 &= b\eta^2+\mu_1\eta\\ G_1 &= c\zeta^2+\nu_1\zeta\end{aligned}\right\},$$

where $a$, $b$, $c$ are constants. With these values, we have

$$\kappa^4-E_1'^2-F_1'^2-G_1'^2 = -4(a^2\xi^2+a\lambda_1\xi+b^2\eta^2+b\mu_1\eta+c^2\zeta^2+c\nu_1\zeta),$$

$$\xi E_1'+\eta F_1'+\zeta G_1'-E_1-F_1-G_1 = a\xi^2+b\eta^2+c\zeta^2;$$

so that the equation to be identically satisfied is

$$(\xi^2+\eta^2+\zeta^2)(a^2\xi^2+a\lambda_1\xi+b^2\eta^2+b\mu_1\eta+c^2\zeta^2+c\nu_1\zeta) = (a\xi^2+b\eta^2+c\zeta^2)(a\xi^2+\lambda_1\xi+b\eta^2+\mu_1\eta+c\zeta^2+\nu_1\zeta).$$

This is the case if, and only if,

$$a = b = c = a_1, \text{ say};$$

and accordingly the value of $x$ is given by

$$x-a_1 = \frac{\lambda_1\xi+\mu_1\eta+\nu_1\zeta}{\xi^2+\eta^2+\zeta^2},$$

where $\lambda_1$, $\mu_1$, $\nu_1$ are constants subject to the relation

$$\kappa^4 = \lambda_1^2+\mu_1^2+\nu_1^2.$$

Similarly, we have

$$y-\alpha_2 = \frac{\lambda_2\xi+\mu_2\eta+\nu_2\zeta}{\xi^2+\eta^2+\zeta^2},$$

$$z-\alpha_3 = \frac{\lambda_3\xi+\mu_3\eta+\nu_3\zeta}{\xi^2+\eta^2+\zeta^2},$$

where the constants are subject to the relations

$$\lambda_2^2+\mu_2^2+\nu_2^2 = \kappa^4$$
$$= \lambda_3^2+\mu_3^2+\nu_3^2.$$

Again, we must have

$$x_\xi y_\xi + x_\eta y_\eta + x_\zeta y_\zeta = 0;$$

substituting the preceding values, we find

$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2 = 0.$$

Similarly, from the other conditions, we have

$$\lambda_1\lambda_3+\mu_1\mu_3+\nu_1\nu_3 = 0,$$
$$\lambda_2\lambda_3+\mu_2\mu_3+\nu_2\nu_3 = 0.$$

Manifestly, we may take

$$\lambda_s,\ \mu_s,\ \nu_s = \kappa^2 l_s,\ \kappa^2 m_s,\ \kappa^2 n_s \qquad (s = 1, 2, 3),$$

where the three sets of quantities $l$, $m$, $n$ are direction-cosines of three perpendicular lines; and now the solutions are

$$\left.\begin{aligned} x-\alpha_1 &= \kappa^2\frac{l_1\xi+m_1\eta+n_1\zeta}{\xi^2+\eta^2+\zeta^2} \\ y-\alpha_2 &= \kappa^2\frac{l_2\xi+m_2\eta+n_2\zeta}{\xi^2+\eta^2+\zeta^2} \\ z-\alpha_3 &= \kappa^2\frac{l_3\xi+m_3\eta+n_3\zeta}{\xi^2+\eta^2+\zeta^2} \end{aligned}\right\}.$$

16. No functional generality is retained by means of the constants $\alpha_1$, $\alpha_2$, $\alpha_3$, because they merely imply a change of origin in their disappearance; nor is any functional generality maintained by the explicit retention of all the constants $l$, $m$, $n$, because a change of axes to the directions

$$\begin{vmatrix} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{vmatrix}$$

leaves the solution substantially the same. Assuming these changes made, we have

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta} = \frac{\kappa^2}{\xi^2+\eta^2+\zeta^2};$$

and therefore also $$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta}{z} = \frac{\kappa^2}{x^2+y^2+z^2}.$$

Hence, finally, we have established the result:

*The only transformations of variables, leading to new solutions of Laplace's equation* $\nabla^2 V = 0$ *when one solution is known, are those which arise from inversion, change of axes, and change of origin.*

Further, it is to be remarked that the preceding investigation, when interpreted in connexion with triply orthogonal surfaces, yields the result:

*The only triply orthogonal surfaces, having their principal curvatures everywhere equal to one another, are the (real) families of triply orthogonal spheres.*

17. The other of the processes, referred to in § 11, is the integration of the equations expressing the property that the curvatures are equal. For this purpose, it is necessary to obtain the complete integral of

$$x_\xi^2+x_\eta^2+x_\zeta^2 = \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2},$$

which, by changes of arbitrary constants, will give also the complete integral of

$$y_\xi^2+y_\eta^2+y_\zeta^2 = \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)},$$

and that of $$z_\xi^2+z_\eta^2+z_\zeta^2 = \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2}.$$

And then such forms, either of the complete integrals or deduced from the complete integrals (that is, general integrals or singular integrals), must be chosen as to satisfy the three equations

$$\left.\begin{aligned} x_\xi y_\xi + x_\eta y_\eta + x_\zeta y_\zeta &= 0 \\ x_\xi z_\xi + x_\eta z_\eta + x_\zeta z_\zeta &= 0 \\ y_\xi z_\xi + y_\eta z_\eta + y_\zeta z_\zeta &= \end{aligned}\right\}.$$

As there are six equations involving three dependent variables,

conditions of coexistence (and of existence) must be satisfied. Three of these are known, viz., the equations of the third order which must be satisfied by $\xi$, by $\eta$, and by $\zeta$, in order that the surfaces may belong to a triply orthogonal system: these are conditions of existence. Three others can be deduced from the conditions of coexistence as follows.

In order that the equations

$$\left.\begin{aligned} U &= y_\xi^2 + y_\eta^2 + y_\zeta^2 - \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2} = 0 \\ V &= y_\xi x_\xi + y_\eta x_\eta + y_\zeta x_\zeta = 0 \end{aligned}\right\}$$

may be satisfied by the same value of $y$, we must have

$$(U, V) = 0,$$

that is, on removing a factor 2,

$$y_\xi^2 x_{\xi\xi} + 2y_\xi y_\eta x_{\xi\eta} + 2y_\xi y_\zeta x_{\xi\zeta} + y_\eta^2 x_{\eta\eta} + 2y_\eta y_\zeta x_{\eta\zeta} + y_\zeta^2 x_{\zeta\zeta}$$

$$-2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}(\xi x_\xi + \eta x_\eta + \zeta x_\zeta) = 0.$$

Similarly, in order that

$$z_\xi^2 + z_\eta^2 + z_\zeta^2 - \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2} = 0,$$

$$z_\xi x_\xi + z_\eta x_\eta + z_\zeta x_\zeta = 0$$

may coexist, we must have

$$z_\xi^2 x_{\xi\xi} + 2z_\xi z_\eta x_{\xi\eta} + 2z_\xi z_\zeta x_{\xi\zeta} + z_\eta^2 x_{\eta\eta} + 2z_\eta z_\zeta x_{\eta\zeta} + z_\zeta^2 x_{\zeta\zeta}$$

$$-2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}(\xi x_\xi + \eta x_\eta + \zeta x_\zeta) = 0.$$

Further, from $x_\xi^2 + x_\eta^2 + x_\zeta^2 - \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2} = 0,$

we have

$$x_\xi x_{\xi\xi} + x_\eta x_{\xi\eta} + x_\zeta x_{\xi\zeta} + 2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}\xi = 0,$$

$$x_\xi x_{\xi\eta} + x_\eta x_{\eta\eta} + x_\zeta x_{\eta\zeta} + 2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}\eta = 0,$$

$$x_\xi x_{\xi\zeta} + x_\eta x_{\eta\zeta} + x_\zeta x_{\zeta\zeta} + 2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}\zeta = 0,$$

which, when multiplied by $x_\xi$, $x_\eta$, $x_\zeta$ respectively, and added, lead to the relation

$$x_\xi^2 x_{\xi\xi} + 2x_\xi x_\eta x_{\xi\eta} + 2x_\xi x_\zeta x_{\xi\zeta} + x_\eta^2 x_{\eta\eta} + 2x_\eta x_\zeta x_{\eta\zeta} + x_\zeta^2 x_{\zeta\zeta}$$
$$+ 2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}(\xi x_\xi + \eta x_\eta + \zeta x_\zeta) = 0.$$

Adding these three equations, and using the original equations in the form

$$\left.\begin{aligned} x_\xi^2 + y_\xi^2 + z_\xi^2 &= \frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2} \\ x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta &= 0 \end{aligned}\right\},$$

we have

$$\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}(x_{\xi\xi}+x_{\eta\eta}+x_{\zeta\zeta}) - 2\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^3}(\xi x_\xi + \eta x_\eta + \zeta x_\zeta) = 0,$$

and therefore

$$x_{\xi\xi}+x_{\eta\eta}+x_{\zeta\zeta} - \frac{2}{\xi^2+\eta^2+\zeta^2}(\xi x_\xi + \eta x_\eta + \zeta x_\zeta) = 0.$$

Similarly,
$$y_{\xi\xi}+y_{\eta\eta}+y_{\zeta\zeta} - \frac{2}{\xi^2+\eta^2+\zeta^2}(\xi y_\xi + \eta y_\eta + \zeta y_\zeta) = 0,$$

$$z_{\xi\xi}+z_{\eta\eta}+z_{\zeta\zeta} - \frac{2}{\xi^2+\eta^2+\zeta^2}(\xi z_\xi + \eta z_\eta + \zeta z_\zeta) = 0,$$

the three new conditions indicated.

18. The object of this part of the paper is, in the main, to give an illustration of Lagrange's classification of the integrals of partial differential equations, applying it to the case where we have to deal with a simultaneous system involving three dependent variables, and accordingly the developments suggested by the preceding analysis will be left on one side, in order to proceed with the direct integration of the equations.

We take the equation, which is the first to be integrated, in the form

$$F = p_1 + p_2^2 + p_3^2 - \frac{\kappa^4}{(x_1^2+x_2^2+x_3^2)^2} = 0,$$

and adopt the Jacobian method of integration.* The subsidiary

* See my *Treatise on Differential Equations*, particularly §§ 215–222.

equations are

$$\frac{dx_1}{-2p_1}=\frac{dx_2}{-2p_2}=\frac{dx_3}{-2p_3}=\frac{dp_1}{\dfrac{4\kappa^4x_1}{(x_1^2+x_2^2+x_3^2)^3}}=\frac{dp_2}{\dfrac{4\kappa^4x_2}{(x_1^2+x_2^2+x_3^2)^3}}=\frac{dp_3}{\dfrac{4\kappa^4x_3}{(x_1^2+x_2^2+x_3^2)^3}}.$$

One integral of these is

$$F_1=x_2p_3-x_3p_2=c;$$

and this equation can be associated with $F=0$.

Another integral of the subsidiary system is

$$F_2=x_3p_1-x_1p_3.$$

We have $$(F, F_2)=0,$$

$$(F_1, F_2)=x_1p_2-x_2p_1=F_3,$$

say; so that, by a known proposition, $F_3$ is another integral of the subsidiary system. Again,

$$(F_1, F_3)=-F_2,$$

so that $$(F_1, F^2+F^2)=0;$$

and $$(F, F_2^2+F_3^2)=0.$$

Hence $$F_2^2+F_3^2=a$$

can be associated with $F=0$, $F_1=c$; and the three determine values of $p_1, p_2, p_3$, which make

$$p_1dx_1+p_2dx_2+p_3dx_3$$

a perfect differential.

Let

$$\left.\begin{aligned}\nabla^2&=a^2(x_2^2+x_3^2)-c^2x_1^2\\ \Phi^2&=\frac{\kappa^4}{x_1^2+x_2^2+x_3^2}-a^2-c^2\\ R^2&=x_1^2+x_2^2+x_3^2\end{aligned}\right\};$$

then we find

$$\left.\begin{aligned}p_1&=\frac{x_1}{R^2}\Phi+\frac{\Delta}{R^2}\\ p_2&=\frac{x_2}{R^2}\Phi-\frac{\Delta}{R^2}\frac{x_1x_2}{x_2^2+x_3^2}-\frac{cx_3}{x_2^2+x_3^2}\\ p_3&=\frac{x_3}{R^2}\Phi-\frac{\Delta}{R^2}\frac{x_1x_3}{x_2^2+x_3^2}+\frac{cx_2}{x_2^2+x^2}\end{aligned}\right\},$$

so that $dZ = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$

$$= \frac{\Phi}{R} dR + \frac{\Delta}{R^2} \left\{ dx_1 - \frac{x_1}{x_2^2 + x_3^2} (x_2 dx_2 + x_3 dx_3) \right\} + cd \tan^{-1} \frac{x_3}{x_2}.$$

Writing $$b^2 = a^2 + c^2,$$

the first term is $$d \left\{ b \tan^{-1} \frac{\Phi}{b} - \Phi \right\},$$

and the second is $$d \left\{ b \tan^{-1} \frac{bx_1}{\Delta} - c \tan^{-1} \frac{cx_1}{\Delta} \right\};$$

so that we have

$$Z - \delta = b \left( \tan^{-1} \frac{\Phi}{b} + \tan^{-1} \frac{bx_1}{\Delta} \right) - c \left( \tan^{-1} \frac{cx_1}{\Delta} - \tan^{-1} \frac{x_3}{x_2} \right) - \Phi.$$

19. Replacing now $x_1$, $x_2$, $x_3$ by $\xi$, $\eta$, $\zeta$; associating constants $b_1$, $c_1$, $a_1$ with $x$; $b_2$, $c_2$, $a_2$ with $y$; and $b_3$, $c_3$, $a_3$ with $z$; and denoting the corresponding values of $\Phi$ and $\Delta$ by $\Phi_1$, $\Delta_1$: $\Phi_2$, $\Delta_2$: $\Phi_3$, $\Delta_3$, respectively; we have

$$\left.\begin{aligned} x - a_1 &= b_1 \left( \tan^{-1} \frac{\Phi_1}{b_1} + \tan^{-1} \frac{b_1 \xi}{\Delta_1} \right) - c_1 \left( \tan^{-1} \frac{c_1 \xi}{\Delta_1} - \tan^{-1} \frac{\eta}{\zeta} \right) - \Phi_1 \\ y - a_2 &= b_2 \left( \tan^{-1} \frac{\Phi_2}{b_2} + \tan^{-1} \frac{b_2 \xi}{\Delta_2} \right) - c_2 \left( \tan^{-1} \frac{c_2 \xi}{\Delta_2} - \tan^{-1} \frac{\eta}{\zeta} \right) - \Phi_2 \\ z - a_3 &= b_3 \left( \tan^{-1} \frac{\Phi_3}{b_3} + \tan^{-1} \frac{b_3 \xi}{\Delta_3} \right) - c_3 \left( \tan^{-1} \frac{c_3 \xi}{\Delta_3} - \tan^{-1} \frac{\eta}{\zeta} \right) - \Phi_3 \end{aligned}\right\}.$$

Retaining $R^2$, so that $$R^2 = \xi^2 + \eta^2 + \zeta^2,$$

the values of the derivatives are

$$\left.\begin{aligned} x_\xi &= \frac{\xi}{R^2} \Phi_1 + \frac{\Delta_1}{R^2} \\ x_\eta &= \frac{\eta}{R^2} \Phi_1 - \frac{\Delta_1}{R^2} \frac{\xi\eta}{\eta^2 + \zeta^2} - \frac{c_1 \zeta}{\eta^2 + \zeta^2} \\ x_\zeta &= \frac{\zeta}{R^2} \Phi_1 - \frac{\Delta_1}{R^5} \frac{\xi\zeta}{\eta^2 + \zeta^2} + \frac{c_1 \eta}{\eta^2 + \zeta^2} \end{aligned}\right\},$$

so that
$$\frac{\Phi_1^2}{R^2}+\frac{\Delta_1^2}{R^2(\eta^2+\zeta^2)}+\frac{c_1^2}{\eta^2+\zeta^2}=\frac{\kappa^4}{R^4};$$

$$\left.\begin{aligned} y_\xi &= \frac{\xi}{R^2}\Phi_2+\frac{\Delta_2}{R^2} \\ y_\eta &= \frac{\eta}{R^2}\Phi_2-\frac{\Delta_2}{R^2}\frac{\xi\eta}{\eta^2+\zeta^2}-\frac{c_2\zeta}{\eta^2+\zeta^2} \\ y_\zeta &= \frac{\zeta}{R^2}\Phi_2-\frac{\Delta_2}{R^2}\frac{\xi\zeta}{\eta^2+\zeta^2}+\frac{c_2\eta}{\eta^2+\zeta^2} \end{aligned}\right\},$$

so that
$$\frac{\Phi_2^2}{R^2}+\frac{\Delta_2^2}{R^2(\eta^2+\zeta^2)}+\frac{c_2^2}{\eta^2+\zeta^2}=\frac{\kappa^4}{R^4};$$

and
$$\left.\begin{aligned} z_\xi &= \frac{\xi}{R^2}\Phi_3+\frac{\Delta_3}{R^2} \\ z_\eta &= \frac{\eta}{R^2}\Phi_3-\frac{\Delta_3}{R^2}\frac{\xi\eta}{\eta^2+\zeta^2}-\frac{c_3\zeta}{\eta^2+\zeta^2} \\ z_\zeta &= \frac{\zeta}{R^2}\Phi_3-\frac{\Delta_3}{R^2}\frac{\xi\zeta}{\eta^2+\zeta^2}+\frac{c_3\eta}{\eta^2+\zeta^2} \end{aligned}\right\},$$

so that
$$\frac{\Phi_3^2}{R^2}+\frac{\Delta_3^2}{R^2(\eta^2+\zeta^2)}+\frac{c_3^2}{\eta^2+\zeta^2}=\frac{\kappa^4}{R^4}.$$

But the equations
$$\left.\begin{aligned} x_\xi y_\xi+x_\eta y_\eta+x_\zeta y_\zeta &= 0 \\ x_\xi z_\xi+x_\eta z_\eta+x_\zeta y_\zeta &= 0 \\ y_\xi z_\xi+y_\eta z_\eta+y_\zeta z_\zeta &= 0 \end{aligned}\right\}$$

must be satisfied; and therefore, substituting the foregoing values and simplifying, we must have

$$\left.\begin{aligned} \frac{\Phi_1\Phi_2}{R^2}+\frac{\Delta_1\Delta_2}{R^2(\eta^2+\zeta^2)}+\frac{c_1c_2}{\eta^2+\zeta^2} &= 0 \\ \frac{\Phi_1\Phi_3}{R^2}+\frac{\Delta_1\Delta_3}{R^2(\eta^2+\zeta^2)}+\frac{c_1c_3}{\eta^2+\zeta^2} &= 0 \\ \frac{\Phi_2\Phi_3}{R^2}+\frac{\Delta_2\Delta_3}{R^2(\eta^2+\zeta^2)}+\frac{c_2c_3}{\eta^2+\zeta^2} &= 0 \end{aligned}\right\}$$

The six equations connecting the quantities $\Delta$, $\Phi$, $c$ show that the tableau

$$\frac{R^2}{\kappa^2}\begin{pmatrix} \frac{\Phi_1}{R}, & \frac{\Delta_1}{R(\eta^2+\zeta^2)^{\frac{1}{2}}}, & \frac{c_1}{(\eta^2+\zeta^2)^{\frac{1}{2}}} \\ \frac{\Phi_2}{R}, & \frac{\Delta_2}{R(\eta^2+\zeta^2)^{\frac{1}{2}}}, & \frac{c_2}{(\eta^2+\zeta^2)^{\frac{1}{2}}} \\ \frac{\Phi_2}{R}, & \frac{\Delta_3}{R(\eta^2+\zeta^2)^{\frac{1}{2}}}, & \frac{c_3}{(\eta^2+\zeta^2)^{\frac{1}{2}}} \end{pmatrix}$$

is a tableau of quantities satisfying the same equations as the direction cosines of three perpendicular straight lines; hence

$$\left.\begin{aligned} \frac{\Phi_1^2}{R^2}+\frac{\Phi_2^2}{R^2}+\frac{\Phi_3^2}{R^2} &= \frac{\kappa^4}{R^4} \\ \frac{1}{\eta^2+\zeta^2}\left(\frac{\Delta_1^2}{R^2}+\frac{\Delta_2^2}{R^2}+\frac{\Delta^2}{R^2}\right) &= \frac{\kappa^4}{R^4} \\ \frac{1}{\eta^2+\zeta^2}(c_1^2+c_2^2+c_3^2) &= \frac{\kappa^4}{R^4} \\ c_1\Delta_1+c_2\Delta_2+c_3\Delta_3 &= 0 \\ c_1\Phi_1+c_2\Phi_2+c_3\Phi_3 &= 0 \\ \Delta_1\Phi_1+\Delta_2\Phi_2+\Delta_3\Phi_3 &= 0 \end{aligned}\right\}.$$

20. Of these equations, the first three are consistent with one another in virtue of the two equations

$$\left.\begin{aligned} c_1^2+c_2^2+c_3^2 &= \frac{\kappa^4}{R^2}\left(1-\frac{x_1}{R^2}\right) \\ b_1^2+b_2^2+b_3^2 &= 2\frac{\kappa^4}{R^2} \end{aligned}\right\},$$

which imply $$a_1^2+a_2^2+a_3^2 = \frac{\kappa^4}{R^2}\left(1+\frac{x_1^2}{R^2}\right).$$

Accordingly, it is manifest that the integrals of the partial differential equations to be chosen are not complete integrals, for which the quantities $b$ and $c$ are arbitrary constants; they must either be instances of the general integrals or be singular integers. The only

other relations which need be retained from the set are

$$\left.\begin{aligned} c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3 &= 0 \\ c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 &= 0 \\ \Delta_1\Phi_1 + \Delta_2\Phi_2 + \Delta_3\Phi_3 &= 0 \end{aligned}\right\};$$

but it can be seen as follows that, owing to the forms of the functions, only the second need be retained.

Let
$$\left.\begin{aligned} c_s &= \frac{\kappa^2}{R^2}(x_2^2+x_3^2)^{\frac{1}{2}}\cos\theta_s \\ b_s &= \frac{\kappa^2}{R}\sin\alpha_s \end{aligned}\right\} \quad (s=1,\ 2,\ 3);$$

then we have
$$\left.\begin{aligned} \Phi_s &= \frac{\kappa^3}{R}\cos\alpha_s \\ \Delta_s &= \frac{\kappa^2}{R}(x_2^2+x_3^2)^{\frac{1}{2}}(\sin^2\alpha_s - \cos^2\theta_s)^{\frac{1}{2}} \end{aligned}\right\}.$$

The values obtained for $c_1^2+c_2^2+c_3^2$ and $b_1^2+b_2^2+b_3^2$ give

$$\cos^2\theta_1+\cos^2\theta_2+\cos^2\theta_3 = 1,$$
$$\cos^2\alpha_1+\cos^2\alpha_2+\cos^2\alpha_3 = 1;$$

and the (zero) value of $c_1\Phi_1+c_2\Phi_2+c_3\Phi_3$ gives

$$\cos\alpha_1\cos\theta_1+\cos\alpha_2\cos\theta_2+\cos\alpha_3\cos\theta_3 = 0.$$

Now we have

$$\sin^2\alpha_1\sin^2\theta_1 = (1-\cos^2\alpha_1)(1-\cos^2\theta_1)$$
$$= \cos^2\alpha_2(1-\cos^2\theta_1)+\cos^2\alpha_3(1-\cos^2\theta_1),$$

and

$$\cos^2\alpha_1\cos^2\theta_1 = \cos^2\alpha_2\cos^2\theta_2+\cos^2\alpha_3\cos^2\theta_3+2\cos\alpha_2\cos\alpha_3\cos\theta_2\cos\theta_3;$$

subtracting the latter from the former, we find

$$1-\cos^2\alpha_1-\cos^2\theta_1$$
$$= \cos^2\alpha_2\cos^2\theta_3+\cos^2\alpha_3\cos^2\theta_2-2\cos\alpha_2\cos\alpha_3\cos\theta_2\cos\theta_3,$$

that is,
$$(\sin^2\alpha_1-\cos^2\theta_1)^{\frac{1}{2}} = \cos\alpha_2\cos\theta_3-\cos\alpha_3\cos\theta_2,$$

and therefore
$$\Delta_1 = \frac{R^2}{\kappa^2}(c_2\Phi_3-c_3\Phi_2).$$

Similarly,
$$\Delta_2 = \frac{R^2}{\kappa^2}(c_3\Phi_1-c_1\Phi_3),$$

and
$$\Delta_3 = \frac{R^2}{\kappa^2}(c_1\Phi_2-c_2\Phi_1);$$

in other words, the equations

$$\left.\begin{aligned} c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3 &= 0 \\ \Phi_1\Delta_1 + \Phi_2\Delta_2 + \Phi_3\Delta_3 &= 0 \end{aligned}\right\}$$

are satisfied in virtue of the equations

$$\left.\begin{aligned} \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 &= 1 \\ \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 &= 1 \\ \cos\alpha_1\cos\theta_1 + \cos\alpha_2\cos\theta_2 + \cos\alpha_3\cos\theta_3 &= 0 \end{aligned}\right\},$$

which, with the definitions of $\alpha$ and $\theta$, are all the independent equations, and accordingly are all that need be retained so far as concerns the preceding system. It is manifest, on the one hand, that $\theta_1$, $\theta_2$, $\theta_3$ and $\alpha_1$, $\alpha_2$, $\alpha_3$ are the direction-angles of two perpendicular lines; and on the other hand, that the three equations may be taken in the form

$$\left.\begin{aligned} c_1^2 + c_2^2 + c_3^2 &= \frac{\kappa^4}{R^2}\left(1 - \frac{x_1^2}{R^2}\right) \\ \Phi_1^2 + \Phi_2^2 + \Phi_3^2 &= \frac{\kappa^4}{R^2} \\ c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 &= 0 \end{aligned}\right\}.$$

21. As our earlier investigation has provided the solution of the equations which is of the widest generality, it is desirable to inquire in what classes of the integrals of the preceding type that solution is included. Now, from the value of $Z$ given by

$$Z - \delta = b\left(\tan^{-1}\frac{\Phi}{b} + \tan^{-1}\frac{bx_1}{\Delta}\right) - c\left(\tan^{-1}\frac{cx_1}{\Delta} - \tan^{-1}\frac{x_3}{x_2}\right) - \Phi,$$

we find, after comparatively simple reductions,

$$\left.\begin{aligned} \frac{\partial Z}{\partial b} &= \tan^{-1}\frac{\Phi}{b} + \tan^{-1}\frac{bx_1}{\Delta} \\ \frac{\partial Z}{\partial c} &= -\tan^{-1}\frac{cx_1}{\Delta} + \tan^{-1}\frac{x_3}{x_2} \end{aligned}\right\}.$$

Denoting $\frac{\partial x}{\partial b_1}$ and $\frac{\partial x}{\partial c_1}$ by $B_1$ and $C_1$ respectively, and likewise for derivatives of $y$ and of $z$, we know (by Lagrange's theory) that the integrals of the differential equations for $x$, $y$, $z$ are derivable from

the complete integrals in the three cases by making the parameters $a_s$, $b_s$, $c_s$ $(s = 1, 2, 3)$ vary subject to the relations

$$\left.\begin{aligned} -\frac{\partial a_s}{\partial \xi} &= B_s \frac{\partial b_s}{\partial \xi} + C_s \frac{\partial c_s}{\partial \xi} \\ -\frac{\partial a_s}{\partial \eta} &= B_s \frac{\partial b_s}{\partial \eta} + C_s \frac{\partial c_s}{\partial \eta} \\ -\frac{\partial a_s}{\partial \zeta} &= B_s \frac{\partial b_s}{\partial \zeta} + C_s \frac{\partial c_s}{\partial \zeta} \end{aligned}\right\};$$

and these variations conserve the form and value of the first derivatives of the dependent variables.

The solution has been proved to be given by

$$\left.\begin{aligned} x - a_1 &= \kappa^2 \frac{l_1 \xi + m_1 \eta + n_1 \zeta}{\xi^2 + \eta^2 + \zeta^2} \\ y - a_2 &= \kappa^2 \frac{l_2 \xi + m_2 \eta + n_2 \zeta}{\xi^2 + \eta^2 + \zeta^2} \\ z - a_3 &= \kappa^2 \frac{l_3 \xi + m_3 \eta + n_3 \zeta}{\xi^2 + \eta^2 + \zeta^2} \end{aligned}\right\},$$

where the constants $l_1, m_1, n_1$; $l_2, m_2, n_2$; $l_3, m_3, n_3$ are the direction-cosines of three lines perpendicular to one another; and $a_1, a_2, a_3$ are constants. Assigning the conditions that the values of the derivatives of $x$, of $y$, and of $z$, with regard to $\xi, \eta, \zeta$ are the same in form and value as those in § 19, we find

$$\left.\begin{aligned} b_s^2 &= \frac{\kappa^4}{\xi^2 + \eta^2 + \zeta^2} \left\{ 1 - \frac{(l_s \xi + m_s \eta + n_s \zeta)^2}{\xi^2 + \eta^2 + \zeta^2} \right\} \\ c_s &= \frac{\kappa^2}{\xi^2 + \eta^2 + \xi^2} (n_s \eta - m_s \zeta) \\ \Phi_s &= -\kappa^2 \frac{l_s \xi + m_s \eta + n_s \zeta}{\xi^2 + \eta^2 + \zeta^2} \\ \Delta_s &= \kappa^2 \left\{ l_s - \frac{\xi (l_s \xi + m_s \eta + n_s \zeta)}{\xi^2 + \eta^2 + \zeta^2} \right\} \end{aligned}\right\}.$$

In the first place, these values verify all the relations of an algebraical type, given in § 20.

In the second place, the value of $B_s$, viz.,

$$\tan^{-1}\frac{\Phi_s}{b_s}+\tan^{-1}\frac{b_s\xi}{\Delta_s},$$

and the value of $C_s$, viz.,

$$\tan^{-1}\frac{\zeta}{\eta}-\tan^{-1}\frac{c_s\xi}{\Delta_s},$$

are both of them distinct from zero; and accordingly, if the three equations for the variations of the parameters coexist, we must have

$$J\left(\frac{a_s,\ b_s,\ c_s}{\xi,\ \eta,\ \zeta}\right)=0.$$

Manifestly there is no functional relation between $b_s$ and $c_s$ alone, in virtue of which this condition is satisfied; consequently, if the three equations coexist, it is in virtue of some functional relation

$$\Theta\,(a_s,\ b_s,\ c_s)=0.$$

The regular theory therefore indicates that the solutions obtained are instances of the general integrals of the equations

$$x_\xi^2+x_\eta^2+x_\zeta^2=y_\xi^2+y_\eta^2+y_\zeta^2=z_\xi^2+z_\eta^2+z_\zeta^2$$
$$=\frac{\kappa^4}{(\xi^2+\eta^2+\zeta^2)^2}.$$

It thus remains to consider the conditions for coexistence of the equations that give the variations of the parameters. These are

$$\frac{dB_s}{d\eta}\frac{\partial b_s}{\partial\xi}-\frac{dB_s}{d\xi}\frac{\partial b_s}{\partial\eta}=-\frac{dC_s}{d\eta}\frac{\partial c_s}{\partial\xi}+\frac{dC_s}{d\xi}\frac{\partial c_s}{\partial\eta},$$

where

$$\frac{dB_s}{d\eta}=\frac{\partial B_s}{\partial\eta}+\frac{\partial B_s}{\partial b_s}\frac{\partial b_s}{\partial\eta}+\frac{\partial B_s}{\partial c_s}\frac{\partial c_s}{\partial\eta},$$

$$\frac{dB_s}{d\xi}=\frac{\partial B_s}{\partial\xi}+\frac{\partial B_s}{\partial b_s}\frac{\partial b_s}{\partial\xi}+\frac{\partial B_s}{\partial c_s}\frac{\partial c_s}{\partial\xi},$$

$$\frac{dC_s}{d\eta}=\frac{\partial C_s}{\partial\eta}+\frac{\partial C_s}{\partial b_s}\frac{\partial b_s}{\partial\eta}+\frac{\partial C_s}{\partial c_s}\frac{\partial c_s}{\partial\eta},$$

$$\frac{dC_s}{d\xi}=\frac{\partial C_s}{\partial\xi}+\frac{\partial C_s}{\partial b_s}\frac{\partial b_s}{\partial\xi}+\frac{\partial C_s}{\partial c_s}\frac{\partial c_s}{\partial\xi}.$$

Substituting these values, the terms in $\frac{\partial B_s}{\partial b_s}$ on the left-hand side

disappear, and likewise those in $\frac{\partial C_s}{\partial c_s}$ on the right-hand side. The terms in $\frac{\partial B_s}{\partial c_s}$ on the left-hand side are

$$\frac{\partial b_s}{\partial \xi}\frac{\partial c_s}{\partial \eta} - \frac{\partial b_s}{\partial \eta}\frac{\partial c_s}{\partial \xi},$$

and those in $\frac{\partial C_s}{\partial b_s}$ on the right-hand side are

$$-\frac{\partial c_s}{\partial \xi}\frac{\partial b_s}{\partial \eta} + \frac{\partial c_s}{\partial \eta}\frac{\partial b_s}{\partial \xi};$$

moreover, $\frac{\partial C_s}{\partial b_s}$ and $\frac{\partial B_s}{\partial c_s}$ are equal, for each of them is equal to $\frac{\partial^2 Z}{\partial b_s \partial c_s}$; hence this set of terms disappears from the equation, which accordingly is

$$\left.\begin{aligned} \frac{\partial B_s}{\partial \eta}\frac{\partial b_s}{\partial} - \frac{\partial B_s}{\partial}\frac{\partial b_s}{\partial \eta} &= -\frac{\partial C_s}{\partial \eta}\frac{\partial c_s}{\partial \xi} + \frac{\partial C_s}{\partial \xi}\frac{\partial c_s}{\partial \eta}. \end{aligned}\right\}$$

Similarly, we have

$$\left.\begin{aligned} -\frac{\partial B_s}{\partial \zeta}\frac{\partial b^s}{\partial \xi} + \frac{\partial B_s}{\partial \xi}\frac{\partial b_s}{\partial \zeta} &= \frac{\partial C_s}{\partial \zeta}\frac{\partial c_s}{\partial \xi} - \frac{\partial C_s}{\partial \xi}\frac{\partial c_s}{\partial \zeta} \\ \frac{\partial B_s}{\partial \xi}\frac{\partial b_s}{\partial \eta} - \frac{\partial B_s}{\partial \eta}\frac{\partial b_s}{\partial \zeta} &= -\frac{\partial C_s}{\partial \zeta}\frac{\partial c_s}{\partial \eta} + \frac{\partial C_s}{\partial \eta}\frac{\partial c_s}{\partial \zeta} \end{aligned}\right\}.$$

Now we have

$$\left.\begin{aligned} \frac{\partial B_s}{\partial \xi} &= \frac{b_s}{\xi^2+\eta^2+\zeta^2}\left\{-\frac{\xi}{\Phi_s} + \frac{\eta^2+\zeta^2}{\Delta_s}\right\} \\ \frac{\partial B_s}{\partial \eta} &= \frac{-b_s\eta}{\xi^3+\eta^2+\zeta^2}\left\{\frac{1}{\Phi_s} + \frac{\xi}{\Delta_s}\right\} \\ \frac{\partial B_s}{\partial \zeta} &= \frac{-b_s\zeta}{\xi^2+\eta^2+\zeta^2}\left\{\frac{1}{\Phi_s} + \frac{\xi}{\Delta_s}\right\} \\ \frac{\partial C_s}{\partial \xi} &= \frac{-c_s}{\Delta_s} \\ \frac{\partial C_s}{\partial \eta} &= \frac{c_s}{\Delta_s}\frac{\xi\eta}{\eta^2+\zeta^2} - \frac{\zeta}{\eta^2+\zeta^2} \\ \frac{\partial C_s}{\partial \zeta} &= \frac{c_s}{\Delta_s}\frac{\xi\zeta}{\eta^2+\zeta^2} + \frac{\eta}{\eta^2+\zeta^2} \end{aligned}\right\}.$$

These values, together with the foregoing values of $b_s$ and $c_s$, satisfy the equations of condition; the verification is somewhat laborious, but not otherwise difficult. The proposition, that *the integrals retained are general integrals*, is accordingly established.

As for the common value of $a_s$ in virtue of which the three equations are satisfied, it is given by

$$a_s + b_s B_s + c_s C_s = \text{constant}$$

$$= a_s;$$

this result following at once from a comparison of the two sets of integrals.

22. The same remark applies of course to the more special form of solution when it is stripped of its non-essential elements of generality. That form is

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta} = \frac{\kappa^2}{\xi^2+\eta^2+\zeta^2};$$

so that, with the notation of § 18, we have

$$p_1 = \kappa^2\left(\frac{1}{R^2} - \frac{2x_1^2}{R^4}\right), \quad p_2 = \kappa^2 \frac{-2x_1x_2}{R^4}, \quad p_3 = \kappa^2 \frac{-2x_1x_3}{R^4}.$$

When these are equated to the general values, we find

$$c_1 = 0, \quad \Phi_1 = -\kappa^2 \frac{x_1}{R^2}, \quad \Delta_1 = \kappa^2\left(1 - \frac{x_1^2}{R^2}\right), \quad a_1^2 = \frac{\kappa^4}{R^2}\left(1 - \frac{x_1^2}{R^2}\right).$$

Similarly, from $$y = \kappa^2 \frac{x_2}{R^2},$$

we have

$$c_2 = -\kappa^2 \frac{x_3}{R^2}, \quad \Phi_2 = -\kappa^2 \frac{x_2}{R^2}, \quad \Delta_2 = -\kappa^2 \frac{x_1x_2}{R^2}, \quad a_2^2 = \kappa^4 \frac{x_1^2}{R^4};$$

and, from $$z = \kappa^2 \frac{x_3}{R^2},$$

we have

$$c_3 = \kappa^2 \frac{x_2}{R^2}, \quad \Phi_3 = -\kappa^2 \frac{x_3}{R^2}, \quad \Delta_3 = -\kappa^2 \frac{x_1x_3}{R^2}, \quad a_3^2 = \kappa^4 \frac{x_1}{R^4}.$$

Moreover, these values give

$$b_1^2 = a_1^2 + c_1^2 = \frac{\kappa^4}{R^2}\left(1 - \frac{x_1^2}{R^2}\right),$$

$$b_2^2 = a_2^2 + c_2^2 = \frac{\kappa^4}{R^2}\left(1 - \frac{x_2^2}{R^2}\right),$$

$$b_3^2 = a_3^2 + c_3^2 = \frac{\kappa^4}{R^2}\left(1 - \frac{x_3^2}{R^2}\right).$$

As regards the classes of integrals which these represent, we have

$$\left.\begin{aligned} B_1 &= \tan^{-1}\frac{\Phi_1}{b_1} + \tan^{-1}\frac{b_1\xi}{\Delta} = 0 \\ C_1 &= \text{a finite variable quantity} \end{aligned}\right\};$$

$$B_2 = \tfrac{1}{2}\pi, \quad C_2 = 0;$$

$$B_3 = \tfrac{1}{2}\pi, \quad C_3 = \tfrac{1}{2}\pi.$$

Hence, for the value of $x$, we have

$$a_1 = \text{constant}, \quad c_1 = 0, \quad B_1 = 0,$$

being an integral apparently intermediate in class between a complete integral and a singular integral; it really is a special form of the general integral.

For the value of $y$, we take

$$a_2 + \tfrac{1}{2}\pi b_2 = \text{constant}, \quad C_2 = 0;$$

this is a form of the general integral.

For the value of $z$, we take

$$a_3 + \tfrac{1}{2}\pi b_3 + \tfrac{1}{2}\pi c_3 = \text{constant};$$

again this is a form of the general integral. Thus the integrals, that satisfy all the conditions and correspond to the case of geometrical inversion, are general integra .

## *On the Continuous Group that is defined by any given Group of Finite Order.* *By* W. Burnside. Received January 10th, 1898. Read January 13th, 1898.

It has been shown by Herr Study that, for every transitive linear homogeneous group in $n$ variables with $n$ independent parameters, the parameters may be so chosen that the finite equations of the group are linear and homogeneous in the parameters as well as in the variables (*Leipziger Berichte*, 1889, p. 201). If $x_1, x_2, \ldots, x_n$ are the original variables, $x_1', x_2', \ldots, x_n'$ the transformed variables, and $y_1, y_2, \ldots, y_n$ the parameters thus chosen, the finite equations of the group are

$$x_s' = \sum_{i,k} a_{sik} x_i y_k \quad (s, i, k = 1, 2, \ldots, n).$$

The quantities $a_{sik}$ are absolute constants, which are connected by a number of relations necessary to ensure that the system of equations shall actually have the group property.

The object of the present paper is to call attention to a special determination of the constants $a_{sik}$, which is obtained by means of a group of finite order $n$; and to investigate some of the properties of the continuous group so defined. The determination in question may be stated thus:

If $S_1, S_2, \ldots, S_n$ are the operations of a group of finite order $n$, then $a_{sik}$ is equal to unity or zero, according as $S_k S_i^{-1} S_s$ is or is not the identical operation.

The matrix of the $n^2$ coefficients in any transformation thus defined has formed the subject of investigation by Herr G. Frobenius ("Über die Primfactoren der Gruppendeterminante," *Berliner Sitzungsberichte*, 1896, pp. 1343–1382); and from a result given by him (*loc. cit.*, p. 1346) it follows at once that such a system of transformations have the group property. The main subject of Herr Frobenius's memoir is, however, entirely different from that of the present paper, and he makes no reference to the continuous group which is connected with the matrix.

In the simplest cases, $n = 2$ and $n = 3$, the sets of transformations are

$$x_1' = x_1 y_1 + x_2 y_2, \qquad x_2' = x_1 y_2 + x_2 y_1;$$

and $$x_1' = x_1y_1+x_2y_2+x_3y_3, \quad x_2' = x_1y_3+x_2y_1+x_3y_2,$$
$$x_3' = x_1y_2+x_2y_3+x_3y_1.$$

It is easily seen, on trial, that these sets of equations have the group property; and it was, in fact, a consideration of these simple cases that suggested to me the form for a system of transformations of $n$ variables, which is given by any finite group of order $n$.

1. Let $S_1, S_2, \ldots, S_n$ denote the operations of a group of finite order $n$, $S_1$ being the identical operation. The multiplication table of the group may be written in the form—

$$\left.\begin{array}{cccccc} S_1 & S_2 & \ldots & S_k & \ldots & S_n \\ S_2^{-1}S_1 & S_1 & \ldots & S_2^{-1}S_k & \ldots & S_2^{-1}S_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_l^{-1}S_1 & S_l^{-1}S_2 & \ldots & S_l^{-1}S_k & \ldots & S_l^{-1}S_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_n^{-1}S_1 & S_n^{-1}S_2 & \ldots & S_n^{-1}S_k & \ldots & S_1 \end{array}\right\}, \qquad (1)$$

so that each place in the principal diagonal is occupied by the symbol of the identical operation. For each suffix $l$, the set of symbols

$$S_l^{-1}S_1, \quad S_l^{-1}S_2, \quad \ldots, \quad S_l^{-1}S_{l-1}, \quad S_l^{-1}S_{l+1}, \quad \ldots, \quad S_l^{-1}S_n$$

is identical with the set

$$S_2, \quad S_3, \quad \ldots, \quad S_n,$$

except as regards the sequence in which the symbols occur.

If we now regard $S_1, S_2, \ldots, S_n$ as $n$ continuously varying parameters capable of taking all real values, and if each compound symbol $S_i^{-1}S_k$ is replaced by the single symbol which is equivalent to it, when the above multiplication table (1) is explicitly written in terms of the original $n$ symbols, then the set of homogeneous linear transformations—

$$\left.\begin{array}{l} x_1' = \quad S_1x_1+ \quad S_2x_2+\ldots+ \quad S_kx_k+\ldots+ \quad S_nx_n \\ x_2' = S_2^{-1}S_1x_1+ \quad S_1x_2+\ldots+S_2^{-1}S_kx_k+\ldots+S_2^{-1}S_nx_n \\ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_l' = S_l^{-1}S_1x_1+S_l^{-1}S_2x_2+\ldots+S_l^{-1}S_kx_k+\ldots+S_l^{-1}S_nx_n \\ \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' = S_n^{-1}S_1x_1+S_n^{-1}S_2x_2+\ldots+S_n^{-1}S_kx_k+\ldots+ \quad S_1x_n \end{array}\right\} \qquad (2)$$

defines a finite continuous group.

To verify this statement let us consider the transformation $TT'$, where $T$ represents the transformation written and $T'$ is the corresponding transformation with accented coefficients. If in $TT'$

$$x'_q = \sum_{p=1}^{p=n} a_{q,p} x_p \quad (q = 1, 2, \ldots, n),$$

then $$a_{q,p} = S'^{-1}_q S_p + S'^{-1}_q S'_2 S^{-1}_2 S_p + \ldots + S'^{-1}_q S'_n S^{-1}_n S_p.$$

Now, if $$S^{-1}_k S_p = S_r,$$

then $$S'^{-1}_p S'_k = S'^{-1}_r.$$

Hence $$a_{p,p} = \sum_{1}^{n} S'^{-1}_r S_r,$$

and therefore the coefficients in the leading diagonal of $TT'$ are all the same.

Suppose now that $$S^{-1}_q S_p = S^{-1}_l S_k.$$

Then, the suffix $y$ being determined uniquely, when $l$, $q$, $x$ are given, by

$$S^{-1}_q S_x = S^{-1}_l S_y;$$

we have $$S^{-1}_p S_x = S^{-1}_k S_y,$$

or $$S^{-1}_x S_p = S^{-1}_y S_k.$$

Hence the relation $$S^{-1}_q S_p = S^{-1}_l S_k$$

implies that, corresponding to every term $S'^{-1}_q S'_x S^{-1}_x S_p$ in $a_{q,p}$, there is an equal term $S'^{-1}_l S'_y S^{-1}_y S_k$ in $a_{l,k}$, and conversely; or that

$$a_{q,p} = a_{l,k}.$$

In particular, for every value of the suffixes $p$ and $q$, there is a single suffix $r$ such that

$$S^{-1}_q S_p = S_r,$$

and corresponding to this we have the relation

$$a_{q,p} = a_{1,r}.$$

The coefficients in the transformation $TT'$ thus follow the same law as those in $T$ and $T'$, which is defined by the multiplication table of the finite group; and therefore the set of transformations form a group. Since its transformations contain a finite number $n$ of continuously varying parameters, the group is a finite continuous group.

2. The infinitesimal transformations of the group are those for which all the parameters except $S_1$ are infinitesimal, while $S_1$ differs from unity by an infinitesimal quantity. Since the group in its finite form (2) contains $n$ independent parameters, it will have $n$ linearly independent infinitesimal transformations; and, in the phraseology which is now generally adopted, it may be described as a continuous group of order $n$. A set of $n$ linearly independent infinitesimal operations is obtained from the finite form (2) of the group by taking, first,

$$S_2 = S_3 = \ldots = S_n = 0;$$

and then

$$S_1 = 1,$$

$$S_2 = S_3 = \ldots = S_{k-1} = S_{k+1} = \ldots = S_n = 0 \qquad (k = 2, 3, \ldots, n).$$

If the corresponding differential operators are denoted by

$$X_k \qquad (k = 1, 2, \ldots, n),$$

then

$$X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots + x_n \frac{\partial}{\partial x_n},$$

and

$$X_k = x_{k_1} \frac{\partial}{\partial x_1} + x_{k_2} \frac{\partial}{\partial x_2} + \ldots + x_{k_n} \frac{\partial}{\partial x_n} \quad (k = 2, 3, \ldots, n),$$

where $x_{k_p}$ denotes the coefficient of $S_k$ in the expression for $x_p'$ given by (2). In fact, if

$$S_1 = 1, \quad S_2 = S_3 = \ldots = S_{k-1} = S_{k+1} = \ldots = S_n = 0,$$

then

$$x_1' = x_1 + S_k x_k,$$
$$x_2' = x_2 + S_k x_{k_2},$$
$$\ldots \quad \ldots \quad \ldots$$
$$x_n' = x_n + S_k x_{k_n};$$

and, if $S_k$ be regarded as infinitesimal, the result of replacing $x_1, x_2, \ldots, x_n$ by $x_1', x_2', \ldots, x_n'$ in any function $f(x_1, x_2, \ldots, x_n)$ is

$$f(x_1 + S_k x_k, \; x_2 + S_k x_{k_2}, \; \ldots, \; x_n + S_k x_{k_n}),$$

or

$$(1 + S_k X_k) f(x_1, x_2, \ldots, x_n).$$

Since

$$x_p' = S_p^{-1} S_1 x_1 + S_p^{-1} S_2 x_2 + \ldots + S_p^{-1} S_q x_q + \ldots + S_p^{-1} S_n x_n,$$

it follows that

$$x_{k_p} = x_q,$$

if

$$S_p^{-1} S_q = S_k,$$

or if

$$S_p S_k = S_q.$$

This immediately suggests a more convenient form of notation than the suffix with a suffix that has been used provisionally. Thus we may write

$$X_k = x_{1k}\frac{\partial}{\partial x_1} + x_{2k}\frac{\partial}{\partial x_2} + \ldots + x_{nk}\frac{\partial}{\partial x_n},$$

on the understanding that the suffixes

$$1, 2, 3, \ldots, n$$

combine among themselves according to the multiplication table (1) of the original group of finite order, the suffix $k$ corresponding to the operation $S_k$.

3. The alternants* $\qquad (X_k X_l) \quad (k, l = 1, 2, \ldots, n),$

of the $n$ differential operators may now be formed by direct calculation. Thus the coefficient of $x_p$ in $X_l$ is $\dfrac{\partial}{\partial x_q}$, if

$$p = ql,$$

or

$$q = pl^{-1};$$

and therefore the only term with a differential coefficient of the first order arising from

$$x_{pk}\frac{\partial}{\partial x_p} X_l$$

is

$$x_{pk}\frac{\partial}{\partial x_{pl^{-1}}}.$$

Hence

$$(X_k X_l) = \sum_{p=1}^{p=n}\left(x_{pk}\frac{\partial}{\partial x_{pl^{-1}}} - x_{pl}\frac{\partial}{\partial x_{pk^{-1}}}\right).$$

If

$$pl^{-1} = t,$$

and

$$pk = tx,$$

then

$$x = t^{-1}pk = lk.$$

Hence

$$\sum_1^n x_{pk}\frac{\partial}{\partial x_{pl^{-1}}} = \sum_{t=1}^{t=n} x_{tlk}\frac{\partial}{\partial x_t} = X_{lk};$$

similarly,

$$\sum_1^n x_{pl}\frac{\partial}{\partial x_{pk^{-1}}} = X_{kl},$$

and therefore

$$(X_k X_l) = X_{lk} - X_{kl}.$$

* One of the referees pointed out to me that the use of the word "alternant," as an equivalent for the German "Klammerausdruck" and French "crochet," to denote the operation $(XY)$ or $XY - YX$, is now generally established.

4. The continuous group of order $n$ that has been co-ordinated to any given group of finite order $n$, whose operations are

$$S_1,\ S_2,\ \ldots,\ S_n,$$

may therefore be defined as generated by $n$ independent infinitesimal operations

$$X_1,\ X_2,\ \ldots,\ X_n,$$

which satisfy the relations

$$(X_k X_l) = X_{lk} - X_{kl} \quad (k, l = 1, 2, \ldots, n);$$

each double suffix $lk$ in these relations being replaced by the single suffix $m$, which is given uniquely by

$$S_l S_k = S_m.$$

That the relations giving the alternants in terms of the $n$ generating infinitesimal operations are self-consistent is a consequence of their having been derived from the finite form of the group. If, however, the definition just given is taken as the starting-point for the continuous group, it is necessary to show that the relations in question are actually self-consistent. The necessary conditions are that

$$(X_k X_l) + (X_l X_k) = 0$$

and

$$\{(X_k X_l) X_i)\} + \{(X_l X_i) X_k\} + \{(X_i X_k) X_l\} = 0$$

hold for all values of $k$, $l$, and $i$.

The first condition is obviously satisfied. Moreover,

$$\begin{aligned}\{(X_k X_l) X_i\} + \{(X_l X_i) X_k\} + \{(X_i X_k) X_l\} &= X_{ilk} - X_{lki} - X_{ikl} + X_{kli} \\ &\quad + X_{kil} - X_{ilk} - X_{kli} + X_{lik} \\ &\quad + X_{lki} - X_{kil} - X_{lik} + X_{ikl} \\ &= 0,\end{aligned}$$

so that the second condition is also satisfied, whatever $i$, $k$, and $l$ may be.

5. For a discussion of the properties of the continuous group, and especially for the determination of its composition, the definition of the group by means of its infinitesimal operations and the expressions for their alternants is more convenient than that which gives the operations of the group in finite form; it will therefore be used almost exclusively. The continuous group itself will be called $G$, the group of finite order $g$; and the suffixes will be used as though they not only correspond to, but actually are, the operations of $g$.

A first step towards determining the composition of $G$ is to find its self-conjugate infinitesimal operations—*i.e.*, those of its infinitesimal operations which are permutable with every operation of the group. Let

$$Y = \sum_1^n e_s X_s$$

be such an operation. Then

$$\begin{aligned}(X_s Y) = 0 &= \sum_1^n e_s (X_t X_s) \\ &= \sum_1^n e_s (X_{st} - X_{st}) \\ &= \sum_1^n (e_{pt^{-1}} - e_{t^{-1}p}) X_p.\end{aligned}$$

Hence, for every value of $p$ and $t$,

$$e_{pt^{-1}} - e_{t^{-1}p} = 0,$$

or

$$e_q - e_{t^{-1}qt} = 0,$$

for all values of $q$ and $t$.

Now every operation of $g$ which is conjugate to $q$, and no others, can be put in the form $t^{-1}qt$. Hence the necessary and sufficient conditions that $Y$ may be a self-conjugate operation are that those coefficients $e_s$, whose suffixes form a set of conjugate operations of $g$, shall be all equal. If, then, $g$ contains $r$ distinct sets of conjugate operations, $Y$ will contain $r$ arbitrary parameters; and $G$ contains $r$ linearly independent self-conjugate infinitesimal operations. These generate a continuous group of order $r$, whose operations are all permutable with each other. This group is obviously a self-conjugate sub-group of $G$; it will be called $H$.

6. Next, the derived group $G_1$ of $G$ may be considered. This is the self-conjugate sub-group which is generated by all the alternants $(X_k X_l)$.

If $q$ and $t^{-1}qt$ are any two conjugate operations of $g$, then $X_q - X_{t^{-1}qt}$ is the alternant of two of the generating operations of $G$; for

$$\begin{aligned}X_q - X_{t^{-1}qt} &= X_{qtt^{-1}} - X_{t^{-1}qt} \\ &= (X_{t^{-1}} X_{qt}).\end{aligned}$$

Hence the generating operations of $G_1$ consist of such of the operations

$$X_q - X_{t^{-1}qt}$$

as are linearly independent.

Suppose that $a_1, a_2, \ldots, a_s$ is a set of conjugate operations of $g$. The $s-1$ operations

$$X_{a_1}-X_{a_2},\ X_{a_1}-X_{a_3},\ \ldots,\ X_{a_1}-X_{a_s}$$

are linearly independent, and by a linear combination of them every operation

$$X_{a_p}-X_{a_q}\quad (p, q = 1, 2, \ldots, s)$$

can be represented.

Hence each set of conjugate operations of $g$ gives a set, one less in number, of linearly independent operations which must appear among the generating operations of $G_1$. If then $r$ still denotes the number of distinct sets of conjugate operations in $g$, $G_1$ is a continuous group of order $n-r$. Moreover, since the operation

$$X_{a_1}+X_{a_2}+\ldots+X_{a_s}$$

cannot be represented as a linear combination of

$$X_{a_1}-X_{a_2},\ X_{a_1}-X_{a_3},\ \ldots,\ X_{a_1}-X_{a_s},$$

the derived group $G_1$ has no operation except identity in common with the group $H$ of the self-conjugate operations. The self-conjugate sub-groups $G_1$ and $H$, which have no common operation except identity, are therefore such that every operation of one is permutable with every operation of the other, while every infinitesimal operation of $G$ can be formed by a linear combination of the infinitesimal operations of $G_1$ and $H$. Adopting a phrase from the theory of groups of finite order, $G$ may be spoken of as the *direct product* of $G_1$ and $H$.

7. This idea of the direct product of two continuous groups may at once be used to verify almost intuitively a property of $G_1$, the direct proof of which is not so obvious. From the definition of the derived group it follows that the derived group of the direct product of two continuous groups is the direct product of the derived groups of the two continuous groups considered separately. Hence the derived group of $G$ is the direct product of the derived groups of $G_1$ and $H$. Since the operations of $H$ are all permutable, the derived group of $H$ consists of the identical operation only; and therefore the derived group of $G$ is the same as the derived group of $G_1$; or, in other words, $G_1$ is its own derived group.

The derived group $G_1$ can only reduce to the identical operation if $r$ is equal to $n$; that is, if every operation of $g$ is self-conjugate, and

$g$ therefore Abelian. Now the necessary and sufficient conditions that a continuous group $H$ may be integrable is that the series of groups $H$, $H_1$, $H_2$, ..., each of which is the derived group of the preceding, shall terminate with the identical operation. If this is the case for the group $G$, then $G_1$ must reduce to the identical operation. Hence $G$ is integrable if, and only if, $g$ is Abelian.

8. When $g$ is not Abelian, it may be shown that $G_1$, the derived group of $G$, is semi-simple; *i.e.*, that it contains no self-conjugate integrable sub-group.

In a memoir with the title *Sur la Structure des Groupes de Transformations finis et continus* (Paris, 1894), M. Cartan has given (p. 109) a method by which the greatest self-conjugate integrable sub-group of any continuous group can be directly determined. The method may be stated thus. Let the characteristic equation of the group $H$, namely,

$$\begin{vmatrix} \Sigma e_i c_{i11}+\omega, & \Sigma e_i c_{i21}, & \ldots, & \Sigma e_i c_{in1} \\ \Sigma e_i c_{i12}, & \Sigma e_i c_{i22}+\omega, & \ldots, & \Sigma e_i c_{in2} \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma e_i c_{i1n}, & \Sigma e_i c_{i2n}, & \ldots, & \Sigma e_i c_{inn}+\omega \end{vmatrix} = 0,$$

be written at length in the form

$$\omega^n+\psi_1(e)\,\omega^{n-1}+\psi_2(e)\,\omega^{n-2}+\ldots=0.$$

Let $X_1$, $X_2$, ..., $X_n$ be the $n$ infinitesimal operations of the group, and suppose that all the infinitesimal transformations of the derived group $H_1$ can be expressed linearly in terms of the $s$ operations

$$a_{i1}X_1+a_{i2}X_2+\ldots+a_{in}X_n \qquad (i=1, 2, \ldots, s).$$

Then, if

$$e_1X_1+e_2X_2+\ldots+e_nX_n$$

is an infinitesimal operation of the greatest self-conjugate integrable sub-group $I$ of $H$, the relations

$$a_{i1}\frac{\partial\psi_2}{\partial e_1}+a_{i2}\frac{\partial\psi_2}{\partial e_2}+\ldots+a_{in}\frac{\partial\psi_2}{\partial e_n}=0 \quad (i=1, 2, \ldots, s)$$

are satisfied; and, conversely, every infinitesimal operation for which these relations are satisfied belongs to $I$.

The coefficients $\psi_1, \psi_2, \ldots$ in the characteristic equation may be calculated by direct expansion of the determinant. In particular*

$$\psi_1 = \sum_{i,k}^{1,2,\ldots,n} e_i c_{ikk},$$

$$\psi_1^2 - 2\psi_2 = \sum_{i,j,k,l}^{1,2,\ldots,n} e_i e_j c_{ikl} c_{jlk}.$$

9. For the group $G$ $\quad c_{ikk} = 0 \quad (i, k = 1, 2, \ldots, n)$;

and therefore $\quad \psi_1 = 0.$

For given values of $i, j, k$, the product $c_{ikl}c_{jlk}$ is different from zero only when one of the relations

$$l = ki \quad \text{or} \quad l = ik,$$

and one of the relations

$$l = kj^{-1} \quad \text{or} \quad l = j^{-1}k$$

are satisfied. Hence either

$$j^{-1} = i,$$

or

$$j^{-1} = kik^{-1},$$

or

$$j^{-1} = k^{-1}ik.$$

The term $e_ie_j$ can therefore only appear in $\psi_2$, if $j^{-1}$ is either conjugate to or identical with $i$.

Suppose first that $j = i$. Then the term $e_i^2$ will occur in $\psi_2$ only if $i$ is an operation of $g$ of order 2, or if $i$ is conjugate to its own inverse. Let $i$ be one of a set of $x$ conjugate operations of order 2 in $g$. For a given $k$, $c_{ikl}c_{ilk}$ is different from zero only when either

$$l = ki \quad \text{or} \quad l = ik,$$

and either

$$k = il \quad \text{or} \quad k = li,$$

are satisfied; $i$ and $k$ not being permutable. Moreover, when these conditions are satisfied, $c_{ikl}$ and $c_{ilk}$ are either both $+1$ or both $-1$. Now $g$ contains $\frac{n(x-1)}{x}$ operations which are not permutable with $i$. Hence the coefficient of $e_i^2$ in $-2\psi_2$ is $\frac{2n(x-1)}{x}$.

Suppose next that $i$ is conjugate to its own inverse, but that it is

* Cartan, *loc. cit.*, p. 25.

not an operation of order 2. For a given $k$, $c_{ikl}c_{ilk}$ is only different from zero if either

$$l = ki \quad \text{or} \quad l = ik,$$

and either

$$l = ki^{-1} \quad \text{or} \quad l = i^{-1}k,$$

are satisfied. Hence $k$ must be an operation which transforms $i$ into its inverse; and each such $k$ gives two admissible values of $l$. For each of these $c_{ikl}$ and $c_{ilk}$ have opposite signs. Moreover the number of operations which transform $i$ into its inverse is equal to the number of operations which are permutable with $i$. Hence, in this case, if $i$ is one of $y$ conjugate operations in $g$, the coefficient of $e_i^2$ in $-2\psi_2$ is $-\frac{2n}{y}$.

By precisely similar reasoning it may be shown that, if $j = i^{-1}$, $i$ not being an operation of order 2, then the coefficient of $e_ie_{i^{-1}}$ in $-2\psi_2$ is $\frac{4n\,(z-1)}{z}$, where $z$ is the number of operations in the conjugate set to which $i$ belongs; and that, if $j$ is conjugate to, but not identical with $i^{-1}$, then the coefficient of $e_ie_j$ in $-2\psi_2$ is $-\frac{4n}{z}$.

It follows from this calculation that the value of $\frac{1}{2n}\frac{\partial\psi_2}{\partial e_i}$ is in general

$$-\frac{z-1}{z}e_{i^{-1}} + \frac{1}{z}\Sigma' e_j,$$

where the summation is extended to all values of $j$ which are conjugate to $i^{-1}$, except $i^{-1}$ itself.

This may be written in the form

$$-e_{i^{-1}} + \frac{1}{z}\Sigma e_j,$$

where the summation is now extended to all values of $j$ which are conjugate to $i^{-1}$.

If, however, $i$ is an operation of $g$ which is conjugate to its own inverse, and at the same time is not of order 2, then

$$\frac{1}{2n}\frac{\partial\psi_2}{\partial e_i} = -e_{i^{-1}} + \frac{1}{z}\Sigma e_j + \frac{1}{z}e_i;$$

and, if $i$ is an operation of order 2, then

$$\frac{1}{2n}\frac{\partial\psi_2}{\partial e_i} = \frac{1}{z}\Sigma e_j - e_i.$$

Now the derived group of $G$ is generated by all infinitesimal operations of the form $X_i - X_{i'}$,

where $i$ and $i'$ are conjugate operations of $g$. Hence, for the infinitesimal operations of the greatest self-conjugate integrable sub-group, the $e$'s must satisfy the relations

$$\frac{\partial \psi_2}{\partial e_i} - \frac{\partial \psi_2}{\partial e_{i'}} = 0.$$

Unless $i$ is an operation which is not of order 2 and is conjugate to its own inverse, this gives

$$e_{i'-1} - e_{i-1} = 0.$$

In the exceptional case

$$e_{i'-1} - e_{i-1} = \frac{1}{z}(e_{i'} - e_i);$$

but corresponding to this there will be the relation

$$e_{i'} - e_i = \frac{1}{z}(e_{i'-1} - e_{i-1}),$$

derived from $$\frac{\partial \psi_2}{\partial e_{i-1}} - \frac{\partial \psi_2}{\partial e_{i'-1}} = 0;$$

and this gives $$e_{i'-1} - e_{i-1} = e_{i'} - e_i = 0.$$

Hence, finally, the necessary and sufficient condition that

$$e_1 X_1 + e_2 X_2 + \ldots + e_n X_n$$

should belong to the greatest self-conjugate integrable sub-group of $G$ is that $e_i = e_{i'}$,

if $i$ and $i'$ are conjugate operations of $g$. The greatest self-conjugate integrable sub-group of $G$ is therefore the sub-group $H$ generated by the self-conjugate operations. Hence $G_1$ contains no integrable self-conjugate sub-group of $G$. Now $G$ is the direct product of $G_1$ and $H$, while every operation of $H$ is permutable with every operation of $G$. If, then, $g_1$ is a self-conjugate sub-group of $G_1$, it is also a self-conjugate sub-group of the group generated by $G_1$ and $H$, that is, of $G$. It follows therefore that $G_1$ contains no integrable self-conjugate sub-group.

The continuous group $G$ therefore, which has been coordinated to every group $g$ of finite order $n$, is the direct product of a group of permutable operations of order $r$, and a semi-simple group of order $n-r$; $r$ being the number of distinct sets of conjugate operations in $g$.

10. In illustration of this result, let $g$ be a dihedral group of order $2p$, where $p$ is an odd prime. Such a group is defined by

$$a^2 = 1, \quad s^p = 1, \quad asa = s^{-1}.$$

There are $\frac{1}{2}(p+3)$ conjugate sets, viz.,

$$1\,;\ s,\ s^{-1}\,;\ s^2,\ s^{-2}\,;\ \ldots\,;\ s^{\frac{1}{2}(p-1)},\ s^{-\frac{1}{2}(p-1)}\,;$$

$$a,\ as,\ as^2,\ \ldots,\ as^{p-1}.$$

The semi-simple group $G_1$ is therefore a group of order $\frac{3}{2}(p-1)$, generated by the infinitesimal operations

$$X_{s^\alpha} - X_{s^{-\alpha}} \quad \left[\alpha = 1, 2, \ldots, \tfrac{1}{2}(p-1)\right],$$

$$X_a - X_{as^\beta} \quad (\beta = 1, 2, \ldots, p-1).$$

To avoid the use of suffixes, these may be represented symbolically by

$$s^\alpha - s^{-\alpha} \quad \text{and} \quad a - as^\beta.$$

If $G_1$ has a self-conjugate sub-group $I_1$, it is easy to see that it must contain infinitesimal operations in the form

$$\Sigma e_\alpha (s^\alpha - s^{-\alpha}) \quad \left[\alpha = 1, 2, \ldots, \tfrac{1}{2}(p-1)\right].$$

Now $$(a,\ s^\alpha - s^{-\alpha}) = 2(as^{-\alpha} - as^\alpha),$$

and $$(as^\beta,\ as^{-\alpha} - as^\alpha) = s^{\alpha+\beta} - s^{-\alpha-\beta} + s^{\alpha-\beta} - s^{-\alpha+\beta}.$$

Hence, since $I_1$ is also a self-conjugate sub-group of $G$, it must contain

$$\Sigma e_\alpha (s^{\alpha+\beta} - s^{-\alpha-\beta} + s^{\alpha-\beta} - s^{-\alpha+\beta}).$$

If $\beta$ is 1, this is

$$e_2(s - s^{-1}) + \sum_r (e_{r-1} + e_{r+1})(s^r - s^{-r}) + (e_{\frac{1}{2}(p-3)} - e_{\frac{1}{2}(p-1)})(s^{\frac{1}{2}(p-1)} - s^{-\frac{1}{2}(p-1)})$$
$$\left[r = 2, 3, \ldots, \tfrac{1}{2}(p-3)\right].$$

Suppose now that $I_1$ contains only one independent infinitesimal operation of the form assumed. Then

$$\frac{e_2}{e_1} = \frac{e_1 + e_2}{e_2} = \ldots = \frac{e_{\frac{1}{2}(p-5)} + e_{\frac{1}{2}(p-1)}}{e_{\frac{1}{2}(p-3)}} = \frac{e_{\frac{1}{2}(p-3)}}{e_{\frac{1}{2}(p-1)}} - 1 = \lambda\,;$$

so that $\lambda$ is determined by

$$-1 = \lambda - \cfrac{1}{\lambda - \cfrac{1}{\lambda - \ldots}} \text{ to } \tfrac{1}{2}(p-1)\ \lambda\text{'s},$$

and the $e$'s are then given by

$$e_1 = 1, \quad e_2 = \lambda, \quad e_{r+1} = \lambda e_r - e_{r-1}.$$

Conversely, if the $e$'s are thus chosen, the self-conjugate sub-group $I_1$ contains only one infinitesimal operation of the form assumed.

Now the roots of the equation for $\lambda$ are all different, for they are

$$\omega^r+\omega^{-r} \quad \left[r = 1, 2, \ldots, \tfrac{1}{2}(p-1)\right],$$

where $\omega$ is a $p^{\text{th}}$ root of unity.

There are therefore $\frac{1}{2}(p-1)$ infinitesimal operations of the form assumed, which are clearly independent. Each of these occurs in a self-conjugate sub-group of $G_1$ which contains none of the others. Suppose now that

$$I_1, I_2, \ldots, I_{\frac{1}{2}(p-1)}$$

are the smallest self-conjugate sub-groups of $G_1$, each of which contains one of the above $\frac{1}{2}(p-1)$ linearly independent infinitesimal operations. No two of these can have a common sub-group; for, if they had, it would be a self-conjugate sub-group of $G_1$ with no operation of the form

$$\Sigma f_a (s^a - s^{-a}),$$

and a consideration of the alternants makes it clear that no such sub-group can exist. Hence the order of none of them can exceed 3; and, since every continuous group of order 2 is integrable, the order of none of them can be less than 3.

Finally, therefore, the semi-simple group $G_1$ is in this case composed of $\frac{1}{2}(p-1)$ simple groups of order 3, no two of which contain a common operation, and each two of which are permutable.

11. The sub-group $H$, formed of the self-conjugate operations of $G$, being generated by $r$ independent infinitesimal operations, is a linear homogeneous group of order $r$ in $n$ variables. Its parameter group is therefore a group of permutable operations in $r$ variables which is completely defined by $g$. It may be easily verified that the sub-group $H$ is given in finite form by the transformations

$$x'_s = a_{s^{-1}1}x_1 + a_{s^{-1}2}x_2 + \ldots + a_{s^{-1}k}x_k + \ldots + a_{s^{-1}n}x_n \qquad (s = 1, 2, \ldots, n),$$

where the parameters are such that

$$a_i = a_j,$$

if $i$ and $j$ are conjugate operations in $g$.

If $a_s''$ $(s = 1, 2, \ldots, n)$ are the parameters of the operation of $G$ resulting from successive transformations with parameters $a_s$ and $a_s'$; then, § 1,

$$a_s'' = a_1' a_s + a_2' a_{2^{-1}s} + \ldots + a_k' a_{k^{-1}s} + \ldots + a_n' a_{n^{-1}s},$$

$$= a_s' a_1 + a_{s2^{-1}}' a_2 + \ldots + a_{sk^{-1}}' a_k + \ldots + a_{sn^{-1}}' a_n;$$

$$(s = 1, 2, \ldots, n).$$

If here $a_s$ be regarded as the original variables, $a_s''$ the transformed variables, and $a_s'$ the parameters, the equations define the so-called *parameter group* of $G$.

If in these equations it be assumed that

$$a_i = a_j,$$

and

$$a_i' = a_j',$$

where $i$ and $j$ are conjugate operations in $g$, then will

$$a_i'' = a_j'',$$

and the equations will define the parameter group of $H$.

Suppose, in fact, that $2, 3, \ldots, t$ is a set of conjugate operations in $g$; and that $s$, $\sigma$ are any two conjugate operations of $g$. Then $a_s''$ contains the set of terms

$$(a_{s2^{-1}}' + a_{s3^{-1}}' + \ldots + a_{st^{-1}}')\, a_2,$$

and $a_\sigma''$ the set

$$(a_{\sigma 2^{-1}}' + a_{\sigma 3^{-1}}' + \ldots + a_{\sigma t^{-1}}')\, a_2.$$

But, since $2, 3, \ldots, t$, and therefore also $2^{-1}, 3^{-1}, \ldots, t^{-1}$, is a set of conjugate operations, it follows that, if

$$r^{-1} s r = \sigma,$$

then the set

$$r^{-1}(s2^{-1}, s3^{-1}, \ldots, st^{-1})\, r$$

is identical except as regards sequence with the set

$$\sigma 2^{-1}, \sigma 3^{-1}, \ldots, \sigma t^{-1}.$$

Hence the written sets of terms in $a_s''$ and $a_\sigma''$ are identical, and therefore

$$a_s'' = a_\sigma''.$$

12. Since only $r$ of the $n$ parameters are independent, the form above written for the parameter group of $H$ is redundant. It may be reduced as follows.

Let $k_1, k_2, \ldots, k_m$ be the $k^{\text{th}}$ of the $r$ conjugate sets, and write

$$a_{k_1} = a_{k_2} = \ldots = a_{k_m} = a_k.$$

Then $$a_s'' = a_s' a_1 + \ldots + (a_{sk_1^{-1}} + a_{sk_2^{-1}}' + \ldots + a_{sk_m^{-1}}) a_k + \ldots.$$

Let the set of operations

$$sk_1^{-1},\ sk_2^{-1},\ \ldots,\ sk_m^{-1}$$

contain $\gamma_{s,\,k^{-1},\,t}$ belonging to the $t^{\text{th}}$ conjugate set. If $s$ is transformed by $r$ into $\sigma$, then each of the above set is transformed into one of the set

$$\sigma k_1^{-1},\ \sigma k_2^{-1},\ \ldots,\ \sigma k_m^{-1}$$

by $r$; and therefore $\gamma_{s,\,k^{-1},\,t}$ does not depend on the particular operation $s$, but only on the conjugate set to which $s$ belongs.* It is an integer, which is completely defined for each value of $s$, $k$, and $t$ from 1 to $r$ by the group $g$. In particular, if $t$ is unity, $\gamma$ must be zero or unity. In terms of this symbol the coefficient of $a_k$ in $a_s''$ may be written

$$\sum_{t=1}^{t=r} \gamma_{s,\,k^{-1},\,t}\, a',$$

and the parameter group is given by

$$a_s'' = \sum_{t,k}^{1,2,\ldots,r} \gamma_{s,\,k^{-1},\,t}\, a_t' a_k \quad (s = 1, 2, \ldots, r).$$

This parameter group is isomorphic with the group $H$; it is therefore a group of permutable operations.

13. A simple example in illustration of the last paragraph is given by the symmetric group of three symbols,

$$A^3 = 1, \quad B^2 = 1, \quad BAB = A^2.$$

There are three conjugate sets of operations, viz.,

$$1; \quad A,\ A^2; \quad B,\ BA,\ BA^2.$$

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* The symbol $\gamma_{s,\,k^{-1},\,t}$ differs only by a factor from the symbol $h_{\alpha\beta\gamma}$ used by Herr Frobenius in a memoir "Uber Grüppencharaktere" (*Berliner Sitzungsberichte*, 1896, p. 987). Herr Frobenius defines the symbol $h_{\alpha\beta\gamma}$ as the number of solutions of the equation $ABC = 1$, when for $A$, $B$, $C$ respectively are put each of the operations belonging to the $\alpha^{\text{th}}$, $\beta^{\text{th}}$, and $\gamma^{\text{th}}$ conjugate sets in a group of finite order. If $n_s$ is the number of operations in the $s^{\text{th}}$ set, and if the inverses of the operations of the $k^{\text{th}}$ and $t^{\text{th}}$ sets form the $k'^{\text{th}}$ and $t'^{\text{th}}$ sets, then

$$n_s \gamma_{s,\,k^{-1},\,t} = h_{s,\,k',\,t'}.$$

Taking suffixes 1, 2, 3 to denote these in the order written, the sets $2^{-1}$ and $3^{-1}$ coincide with 2 and 3. Then by the multiplication table,

$$\begin{aligned}
&\gamma_{1,1,1}=1, \quad \gamma_{2,1,1}=0, \quad \gamma_{3,1,1}=0,\\
&\gamma_{1,2,1}=0, \quad \gamma_{2,2,1}=1, \quad \gamma_{3,2,1}=0,\\
&\gamma_{1,3,1}=0, \quad \gamma_{2,3,1}=0, \quad \gamma_{3,3,1}=1,\\
&\gamma_{1,1,2}=0, \quad \gamma_{2,1,2}=1, \quad \gamma_{3,1,2}=0,\\
&\gamma_{1,2,2}=2, \quad \gamma_{2,2,2}=1, \quad \gamma_{3,2,2}=0,\\
&\gamma_{1,3,2}=0, \quad \gamma_{2,3,2}=0, \quad \gamma_{3,2,2}=2,\\
&\gamma_{1,1,3}=0, \quad \gamma_{2,1,3}=0, \quad \gamma_{3,1,3}=1,\\
&\gamma_{1,2,3}=0, \quad \gamma_{2,2,3}=0, \quad \gamma_{3,2,3}=2,\\
&\gamma_{1,3,3}=3, \quad \gamma_{2,3,3}=3, \quad \gamma_{3,3,3}=0.
\end{aligned}$$

Hence the parameter group of the sub-group of self-conjugate operations is expressible in the form

$$\begin{aligned}
\alpha'' &= \alpha'\alpha+2\beta'\beta+3\gamma'\gamma,\\
\beta'' &= \beta'\alpha+(\alpha'+\beta')\,\beta+3\gamma'\gamma,\\
\gamma'' &= \gamma'\alpha+2\gamma'\beta+(\alpha'+2\beta')\,\gamma.
\end{aligned}$$

It is easy to verify that all the operations of this group are, as they should be, permutable.

[*Note* (*February 3rd*, 1898).—It was suggested by one of the referees that it might be desirable to add the proof of the theorem due to M. Cartan quoted in § 8. This would, however, involve too long a quotation from M. Cartan's memoir. The theorem appears to me of so great importance in the discussion of the composition of a group that I have allowed §§ 8, 9 to remain unaltered. The result to which it is applied may however be arrived at by means of another theorem given by M. Cartan in the same memoir (p. 52) which admits of ready quotation.

The theorem is that, if $H$, a continuous group of order $n$, has a self-conjugate integrable sub-group, then the discriminant of $\psi_2(e)$ (§ 8) must be zero; and therefore that, if the discriminant of $\psi_2(e)$ is not zero, $H$ must be semi-simple.

M. Cartan's proof is as follows. Let $h$ be an integrable self-conjugate sub-group of $H$; and form the series of derived groups

of $h$. Each of these is a self-conjugate sub-group of $H$, and the last but one is certainly a group all of whose operations are permutable. Let

$$X_{m+1},\ X_{m+2},\ \ldots,\ X_n$$

be the independent infinitesimal operations of this self-conjugate sub-group of permutable operations, and take these for the last $n-m$ of the independent infinitesimal operations of $H$. When the characteristic equation of $H$ is formed for this set of independent infinitesimal operations, it may be at once verified that it is independent of

$$e_{m+1},\ e_{m+2},\ \ldots,\ e_n.$$

Hence $\psi_2(e)$ is a function of $e_1, e_2, \ldots, e_m$ only, and therefore, regarded as a function of $n$ symbols, its discriminant vanishes. If, therefore, the discriminant of $\psi_2(e)$ does not vanish, there can be no such self-conjugate sub-group as $h$.

From this result it may be shown at once that $G_1$ (§ 8) is a semi-simple group. From the result of § 9 the value of $\psi_2$ for $G$ may be written down. If in this the number of $e$'s is reduced to $n-r$, by taking

$$e_{i_1}+e_{i_2}+\ldots+e_{i_k}=0$$

for each set of suffixes $i_1, i_2, \ldots, i_k$ which form a conjugate set in $g$, the resulting value of $\psi_2$ is the coefficient of $\omega^{n-r-2}$ in the characteristic equation of $G_1$. If $i_1$ is an operation of $g$ which is not conjugate to its inverse, the terms in $\psi_2$ which contain suffixes conjugate to $i_1$ are

$$-\frac{1}{n}(e_{i_1}e_{i_1^{-1}}+\ldots+e_{i_{k-1}}e_{i_{k-1}^{-1}})-\frac{1}{n}(e_{i_1}+\ldots+e_{i_{k-1}})(e_{i_1^{-1}}+\ldots+e_{i_{k-1}^{-1}}),$$

and these can clearly be expressed as the sum of $2(k-1)$ linearly independent squares. A similar result is easily shown to hold for the set of terms in $\psi_3$ which arise from a suffix which is conjugate to its inverse; and $\psi_2$ is thus expressed as the sum of $(n-r)$ linearly independent squares.—W. B.]

## *On the general Theory of Stationary Motion in an Infinite System of Molecules.*

*By* S. H. Burbury, M.A., F.R.S. Received and read January 13th, 1898.

I begin this paper with a confession of error made in a former paper of mine "On Boltzmann's Law of the Equality of Mean Kinetic Energy." I obtained a correct result, and then drew a wrong inference. I considered, namely, the case of two sets of elastic spheres having masses $M$ and $m$. The velocities in $x$ of the $M$ spheres are $U_1$, $U_2$, &c., and of the $m$ spheres $u_1$, $u_2$, &c. Then I assumed the law of distribution of the velocities to be

$$\epsilon^{-hQ} dU_1 \dots du_1 \dots dw_n,$$

and $$Q = A\,(U_1^2 + U_2^2 + \&c.) + a\,(u_1^2 + u_2^2 + \&c.) + B\Sigma\Sigma UU' + b\Sigma\Sigma uu' + \beta\Sigma\Sigma Uu,$$

with a similar expression for the velocities in directions $y$ and $z$. And I proved that in stationary motion the relation between the coefficients $A$, $a$, $\beta$ must be

$$2Am - 2aM + \beta\,(m - M) = 0;$$

and therefore $$\frac{A}{a} \neq \frac{M}{m}.$$

That is correct. I inferred that

$$A\overline{U}^2 = a\overline{u}^2;$$

and therefore $$M\overline{U}^2 \neq m\overline{u}^2,$$

which was incorrect.

### *Plan of this Paper.*

Boltzmann, Watson, and other writers take for the foundation of their theory of gases a certain assumption, which may be expressed as follows. The chance of any molecule having velocities $u_1, \dots u_1 + du_1$, &c., is independent of the velocities which any other molecule may have, unless the two molecules are, or recently have been, within one another's sphere of action. I call this assumed state of things "Condition A."

If the chance of a group of $n$ molecules having velocities respectively $u_1, \ldots u_1+du_1, \ldots w_n \ldots w_n+dw_n$ be denoted by

$$\epsilon^{-hQ} du_1 \ldots dw_n,$$

then on this assumption $Q$ must contain squares only, in the form

$$Q = \Sigma m (u^2+v^2+w^2).$$

Condition A implies that, and that implies Condition A.

It is proved by what is known as the $H$ theorem that this is the only distribution which can exist consistently with the continued existence of Condition A.

It is not proved, but it is always assumed, that the state of things described by Condition A can and does continue to exist.

I proved in my last paper that, if the molecules be equal elastic spheres having finite diameter, the motion cannot be stationary unless $Q$ contains products as well as squares of the velocities; in other words, Condition A cannot continue to exist. I admit, of course, that, if it exists in fact—if, for instance, it could be maintained artificially—the $H$ theorem and all its consequences follow.

I propose to show in the present paper (see Arts. 1–10) that the assumption of Condition A is unnecessary, because all the theoretical results obtained by assuming it can be obtained by the more general assumption of Art. 2, namely,

$$Q = \Sigma a (u^2+v^2+w^2) + \Sigma\Sigma b (uu'+vv'+ww').$$

It is unnecessary (see Art. 11), because all the practical results obtained by assuming it, *e.g.*, for diffusion, viscosity, &c., depend on the calculation of mean free path, which for gases under ordinary condition would have very approximately the same value whether the coefficients $b$ exist or not.

Arts. 12–25.—I propose to justify the above general assumption by extending the method of my former paper, which deals only with the case where the molecules are equal elastic spheres, to other forms of molecule.

### *The Assumed Form of Molecule.*

1. Let the molecules be material particles, between any two of which acts a finite force. This force is assumed to be a continuous function of the distance $r$ between the two molecules, acting in the line $r$, becoming repulsive and infinite, either at some finite distance, or ultimately as $r$ is indefinitely diminished, and becoming

insensible when $r$ exceeds a certain very small distance, which may be called the *radius of action* of a molecule. This includes the case of elastic spheres as a limiting case. If it be true that a molecule has finite size, or that there is a distance within which two molecules cannot approach one another, that appears to be inconsistent with the ultimate force when the distance is indefinitely diminished being attractive.

*The Law of Distribution assumed.*

2. I assume the law of distribution of the coordinates and the velocities to be as follows, viz., the chance that the $n$ molecules in any infinite space shall at a given instant have coordinates

$$x_1, \dots x_1+dx_1, \dots z_n \dots z_n+dz_n,$$

and component velocities $u_1, \dots u_1+du_1$, &c., is

$$AC\epsilon^{-hQ}dx_1 \dots dz_n du_1 \dots dw_n,$$

in which

$$Q = a_1u_1^2+a_2u_2^2+\&c.+a_1v_1^2+\&c.$$
$$+b_{12}(u_1u_2+v_1v_2+w_1w_2)+b_{13}(u_1u_3+v_1v_3+w_1w_3)+\&c.$$

Here the $a$ coefficients are supposed to be independent of the position of the molecules and of the time, but every $b$, as $b_{pq}$, is a function of $r_{pq}$, the distance at the given instant between the molecules whose velocities are $u_p$, &c., and $u_q$, &c., and, moreover, is a continuous function of $r_{pq}$, which diminishes as $r_{pq}$ increases, and becomes evanescent when $r_{pq}$ exceeds a certain very small finite value.

Further, $C$ is that function of $h$, and of the coefficients $a$, $b$ which makes

$$\iiint_{-\infty}^{+\infty} \dots C\epsilon^{-hQ}du_1 \dots dw_n = 1.$$

That is,
$$C = \sqrt{D}\left(\frac{h}{\pi}\right)^{\frac{3}{2}n},$$

$3n$ being the number of the variables $u$, $v$, $w$, and $D$ the determinant of the coefficients $a$, $b$ set out in Art. 3. It is a function of the coordinates only as contained in the $b$s.

Further, $A$ is a constant or an explicit function of the coordinates. With these meanings of $A$, $C$, $Q$, let

$$f = AC\epsilon^{-hQ}.$$

In stationary motion

$$\left(\frac{dx_1}{dt}\frac{d}{dx_1}+\dots+\frac{dz_n}{dt}\frac{d}{dz_n}+\frac{du_1}{dt}\frac{d}{du_1}+\dots+\frac{dw_n}{dt}\frac{d}{dw_n}\right)f = 0.$$

But also in stationary motion $Q$ must be constant, or

$$\Sigma \frac{dQ}{du}\frac{du}{dt} = 0$$

and

$$\Sigma \frac{dQ}{db}\frac{db}{dt} = 0.$$

3. We now prove that on average, given $b_{12}$, &c.,

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} = \&c. = \frac{1}{h}.$$

For

$$u_1 \frac{dQ}{du_1} = u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c.).$$

Let us now find the mean value, given $u_1$, of

$$u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c.).$$

To that end we perform the integration

$$A \iiint \ldots du_2 \ldots dw_n \epsilon^{-hQ} u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c.), \qquad (a)$$

and divide the result by

$$A \iiint \ldots du_2 \ldots dw_n \epsilon^{-hQ}. \qquad (b)$$

As the result of the integration, the index becomes in both $(a)$ and $(b)$, $h\frac{D}{2D_{11}}u_1^2$, where $D$ is the determinant

$$D = \begin{vmatrix} 2a_1 & b_{12} & b_{13} & \ldots \\ b_{12} & 2a_2 & b_{23} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

and $D_{11}$ its first coaxial minor.

It follows that the mean value required is what

$$u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \&c.)$$

becomes when all the integrations are effected.

4. The integration can be effected thus :—Let

$$2a_1 u_1 + b_{12} u_2 + \&c. = P.$$

Then, before the integration begins, we see that the coefficient of $u_1$ in $P$ is twice the coefficient of $u_1^2$ in $Q$, but the coefficient of any other $u$, as $u_2$ in $P$, is equal to the coefficient of $u_1 u_2$ in $Q$. It will be found

that this relation holds after the integration according to $u_n$, and therefore after all the integrations. And so, finally, when $Q$ is reduced to $\frac{D}{2D_{11}}u_1^2$, $P$ must be $\frac{D}{D_{11}}u_1$, and

$$\overline{u_1 \frac{dQ}{du_1}} = \frac{D}{D_{11}}\overline{u_1^2},$$

which is the mean value sought.

For, to integrate according to $u_n$, we proceed as follows. The terms in $Q$ which contain $u_n$ are

$$a_n u_n^2 + (b_{1n}u_1 + b_{2n}u_2 + \&c.)\, u_n,$$

which may be put in the form

$$\left(u_n\sqrt{a_n} + \frac{b_{1n}u_1 + b_{2n}u_2 + \&c.}{2\sqrt{a_n}}\right)^2 - \frac{(b_{1n}u_1 + b_{2n}u_2 + \&c.)^2}{4a_n},$$

or, as we may write for brevity,

$$\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}}\right)^2 - \frac{\beta^2}{4a_n},$$

and $$P = 2a_1 + b_{12}u_2 + \&c. + b_{1n}u_n$$

$$= 2a_1 + b_{12}u_2 + \&c. + \frac{b_{1n}}{\sqrt{a_n}}\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}} - \frac{\beta}{2\sqrt{a_n}}\right).$$

Then we take $\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}}\right)$ for variable, and integrate for it between $+\infty$ and $-\infty$. The result is, omitting the numerical factors, that, in $Q$, all the terms involving $u_n$ disappear, and are replaced by

$$-\frac{(b_{1n}u_1 + b_{2n}u_2 + \&c.)^2}{4a_n},$$

and, in $P$, $b_{1n}u_n$ is replaced by

$$-b_{1n}\frac{b_{1n}u_1 + b_{2n}u_2 + \&c.}{2a_n}.$$

Expanding the new term in $Q$, we find that the new coefficient of $u_1u_q$ is

$$b_{1q} - \frac{b_{1n}b_{qn}}{2a_n},$$

which is the coefficient of $u_q$ in $P$. Also the new coefficient of $u_1^2$ in $Q$ is $a_1 - \frac{b_{1n}^2}{4a_n}$, and the new coefficient of $u_1$ in $P$ is $2a_1 - \frac{b_{1n}^2}{2a_n}$. The same relation then subsists after integration as before between the

respective coefficients in $Q$ and in $P$; and this holds for every subsequent integration. And ultimately, when all the integrations are effected to $u_2$ inclusive,

$$Q = \frac{D}{2D_{11}} u_1^2, \quad u_1 P = \frac{D}{D_{11}} u_1^2,$$

or
$$u_1 (\overline{2a_1 u_1 + b_{12} u_2} + \&c.) = \frac{D}{D_{11}} u_1^2.$$

We have then on average

$$\overline{u_1 \frac{dQ}{du_1}} = \frac{D}{D_{11}} \overline{u_1^2}, \quad \overline{u_2 \frac{dQ}{du_2}} = \frac{D}{D_{22}} \overline{u_2^2}, \ \&c.$$

5. This is true on average of all possible values of $u_1 \ldots u_n$, treating all the $b$'s as constants. That is, it is true for every separate set of values of $b_{12} \ldots b_{1n}$, &c. But it is also true that for each set of values of $b_{12} \ldots b_{1n}$, &c.,

$$\overline{u_1^2} = \frac{D_{11}}{hD}, \quad \overline{u_2^2} = \frac{D_{22}}{hD}, \ \&c.;$$

and therefore
$$\overline{u_1 \frac{dQ}{du_1}} = \frac{1}{h} = \overline{u_2 \frac{dQ}{du_2}} = \overline{u_3 \frac{dQ}{du_3}} = \&c.$$

This is the general proposition of which

$$m_1 \overline{u_1^2} = m_2 \overline{u_2^2} = \&c.,$$

with
$$Q = \Sigma m (u^2 + v^2 + w^2),$$

is a particular case.

6. But, again, in the system of an infinite number of molecules, not only is $Q$ constant, but the kinetic energy, $T = \frac{1}{2}\Sigma m u^2$, is also separately constant; and therefore we may write

$$Q = \mu T,$$

where $\mu$ is constant, and

$$\overline{Q} = \frac{3n}{2h}, \quad T = \frac{3n}{2\mu h}.$$

We have then
$$\Sigma \frac{dQ}{du} \frac{du}{dt} = 0,$$

and
$$\Sigma m u \frac{du}{dt} = 0,$$

and, if we write $\frac{dQ}{du_1} = \mu m_1 u_1, \quad \frac{dQ}{du_2} = \mu m_2 u_2,$ &c.,

and $$m_1 \overline{u_1^2} = m_2 \overline{u_2^2} = \text{\&c.},$$

we obtain consistent results.

*On Stationary Motion as affected by the b-Coefficients.*

7. Let us next consider the second condition of Art. 2 for stationary motion, namely,

$$\Sigma \frac{dQ}{db} \frac{db}{dt} = 0.$$

That asserts that, if we regard the system at time $t$, and again at time $t+dt$, there will in the infinite number of molecules be as many pairs of molecules at the second instant as at the first whose distances from each other are between $r$ and $r+dr$, and as many molecules at the second instant as at the first whose neighbours are distant from them respectively $r_1 \ldots r_1+dr_1$, $r_2 \ldots r_2+dr_2$, &c., and that the distribution of the velocities has generally the same relation to the distances at the second as at the first instant.

In my last paper I found a solution for the case of equal elastic spheres as follows. Let $f$ be a function of $r$, the distance from any point $P$, which satisfies the conditions

$$f = 1, \quad \text{if} \quad r < r_0,$$

$$f = \frac{r_0}{r}, \quad \text{if} \quad r > r_0,$$

or, instead of $f = \frac{r_0}{r}$, we may use any continuous function of $r$ for which $\frac{df}{dr}$ is negative. Then, at any point $P$, let

$$\xi = \frac{\Sigma fu}{\Sigma f}, \quad \eta = \frac{\Sigma fv}{\Sigma f}, \quad \zeta = \frac{\Sigma fw}{\Sigma f},$$

the summation including all the molecules of the system. When the molecules are material points between which no collisions occur, let $\overline{\xi^2}$ be the mean value of $\xi^2$. Then, when the molecules become

spheres of finite diameter $c$, stationary motion requires that $\overline{\xi^2}$ shall become

$$\overline{\xi^2}+\overline{\xi'^2}, \quad \text{and} \quad \overline{\xi'^2}=\frac{\kappa}{1-\kappa}\overline{\xi^2}.$$

Here $\rho$ is the number of molecules per unit volume, and

$$\kappa = \tfrac{2}{3}\pi c^3\rho.$$

Further, $\overline{\xi'^2}$ necessarily introduces into $Q$ the term

$$\Sigma\Sigma b\,(uu'+vv'+ww').$$

If the $b$'s are on the average of all molecules invariable with the time, then $\overline{\xi'^2}$ is invariable with the time, and *vice versa*. The constancy of $\overline{\xi'^2}$ involves the condition

$$\Sigma\frac{dQ}{db}\frac{db}{dt}=0.$$

Further, I found that, if

$$b=-\tfrac{2}{3}\frac{c^5}{r^5},$$

then the condition $\xi'^2=\kappa\xi^2$,

or, so long as $\kappa$ is very small,

$$\overline{\xi'^2}=\frac{\kappa}{1-\kappa}\overline{\xi^2}$$

is satisfied.

*Of the System under Conservative External Forces and of the Coefficient A.*

8. Let us now suppose our system to be placed in a field of conservative external forces, whose components at any point are $X$, $Y$, $Z$, and let us consider what modifications are required in the above results. We have

$$\frac{d}{dt}(AC\epsilon^{-hQ})=0,$$

in which $A$ is now to be regarded as an explicit function of the coordinates, $C$ as a function of the coordinates as they are involved in the $b$s. We have then

$$\frac{d}{dt}(AC\epsilon^{-hQ})=\Sigma\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}+X\frac{d}{du}+Y\frac{d}{dv}+Z\frac{d}{dw}\right)AC\epsilon^{-hQ}.$$

That gives
$$\left(\frac{dA}{dx_1}u_1 - hAX\frac{dQ}{du_1}\right)C\epsilon^{-hQ}$$
$$+\left(\frac{dA}{dx_2}u_2 - hAX\frac{dQ}{du_2}\right)C\epsilon^{-hQ}$$
$$+\&c.$$
$$+u_1\left\{\frac{dC}{db_{12}} - hC(u_1u_2+v_1v_2+w_1w_2)\right\}\frac{db_{12}}{dx_1}A\epsilon^{-hQ}$$
$$+u_2\left\{\frac{dC}{db_{12}} - hC(u_1u_2+v_1v_2+w_1w_2)\right\}\frac{db_{12}}{dx_2}A\epsilon^{-hQ}$$
$$+\&c. = 0.$$

The terms derived from differentiation of the $b$ coefficients vanish in pairs. For any $b$, as $b_{pq}$, appears twice only, namely, in $u_p\frac{db_{pq}}{dx_p}$, and in $u_q\frac{db_{pq}}{dx_q}$, and, since every $b$ is a function of $r$ only, we have generally

$$\frac{db}{dx_1} = \frac{db}{dr}\frac{dr}{dx_1} = \frac{x_2-x_1}{r}\frac{db}{dr},$$
$$\frac{db}{dx_2} = \frac{x_1-x_2}{r}\frac{db}{dr};$$

therefore the terms in $\frac{df}{dt}$ derived from differentiation of the $b$s according to the coordinates resolve themselves into pairs of the form

$$Ch(u_1-u_2)(u_1u_2+v_1v_2+w_1w_2)\frac{db}{dr}\frac{x_1-x_2}{r}$$

and
$$(u_1-u_2)\frac{db}{dr}\frac{x_1-x_2}{r},$$

all which pairs vanish, because there will be on average just as many cases in which any given velocity $u$ is at $x_1, y_1, z_1$ as at $x_2, y_2, z_2$.

We have, then, dividing by $AC$,

$$\frac{1}{AC}\frac{d}{dt}(AC\epsilon^{-hQ})$$
$$= \left\{\frac{d\log A}{dx_1}u_1 - hX(2a_1u_1+b_{12}u_2+\&c.)\right\}\epsilon^{-hQ}$$
$$+\left\{\frac{d\log A}{dx_2}u_2 - hX(2a_2u_2+b_{12}u_1+\&c.)\right\}\epsilon^{-hQ}$$
$$+\&c.$$
$$= 0$$

$$= \left(\frac{d\log A}{dx_1} - hm_1 X \frac{D}{D_{11} m_1}\right) u_1 \epsilon^{-hQ}$$

$$+ \left(\frac{d\log A}{dx_2} - hm_2 X \frac{D}{D_{22} m_2}\right) u_2 \epsilon^{-hQ} + \&c.$$

$$= \left\{\left(\frac{d\log A}{dx_1} + h\mu \frac{d\chi}{dx_1}\right) u_1 + \left(\frac{d\log A}{dx_2} + h\mu \frac{d\chi}{dx_2}\right) u_2 + \&c.\right\} \epsilon^{-hQ},$$

if $\chi$ be the potential of the external forces, and

$$\frac{1}{A}\frac{d}{dt}(AC\epsilon^{-hQ}) = 0,$$

if $$A = \epsilon^{-h\mu\chi}, \quad f = \epsilon^{-h\mu\chi} C\epsilon^{-hQ}.$$

Comparing the state of the system at two different levels of potential, we may take the average of $Q$, or $Q = \mu T$. Then the law becomes

$$C\epsilon^{-h\mu(\chi+T)},$$

agreeing with Boltzmann's result, with $h\mu$ for his $h$.

*Application to Intermolecular Forces.*

9. The method of Art. 8 would apply equally to the case where $\chi$ is the potential of intermolecular forces instead of external forces only. The following difficulty has, however, been raised:—

Unless in the case of very small density, where the encounters are binary, $\chi$ can be completely expressed only as a single function involving the coordinates of all the molecules. It cannot be treated as the sum of the potentials of distinct pairs or groups. It seems to me that the following method (which is substantially that used by Dr. Watson in his *Kinetic Theory of Gases*, 2nd edition, p. 70) enables us to remove the difficulty, if any.

Consider any molecule $m_1$, whose coordinates are $x_1$, $y_1$, $z_1$. Let $\chi_{12}$ be the potential at $x_1$, $y_1$, $z_1$ of the forces due to the molecule $m_2$ at $x_2$, $y_2$, $z_2$; and so on. Then $\chi_{12} = 0$ unless the two molecules are very near each other.

Let $$\chi_{12} + \chi_{13} + \&c. = \chi_1.$$

Then we may write $$\chi = \chi_1 + \chi',$$

where $\chi'$ is the potential of mutual action of all the molecules except $m_1$. Given $\chi_1$, $\chi'$ may have many different values according to the relative positions *inter se* of the other molecules. But on average it will be constant.

Then we obtain consistent results if we assume that the chance of $m_1$ being in a position relative to the other molecules in which its potential is $\chi_1$ is, *cæteris paribus*, proportional to

$$\epsilon^{-h\mu\chi_1}.$$

That assumption enables us to prove that our gas obeys the Second Law of Thermodynamics, which, in case of intermolecular forces, has never, I believe, been proved on any other assumption.

10. The proof that I have given of the proposition of Art. 8 is, with slight modifications, the same which Boltzmann gives in his *Vorlesungen* for the case in which $Q$ is a sum of squares only ; that is, in which Condition A is assumed. It is necessary to extend the proof to my case, where I have begun by denying that assumption, because Boltzmann has said (*Vorlesungen*, p. 20) that without making that assumption it is impossible to prove any theorem in the kinetic theory of gases. I submit that I have proved two, at least, of these theorems without making Boltzmann's assumption, and without making any assumption except that $Q$ is always positive.

11. Boltzmann has also calculated the rate of diffusion, &c., on his own assumption, such calculations being based on the determination of the "mean free path." Now, for gases under the ordinary conditions, to which Boltzmann's calculation of mean free path apply, my coefficients $b$ would be very small, and the mean free path determined from my formula would differ so little from the mean free path determined from his formula that the results arrived at for diffusion, &c., would agree with his, at least, within the limits of experimental error.

### *Elastic Spheres of Unequal Masses.*

12. I proved in my former paper the necessary existence of the $b$ coefficients in a system of equal elastic sphere molecules. Before extending the theory to any more general form of molecule, I worked out the corresponding problem for two sets of elastic spheres of unequal masses and diameters. As the process is laborious, I here give the results only. Let $m_1$, $m_2$ be the masses, $c_{11}$, $c_{22}$ the diameters, of the spheres of the two classes respectively. Also let

$$c_{12} = \frac{c_{11} + c_{22}}{2}.$$

Let $\rho_1$ be the number per unit of volume of the $m_1$s, $\rho_2$ of the $m_2$s. Let $\xi_1$, $\eta_1$, $\zeta_1$ be the functions $\xi$, $\eta$, $\zeta$ for the $m_1$s, $\xi_2$, $\eta_2$, $\zeta_2$ the same functions for $m_2$. If the spheres were all material points, so that no collisions should occur, the motion of the $m_1$ system would be independent of that of the $m_2$ system, so that

$$\overline{\xi_1\xi_2} = \overline{\eta_1\eta_2} = \overline{\zeta_1\zeta_2} = 0.$$

I then found that, as the condition for stationary motion, $\xi_1$ must become $\xi_1+\xi_1'$, &c., $\xi_2$ must become $\xi_2+\xi_2'$, &c., and, as in my former paper, $\xi_1'$, $\xi_2'$, &c., must satisfy the equations

$$\overline{\xi_1'^2} = \tfrac{2}{3}\pi c_{11}^3\rho_1\overline{\xi_1^2},$$

$$\overline{\xi_2'^2} = \tfrac{2}{3}\pi c_{22}^3\rho_2\overline{\xi_2^2}.$$

These result from collisions between $m_1$ and $m_1$, or $m_2$ and $m_2$; and, for collisions between $m_1$ and $m_2$, we have

$$\overline{\xi_1'\xi_2'} = \tfrac{2}{3}\pi c_{12}^3\frac{2m_1m_2}{m_1+m_2}\frac{\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}(\overline{\xi_1^2}+\overline{\xi_2^2}).$$

The two systems cease to be independent when collisions take place between $m_1$ and $m_2$, so that

$$\overline{\xi_1'\xi_2'} \neq 0.$$

The above expressions are accurate only on the assumption that squares and products of $\pi c_{11}^3\rho_1$, $\pi c_{22}^3\rho_2$, and $\pi c_{12}^3\overline{\rho_1+\rho_2}$ are to be neglected.

If $B$ denote the coefficient of $u_1u_1'$ in $Q$, $b$ that of $u_2u_2'$, and $\beta$ that of $u_1u_2$, then the values of $B$, $b$, $\beta$, which lead to the above results, are

$$B = -\tfrac{2}{3}\frac{c_{11}^5}{r^5},$$

$$b = -\tfrac{2}{3}\frac{c_{22}^5}{r^5}.$$

$$\beta = -\frac{4m_1m_2}{(m_1+m_2)^2}\tfrac{2}{3}\frac{c_{12}^5}{r^5}.$$

*Consideration of the Case in which the Molecules are Centres of Force.*

13. Next let us assume the molecules to be centres of force under the conditions described in Art. 1; and, first, let the forces be repulsive only.

Further, we will begin by assuming the density to be so small that the encounters are binary.

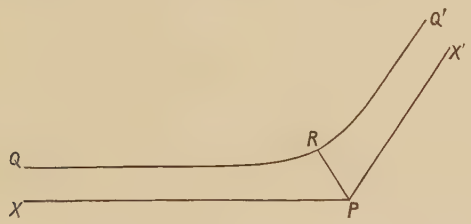
As in my former paper, the functions $\xi$, $\eta$, $\zeta$, and $M$ having the same meanings as before, we find, for a system of material points between which no mutual action takes place, the equation for stationary motion

$$\frac{dM}{dt} = 0 = uw\frac{d}{dt}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) - \frac{1}{2h}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2.$$

We shall find that, when the molecules exert mutual action on each other, the effect is to introduce into $\frac{dM}{dt}$ new terms, which are necessarily positive, and so to disturb the stationary character of the motion; and stationary motion can be restored only by increasing $\xi^2$, &c., in a certain ratio, which increase involves the introduction into $Q$ of the terms $b(uu' + vv' + ww')$. But we shall find that in this case the $b$'s are functions of density and temperature.

*Centres of Force with Binary Encounters.*

14. First, suppose a molecule $P$ fixed in space.




Let $PX$ be any line through $P$. Let $Q$ be a point whose distance from $PX$ is $\beta$, and is less than the radius of action, but whose distance from $P$ is greater than the radius of action. A molecule is projected from $Q$ with velocity $q$ in direction parallel to $XP$ towards $P$. An encounter then occurs, and, $P$ being fixed, the molecule $Q$ describes a symmetrical curved path, $QRQ'$, as in the figure, $R$ being the apse, and the asymptotes are parallel to $PX$ and $PX'$. Let the angle

$$RPX = \psi = \tfrac{1}{2}XPX',$$

and $\psi$ must be less than $\frac{\pi}{2}$.

The integral effect of the repulsive central force at $P$, or, as we

may call it, the effect of the encounter, is equivalent, so far as the change of direction of the moving particle is concerned, to an impulse on the moving particle in direction $PR$, whose value is

$$mq\sqrt{2}\sqrt{1+\cos 2\psi} = 2mq\cos\psi,$$

$m$ being the mass of the particle. Therefore, also, the resultant force on $P$ is $2mq\cos\psi$ in direction $RP$; and the resultant force on $P$ in direction perpendicular to $XP$ is $2mq\cos\psi\sin\psi$.

Let, now, $\rho f(q)\,dq$ be the number per unit of volume of molecules whose velocities parallel to $XP$ are between $q$ and $q+dq$. Then the number of such molecules which in unit of time cross an element of unit area at right angles to $XP$ at $Q$ is $\rho q f(q)\,dq$.

And so the mean force on $P$ in direction perpendicular to $XP$ is, for all the molecules which pass through the element of area with velocity $q \ldots q+dq$, in $XP$, equal to

$$F = \rho\, 2mq^2 f(q)\,dq\cos\psi\sin\psi,$$

which is necessarily positive.

15. As $q^2$ increases, $\beta$ remaining unchanged, $\psi$ increases, and therefore $\cos\psi$ diminishes. Also, as $\psi$ increases, if $\psi < \frac{\pi}{4}$, $\cos\psi\sin\psi$ increases; if $\psi > \frac{\pi}{4} < \frac{\pi}{2}$, $\cos\psi\sin\psi$ diminishes. Now, in order that $\psi$ may be less than $\frac{\pi}{4}$, $\beta$ must be less than a certain value $\beta_0$, known if the law of force is known, and $\beta_0$ diminishes as $q^2$ increases. It follows that, for sufficiently great values of $\overline{q^2}$, $\cos\psi\sin\psi$ diminishes on average as $\overline{q^2}$ increases; and therefore $F$ increases less rapidly than $\overline{q^2}$. If we write

$$F = \kappa_1\overline{q^2},$$

$\kappa_1$ is a quantity which diminishes as $\overline{q^2}$, *i.e.*, the temperature, increases.

16. If the molecules are in stationary motion with $\frac{3}{2h}$ for mean square velocity, we must write $\frac{1}{2h}$ for $\frac{m\overline{q^2}}{2}$ in the above expressions (I here use $h$ instead of $\mu h$ above).

If the molecule $P$, instead of being fixed, is itself moving relatively to the system of molecules with velocity $u$ in direction $PX$, we must write $\frac{1}{2h} + \frac{1}{2}mu^2$ for $\frac{m\overline{q^2}}{2}$ in the above expressions.

17. Now let us suppose that at $P$ taken for origin $\xi = \xi_0$, and at any point whose $z$ coordinate is $\partial z$

$$\xi = \xi_0 + \frac{d\xi}{dz}\partial z.$$

That means that the molecules at positive $\partial z$, say at $P'$, have a mean velocity $\frac{d\xi}{dz}\partial z$ in $x$ relative to the molecules for which $\partial z$ is negative, say at $P''$ (not shown in figure).

Therefore the mean force on $P$, moving in direction $PX$ with velocity $u$, due to the molecules at $P'$ is

$$\frac{\kappa_1}{2h} + \kappa_1 \frac{m}{2}\left(u + \frac{d\xi}{dz}\partial z\right)^2,$$

and the mean force on $P$ due to the molecules at $P''$ is

$$\frac{\kappa_1}{2h} + \kappa_1 \frac{m}{2}\left(u - \frac{d\xi}{dz}\partial z\right)^2$$

in the opposite direction. The resultant force on $P$ in direction $z$ in plane of the motion is proportional to

$$2\kappa_1 mu \frac{d\xi}{dz}\partial z.$$

If the plane of the motion make angle $\phi$ with that of $xz$, we multiply the last expression by $\cos\phi$. Also, if $XQ = \beta$, we have

$$\partial z = \beta \cos\phi,$$

and the resultant force due to all encounters, given $\beta$, is

$$2\kappa_1 mu\beta \frac{d\xi}{dz}\overline{\cos^2\phi} = \kappa_1 mu\beta \frac{d\xi}{dz}.$$

Since $\kappa_1$ is an unknown function of $\beta$, we will now write $\kappa$ for $\kappa_1\beta$.

18. We have, then, if $v$, $w$ be the velocities of $P$ in $y$ and $z$ respectively,

$$\frac{dw}{dt} = \kappa mu \frac{d\xi}{dz},$$

whence

$$u\frac{dw}{dt} = \kappa mu^2 \frac{d\xi}{dz}.$$

Similarly,

$$w\frac{du}{dt} = \kappa mw^2 \frac{d\zeta}{dx};$$

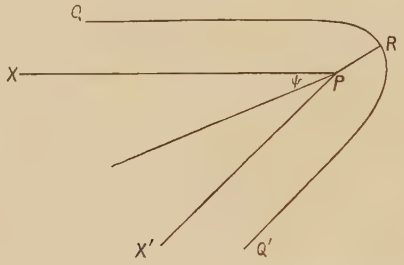
and therefore

$$\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\frac{d}{dt}(uw) = \kappa m \frac{1}{2h}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2.$$

This expression is of the same form as I obtained in the case of elastic spheres, except that $\kappa$ is now an unknown function depending on the law of force, and is a function of temperature as well as density, and generally diminishes as temperature increases.

*Case of Attractive Forces considered.*

19. If, for some values of $r$, the distance between two molecules, the force be attractive instead of repulsive, these conclusions require modification as follows: Let us use the notation of Art. 14. The force being attractive, it may be that, for some values of $q$, the molecules $P$ and $Q$, if alone in space, would form a permanent union, revolving round one another in a circular or other orbit. But, if $q$ be great enough, the $Q$ molecule, $P$ being fixed, will describe a path as shown in the figure.




All that has been said concerning the path described in Art. 14 applies equally to this, except that the force on $P$ will now be in direction $PR$, and its resolved part, at right angles to $PX$, will be $-2mq\cos\psi\sin\psi$, instead of $2mq\cos\psi\sin\psi$. Consequently we shall have in this case, so far as regards the result of these encounters, $-\xi'^2$ instead of $\xi'^2$, and the $b$ coefficients would, were encounters of this kind alone concerned, be positive instead of negative.

We may assume, however, that, for sufficiently small values of $r$, the mutual force is repulsive, and, ultimately, as $r$ diminishes indefinitely, infinite. It follows that, as the density increases and the temperature increases, the repulsive forces must become predominant. And the general effect will be of the same form as if only repulsive forces existed; the attractive forces, if they exist, only serving to make the increase in absolute value of the $b$ coefficients with increasing density less than it otherwise would be.

*Another Method of Proving the Terms which appear in* $\frac{dM}{dt}$ *as the Effect of Mutual Action, to be Positive.*

20. I will now treat the case in the same way as in my former paper I treated the case of elastic spheres in collision, in order to show that the new terms in $\frac{dM}{dt}$ are positive.

Let $V$ be the half relative velocity of two molecules immediately before their encounter, $\lambda$, $\mu$, $\nu$ its direction cosines before the encounter commences, $\lambda'$, $\mu'$, $\nu'$ after it has ceased. Let

$$\lambda V = V_x, \quad \mu V = V_y, \quad \nu V = V_z.$$

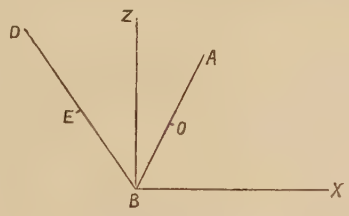
Let $c$ be the distance of their centres apart when the encounter commences. Then the number of encounters per unit of volume and time is $2\pi c^2 \rho V$. Hence, for the change with the time of $V_x V_z$, due to encounters with given $V$, we have

$$2\pi c^2 \rho V^3 (\overline{\lambda'\nu'} - \overline{\lambda\nu}),$$

the bar denoting mean values. We have now to find the mean values of $\overline{\lambda'\nu'}$ and $\overline{\lambda\nu}$, given $V$. At the beginning of the encounter, let the centres of the two encountering molecules be $A$, $B$. Let $BD$ be the relative velocity. Or, if $DE = BE$, one molecule has velocity $DE$, and the other $BE$, in addition to the velocity of their common centre of inertia, whatever that velocity may be. Let the angle

$$DBA = \theta;$$

let $O$ be the middle point of $AB$. Let $BX$, $BZ$ be the directions of




the axes of $x$ and $z$, and let the angle between the planes $DBA$ and $DBZ$ be $\phi$. When the plane $DBA$, turning round $DB$, contains the axis of $z$, $\phi = 0$ for that position of $BA$ which makes the least angle with the axis of positive $z$.

Similarly, let the angle between the planes $DBA$ and $DBX$ be $\phi'$. Then, if $x_0$, $z_0$ be the $x$ and $z$ coordinates of $A$ referred to $O$ as origin,

and if the direction $DE$ be that of $\lambda$, $\mu$, $\nu$,

$$z_0 = -\nu\frac{c}{2}\cos\theta + \frac{c}{2}\sqrt{1-\nu^2}\sin\theta\cos\phi,$$

$$x_0 = -\lambda\frac{c}{2}\cos\theta + \frac{c}{2}\sqrt{1-\lambda^2}\sin\theta\cos\phi'.$$

Now let $\xi_0$, $\zeta_0$ be the values of $\xi$, $\zeta$ at $O$. Then at $A$

$$\xi - \xi_0 = \frac{c}{2}\left(-\nu\cos\theta + \sqrt{1-\nu^2}\sin\theta\cos\phi\right)\frac{d\xi}{dz},$$

$$\zeta - \zeta_0 = \frac{c}{2}\left(-\lambda\cos\theta + \sqrt{1-\lambda^2}\sin\theta\cos\phi'\right)\frac{d\zeta}{dx}.$$

Then we find, as in my former paper, that the mean value of $V_xV_z$, for all encounters with given $V$, is

$$\overline{V_xV_z} = \overline{\lambda\nu}\,V^2 = -\frac{2hc}{3}V^3\frac{1}{3.5}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right).$$

21. We have next to calculate $\overline{\lambda'\mu'}$. When the plane $DBA$, turning round $DB$, contains the axis of $z$, let $A_z$, $A'_z$ be the two positions in which that plane is cut by the circle which $A$ describes. Let $A_z$ be nearer to the positive axis of $z$ than $A'_z$. Then, for $A_z$, $\phi = 0$; for $A'_z$, $\phi = \pi$. Similarly, when the plane $DBA$, turning round $DB$, contains the axis of $x$, $A_x$ and $A'_x$ are the two positions in which that plane is cut by the circle which $A$ describes. And, if $A_x$ is nearer to the positive axis of $x$ than $A'_x$, for $A_x$ $\phi' = 0$; for $A'_x$ $\phi' = \pi$.

Let $\epsilon$ be the angle between the plane of $DB$ and $z$ and the plane of $DB$ and $x$. Then

$$\cos\epsilon = \frac{-\lambda\nu}{\sqrt{1-\lambda^2}\sqrt{1-\nu^2}}, \quad \sin\epsilon = \frac{\mu}{\sqrt{1-\lambda^2}\sqrt{1-\nu^2}}.$$

In order to find $\lambda'$ and $\nu'$ we must now use $2\psi$ of Art. 14 instead of $2\theta$, which we used in the case of elastic spheres. We have the two symmetrical systems

$$\left.\begin{aligned}\nu' &= -\nu\cos 2\psi + \sqrt{1-\nu^2}\sin 2\psi\cos\phi\\ \lambda' &= -\lambda\cos 2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\end{aligned}\right\},$$

$$\left.\begin{aligned}\nu' &= -\nu\cos 2\psi - \frac{\lambda\nu}{\sqrt{1-\lambda^2}}\sin 2\psi\cos\phi'\\ \lambda' &= -\lambda\cos 2\psi + \sqrt{1-\lambda^2}\sin 2\psi\cos\phi'\end{aligned}\right\}.$$

The term involving $\sin\epsilon$ will disappear because $\sin\epsilon$ contains the factor $\mu$. We find then that, for given $V$,

$$V^2\overline{\lambda'\nu'} = V^2\int_0^{\frac{1}{2}\pi} d\theta \int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,(1+2h\lambda V\overline{\xi-\xi_0}+2h\nu V\overline{\zeta-\zeta_0})\,\lambda'\nu';$$

but, with the above values of $\lambda'\nu'$,

$$\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\lambda'\nu' = \lambda\nu\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\sin\theta\cos\theta(\cos^2 2\psi-\tfrac{1}{2}\sin^2 2\psi),$$

$\psi$ being a function of $\theta$ and $V$ known if the law of force is known. This is zero on average, because

$$\overline{\lambda\nu} = 0.$$

Also
$$\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,\lambda\overline{\xi-\xi_0}\lambda'\nu'$$

$$= \frac{d\xi}{dz}\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,\frac{c}{2}\lambda\,(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi)$$
$$\times(-\nu\cos 2\psi+\sqrt{1-\nu^2}\sin 2\psi\cos\phi)$$
$$\times\left(-\lambda\cos 2\psi-\frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right)$$

$$= \frac{d\xi}{dz}\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\lambda^2\nu^2\,(-\cos\theta\cos^2 2\psi+\tfrac{1}{2}\sin\theta\sin 2\psi\cos 2\psi$$
$$+\tfrac{1}{2}\cos\theta\sin^2 2\psi)$$
$$-\frac{d\xi}{dz}\lambda^2(1-\nu^2)\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\sin\theta\,(\tfrac{1}{2}\sin 2\psi\cos 2\psi)$$

$$= \frac{d\xi}{dz}\lambda^2\nu^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi+\sin\theta\sin 2\psi\cos 2\psi)$$
$$-\frac{d\xi}{dz}\lambda^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\sin\theta\,\tfrac{1}{2}\sin 2\psi\cos 2\psi,$$

and, since on average
$$\lambda^2 = 5\lambda^2\nu^2,$$

this becomes

$$\frac{d\xi}{dz}\lambda^2\nu^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi-\tfrac{3}{2}\sin\theta\sin 2\psi\cos 2\psi).$$

Hence, by symmetry between $x$ and $z$,

$$V^2(\overline{\lambda'\nu'}-\overline{\lambda\nu}) = 2hcV^3\lambda^2\nu^2\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)$$
$$\times\left\{\tfrac{2}{3}+\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi-\tfrac{3}{2}\sin\theta\sin 2\psi\cos 2\psi)\right\}.$$

The last factor is equal to

$$1-\tfrac{3}{2}\int_0^{\frac{1}{2}\pi} 2\sin\theta\cos^2\theta\cos^2 2\psi\,d\theta-\tfrac{3}{2}\int_0^{\frac{1}{2}\pi} 2\sin^2\theta\cos\theta\sin 2\psi\cos 2\psi\,d\theta.$$

22. Now $\psi$ cannot be less than $\theta$, and cannot be greater than $\frac{\pi}{2}$. Therefore the greatest possible negative value which the two last terms can have is when

$$\psi=\frac{\pi}{2},$$

and that greatest possible negative value is $-1$. And this is a limiting case that can never occur, because

$$\psi=\frac{\pi}{2},\quad 2\psi=\pi$$

expresses the fact that neither molecule suffers from the encounter any deflexion. It follows that on the average of all encounters which do produce deflexion $V^2(\overline{\lambda'\nu'}-\overline{\lambda\nu})$ is positive, and proportional to $\frac{d\xi}{dz}+\frac{d\zeta}{dx}$; and therefore $\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)\frac{\partial}{\partial t}\overline{V_x V_z}$ is $\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2$ multiplied by a necessarily positive coefficient.

23. Next we must consider in the same way the term

$$uw\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right).$$

We have now to take into account not only the variation of $\lambda, \mu, \nu$ due to the encounter, but also the variation of position of the molecules, that is, of $f$. It is sufficient to deal with $\frac{d}{dt}\frac{d\xi}{dz}$. Suppose, then, an encounter to take place between two molecules, their centre of inertia being at the origin. About the origin describe a sphere of radius $a$, and let $PP'$ be an ordinate parallel to $z$, $P$ and $P'$ being on the surface of the sphere. Let $\xi$, $\xi'$ be the values of $\xi$ at $P$ and $P'$. If $\Omega$ be the volume of that sphere,

$$\Omega=\iint PP'\,dx\,dy,$$

and, if $\frac{d\xi}{dz}$ be the mean value of that function throughout the space

enclosed by the sphere,

$$\frac{d\xi}{dz} = \iint dx\,dy\,(\xi - \xi') \div \Omega;$$

and therefore

$$\frac{d}{dt}\frac{d\xi}{dz} = \iint dx\,dy\left(\frac{d\xi}{dt} - \frac{d\xi'}{dt}\right) \div \Omega.$$

24. Let $\lambda$, $\mu$, $\nu$ be the direction cosines of the relative velocity before the encounter; $\lambda'$, $\mu'$, $\nu'$ the same after the encounter has ceased. Let $f$ be the value at the origin of that function referred to $P$ as centre; $f_1$, $f_2$ its values at the centres of the two molecules respectively. At the end of the encounter let $f_1$, $f_2$ become $f_1'$, $f_2'$. Then, in the notation above employed, we have for the change in the value of $\xi$ due to the encounter, omitting the factor $\frac{1}{\Sigma f}$,

$$\partial\xi = (f_1' - f_2')\,\lambda' V - (f_1 - f_2)\,\lambda V,$$

if $f$ be referred to $P$. Also

$$\lambda' V = \lambda V + \left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right) V.$$

Therefore

$$\partial\xi = +(f_1' - f_2')\,\lambda V - (f_1 - f_2)\,\lambda V + (f_1' - f_2')\left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right).$$

Also

$$f_1' - f_2' = -c\,(-\nu\cos 2\psi + \sqrt{1-\nu^2}\sin 2\psi\cos\phi)\frac{df}{dz},$$

$$f_1 - f_2 = -c\,(-\nu\cos\theta + \sqrt{1-\nu^2}\sin\theta\cos\phi)\frac{df}{dz};$$

therefore, since, on integrating with reference to $\phi$, $\cos\phi$ disappears,

$$\partial\xi = -c\lambda\nu\,(\cos\theta - \cos 2\psi)\,V\frac{df}{dz} - c\int_0^\pi d\phi\,(-\nu\cos 2\psi + \sqrt{1-\nu^2}\sin 2\psi\cos\phi) \times\left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right) V\frac{df}{dz};$$

that is,

$$\partial\xi = -c\lambda\nu\,(\cos\theta - \cos 2\psi)\,\frac{z}{r}\,\frac{df}{dr}\,V - c\lambda\nu\,(2\cos^2\psi\cos 2\psi - \tfrac{1}{2}\sin^2 2\psi)\,\frac{z}{r}\,\frac{df}{dr}\,V.$$

The number of encounters per unit of volume and time is $\pi c^2 \rho V$. So, given $V$,

$$\frac{\partial}{\partial t}\frac{d\xi}{dz} = -2\pi c^3 \rho V^2 \lambda\nu \frac{z}{r}\frac{df}{dr}(\cos\theta - \cos 2\psi + 4\cos^4\psi - 4\cos^2\psi + 2\cos^4\psi)$$

$$= -2\pi c^3 \rho V^2 \lambda\nu \frac{z}{r}\frac{df}{dr}(\cos\theta - 6\cos^2\psi + 1 + 6\cos^4\psi),$$

and $\frac{\partial}{\partial t}\frac{d\xi'}{dz}$ is the same expression with $-z$ for $z$; therefore

$$\frac{\partial}{\partial t}\frac{d\xi}{dz} = -4\pi c^3 \rho V^2 \lambda\nu \frac{z}{r}\frac{df}{dr}(\cos\theta - 6\cos^2\psi + 6\cos^4\psi + 1),$$

and the expression $\cos\theta - 6\cos^2\psi + 6\cos^4\psi + 1$,

where $\psi > \theta$ and $\psi < \frac{\pi}{2}$, is necessarily positive. For it is equal to

$$(\cos\theta - \cos^2\psi) + (6\cos^4\psi - 5\cos^2\psi + 1).$$

The first term is always positive. The second is negative between certain limits. But it will be found that the greatest negative value of the second term between those limits is less in absolute magnitude than the least value of the positive term between the same limits. Therefore also $\left(\psi \text{ being a known function of } \theta \text{ always } > 0 < \frac{\pi}{2}\right)$

$$\int_0^{\frac{1}{2}\pi} 2\sin\theta\cos\theta\, d\theta\,(1 + \cos\theta + 6\cos^4\psi - 6\cos^2\psi)$$

is positive. Therefore, also $\frac{\partial}{\partial t}\frac{\partial\xi}{\partial z}$ is of the opposite sign to $\lambda\nu\frac{df}{dr}$, *i.e.*, to $uw\frac{df}{dr}$. Therefore also $uw\frac{\partial}{\partial t}\frac{d\xi}{dz}$ is of the opposite sign to $\frac{df}{dr}$, and is therefore positive because $\frac{df}{dr}$ is negative. The same is true of $uw\frac{\partial}{\partial t}\frac{d\zeta}{dx}$.

25. We see, then, that, whereas for a system of material points without mutual action

$$\frac{dM}{dt} = 0 = uw\frac{d}{dt}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) - \frac{1}{2h}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2,$$

when they exert finite forces on each other $\frac{dM}{dt}$ is increased by necessarily positive terms. In order to make $\frac{dM}{dt}$ again zero, and so restore the stationary character of the motion, we must, as in the case of elastic spheres, use $\overline{\xi^2}+\overline{\xi'^2}$ instead of $\overline{\xi^2}$, and this requires that $Q$ shall contain the terms

$$\Sigma\Sigma b\,(uu'+vv'+ww').$$

26. In my former paper I drew the conclusion, from the form assumed in stationary motion by the function $Q$, that molecules which at any instant happen to be near to one another are, for that reason, more likely to be moving in the same direction; hence that any mass of contiguous molecules have on the average greater energy of the motion of their common centre of inertia, and correspondingly less energy of relative motion, than they would have as material particles without mutual action in Maxwell's system. But the following conclusion may also be drawn, namely, that local inequalities of density are more probable, or greater and more frequent, in the system of finite molecules than they are in Maxwell's system. For, in Maxwell's system, whatever at any instant be the velocities, the molecules are as likely to have any assigned positions in space as any other assigned positions. Suppose two spheres, each of radius $a$, not intersecting each other, and the average number of molecules which each contains is $n$. The chance that, at a given instant, one shall contain $n+q$, and the other $n-q$, molecules can be calculated; and, if $q$ be considerable, is much smaller than that each shall contain $n$. But, whatever it be, the effect of introducing the terms $\Sigma\Sigma b(uu'+vv'+ww')$ into $Q$ is to increase it in a finite ratio. For, if $b$ vary as $\frac{x^5}{\gamma^5}$ and is negative, the mean value of $b$, for all pairs of points within one of the $a$ spheres, varies as $\frac{x^3}{a^3}$.

We may assume the two spheres to be so far apart that, for any pair of points, one in one $a$ sphere and one in the other, $b$ is negligible. Then, of all the pairs which can be formed out of the $2n$ molecules in the two spheres, $\frac{\overline{n+q}\,\overline{n+q-1}}{2}$ pairs in one sphere, and $\frac{\overline{n-q}\,\overline{n-q-1}}{2}$ in the other sphere, have on average $-\frac{c^3}{a^3}$ for $b$, and all the other pairs have zero $b$. Now

$$\frac{\overline{n+q}\,\overline{n+q-1}}{2}+\frac{\overline{n-q}\,\overline{n-q-1}}{2}=n^2+q^2-n\,,$$

and increases as $q$ increases, and has its greatest value when $q = n$. The effect, therefore, of the inequality of density is to increase the absolute value of $\Sigma\Sigma b(uu' + vv' + ww')$, and, therefore, to diminish $Q$. And, to that extent, the inequalities become more probable. So that, as the general density increases, the system tends more and more to assume the form of a number of denser masses moving through a comparatively rare medium.

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*Thursday, February 10th*, 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Thirteen members and a visitor present.

Mr. B. Hopkinson was admitted into the Society.

Lt.-Col. Cunningham read a paper "On Aurifeuillians," which Mr. Bickmore supplemented at some length by a treatment of the subject from another point of view. A discussion took place on the two communications.

Lt.-Col. Cunningham having (*pro tem.*) taken the Chair, the President communicated a paper by Mr. J. E. Campbell, entitled "On the Transformations which leave the Lengths of Arcs on Surfaces unaltered."

Mr. Hargreaves made a brief impromptu communication.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LXII., Nos. 382, 383.

"Proceedings of the Royal Irish Academy," Vol. IV., No. 4; Dublin, 1897.

"Bulletin de la Société Mathématique de France," Tome XXV., Nos. 8, 9, et dernier; Paris, 1897.

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. 6; January, 1898.

"Monatshefte für Mathematik und Physik," 1898, Pt. 1; Wien, 1898.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIII., No. 3; Coimbra, 1897.

"Bulletin des Sciences Mathématiques," Tome XXI., December, 1897; Tome XXII., January, 1898.

Laisant, C. A.—"La Mathématique, Philosophie-Enseignement," 8vo; Paris, 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 11, 12, 1897; Bd. XXII., St. 1, 1898; Leipzig.

"Proceedings of the Physical Society," Vol. xv., Pt. 12, No. 87; December, 1897.

"Wiskundige Opgaven met de Oplossingen door de Leden van het Wiskundige Genootschap Amsterdam," Deel vii., St. 4; Amsterdam, 1898.

"Nieuw Archief voor Wiskunde," Deel iii., St. 3; Amsterdam, 1897.

"Revue Semestrielle des Publications Mathématiques," Tome vi., 1897, Av.–Oct.; Amsterdam, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. iii., Fasc. 12; Napoli, 1897.

"Bulletin of the American Mathematical Society," Series 2, Vol. iv., No. 4; January, 1898.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Geschäftliche Mittheilungen, Heft 2, Mathematisch-Physikalische Klasse, 1897, Heft 3.

"Annali di Matematica," Tomo xxvi., Fasc. 4; Milano, December, 1897.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. vi., Fasc. 12; Sem. 1, Vol. vii., Fasc. 1; Roma.

Wundt, W.—"Die Geometrisch Optischen Täuschungen," 8vo; Leipzig, 1898.

"Transactions of the Cambridge Philosophical Society," Vol. xvi., Pt. 3; 1898.

"Proceedings of the Cambridge Philosophical Society," Vol. ix., Pt. 7; 1898.

"Journal für die reine und angewandte Mathematik," Bd. cxix., Heft 1; Berlin, 1898.

"Educational Times," February, 1898.

"Indian Engineering," Vol. xxii., No. 26; Vol. xxiii., Nos. 1–3; Dec. 25, 1897–Jan. 15, 1898.

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## *Transformations which leave the Lengths of Arcs on Surfaces unaltered.* *By* J. E. Campbell. Received and read February 10th, 1898.

In this paper it is shown, by an application of Lie's method of contact transformations, that the infinitesimal transformations which have the property of leaving unaltered the lengths of arcs on any given surface, $z = f(x_1, \ldots x_n)$, in space of $n+1$ $(n>2)$ dimensions are in general mere motions in $n+1$ dimensional space (§ 3). For a special class of surfaces there is a more general transformation than a mere motion; such special surfaces must be the envelopes of planes

$$z = ax_1 + bx_2 + {}_3f(a, b)\, x_3 + \ldots + {}_nf(a, b)\, x_n + {}_{n+1}f(a, b),$$

and the functions ${}_3f, \ldots {}_{n+1}f$ which appear in the equation of the plane are connected in a remarkable manner with the problem of determining the surfaces applicable on a given surface in ordinary space (§ 6).

It is also shown (§ 7) that in every case, if $H$ is the Hessian of $f(x_1, \ldots x_n)$, then

$$H \div \left\{1+\left(\frac{dz}{dx_1}\right)^2+\ldots+\left(\frac{dz}{dx_n}\right)^2\right\}^{\frac{1}{2}(n+2)}$$

is an invariant for such transformations; this is a generalization of the well known theorem that the measure of curvature on a surface in ordinary space is unaltered by transformations which leave the length of arcs unaltered.*

1. Let $x_1, \ldots x_n$ be $n$ independent variables, and $z$ the dependent variable, and let

$$ds^2 = dx_1^2+\ldots+dx_n^2+dz^2.$$

Let $\xi_1(x_1, \ldots x_n)$, $\ldots \xi_n(x_1, \ldots x_n)$, $\zeta(x_1, \ldots x_n)$ denote any $n$ independent functions of $x_1, \ldots x_n$. Understanding by $\frac{du}{dx_i}$ the ratio of the increment of $u$ to that of $x_i$ when all the other independent variables are kept constant, let us write $\frac{dz}{dx_i}$, $\frac{d\zeta}{dx_i}$, $\frac{d^2z}{dx_i dx_\kappa}$, $\frac{d^2\zeta}{dx_i dx_\kappa}$, shortly $z_i$, $\zeta_i$, $z_{i\kappa}$, $\zeta_{i\kappa}$; and let

$$2\lambda_{i\kappa} = \frac{d}{dx_i}\xi_\kappa+\frac{d}{dx_\kappa}\xi_i+z_i\zeta_\kappa+z_\kappa\zeta_i;$$

and therefore

$$\lambda_{ii} = \frac{d}{dx_i}\xi_i+z_i\zeta_i.$$

Consider the infinitesimal transformation which consists in substituting for $x_i$, $x_i+\xi_i t$ $(i=1, \ldots n)$, and for $z$, $z+\zeta t$, where $t$ is a constant so small that its square may be neglected. We can now calculate the corresponding increments in $z_i$, $z_{i\kappa}$, as in Lie (*Transformationsgruppen*, II., § 95), and thus obtain what he calls an "erweiterte Transformation."

Let us suppose that $z_i$ becomes $z_i+\pi_i t$, and $z_{i\kappa}$, $z_{i\kappa}+\pi_{i\kappa}t$ $(i, \kappa = 1, \ldots n)$,

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* The author wishes to acknowledge here the help which he has received from the criticism and suggestions of one of the referees, particularly with regard to what he has called the special surfaces; it was owing to his criticism of the result first given that an error was detected and the much more complete solution of the problem obtained which is given on p. 262.

where $\pi_i$, $\pi_{i\kappa}$ are functions of $x_1, \dots x_n, z$, whose forms are to be calculated.

By the point transformation which we are considering, each member of the system of equations

$$dz - \sum_{i=1}^{i=n} z_i dx_i = 0, \quad dz_i - \sum_{\kappa=1}^{\kappa=n} z_{i\kappa} dx_\kappa = 0 \quad (i = 1 \dots n),$$

is to be unaltered in form. Now the increment of the first is

$$d\zeta - \sum_{i=1}^{i=n} (z_i d\xi_i + \pi_i dx_i);$$

putting this equal to zero, and remembering that

$$d\zeta = \sum_{i=1}^{i=n} \zeta_i dx_i,$$

we get, by equating the coefficient of $dx_i$ to zero,

$$\pi_i = \zeta_i - \sum_{\kappa=1}^{\kappa=n} z_\kappa \frac{d\xi_\kappa}{dx_i} \qquad (i = 1 \dots n).$$

Similarly, we obtain

$$d\pi_i - \sum_{\kappa=1}^{\kappa=n} (\pi_{i\kappa} dx_\kappa + z_{i\kappa} d\xi_\kappa) = 0 \quad (i = 1 \dots n),$$

and deduce that $$\pi_{i\kappa} = \frac{d\pi_i}{dx_\kappa} - \sum_{r=1}^{r=n} z_{ir} \frac{d\xi_r}{dx_\kappa};$$

or, using the value of $\pi_i$ already obtained,

$$\pi_{i\kappa} = \zeta_{i\kappa} - \sum_{r=1}^{r=n} z_{\kappa r} \frac{d\xi_r}{dx_i} - \sum_{r=1}^{r=n} z_{ir} \frac{d\xi_r}{dx_\kappa} - \sum_{r=1}^{r=n} z_r \frac{d^2\xi_r}{dx_i\, dx} \quad (i, \kappa = 1 \dots n).$$

The extended infinitesimal transformation is then

$$\sum_{i=1}^{i=n} \xi_i \frac{d}{dx_i} + \zeta \frac{d}{dz} + \sum_{i=1}^{i=n} \pi_i \frac{d}{dz_i} + \sum_{\substack{i=1 \\ \kappa=1}}^{\substack{i=n \\ \kappa=n}} \pi_{i\kappa} \frac{d}{dz_{i\kappa}};$$

the first two terms in this expression forming the original infinitesimal transformation.

The object of this paper is to consider the particular forms of these transformations which have the property of leaving the expression for $ds$ unaltered in form.

2. If $\delta ds$ denotes the increment of $ds$, owing to the infinitesimal transformation, we easily obtain

$$ds\frac{\delta ds}{\delta t} = \sum_{r=1}^{r=n} \lambda_{rr} dx_r^2 + \Sigma 2\lambda_{i\kappa} dx_i dx_\kappa,$$

where in the second expression the summation is to be taken for all unequal values of $i$ and $\kappa$ from 1 to $n$ inclusive.

The form of the infinitesimal transformation which leaves $ds$ unaltered is therefore given by equating all the $\lambda$ functions to zero.

The following identity can be immediately verified :—

$$\frac{d^2\xi_\kappa}{dx_r dx_s} + z_{rs}\zeta_\kappa + z_\kappa \zeta_{rs} = \frac{d}{dx_r}\lambda_{\kappa s} + \frac{d}{dx_s}\lambda_{r\kappa} - \frac{d}{dx_\kappa}\lambda_{rs}.$$

If, then, the infinitesimal transformation leaves $ds$ unaltered, we must have, for all values of $\kappa$, $r$, and $s$,

$$\frac{d^2\xi_\kappa}{dx_r dx_s} + z_{rs}\zeta_\kappa + z_\kappa \zeta_{rs} = 0.$$

Since

$$\frac{d^3\xi_\kappa}{dx_h dx_r dx_s} = \frac{d^3\xi_\kappa}{dx_r dx_h dx_s},$$

we at once deduce

$$z^{hs}\zeta_{\kappa r} + z_{\kappa r}\zeta_{sh} - z_{sr}\zeta_{\kappa h} - z_{\kappa h}\zeta_{rs} = 0.* \qquad (A)$$

This set of equations must therefore hold for all values of $\kappa$, $h$, $r$, $s$, if the infinitesimal transformation has the required property. Let us now write down the equations of the system (A) which only involve three suffixes, say 1, 2, 3 : they are

$$2z_{23}\zeta_{23} = z_{22}\zeta_{33} + z_{33}\zeta_{22},$$
$$2z_{31}\zeta_{31} = z_{33}\zeta_{11} + z_{11}\zeta_{33},$$
$$2z_{12}\zeta_{12} = z_{11}\zeta_{22} + z_{22}\zeta_{11},$$

and

$$z_{23}\zeta_{11} = z_{12}\zeta_{13} + z_{13}\zeta_{12} - z_{11}\zeta_{23},$$
$$z_{31}\zeta_{22} = z_{21}\zeta_{23} + z_{23}\zeta_{21} - z_{22}\zeta_{31},$$
$$z_{12}\zeta_{33} = z_{31}\zeta_{32} + z_{32}\zeta_{31} - z_{33}\zeta_{12}.$$

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* If $n = 2$, there is only one equation,

$$z_{11}\zeta_{22} - 2z_{12}\zeta_{12} + z_{22}\zeta_{11} = 0,$$

the others reducing to mere identities. This equation, which is fundamental in the theory of the infinitesimal deformation of a surface applicable on a given surface, is obtained and discussed in Darboux, *Théorie générale des Surfaces* (Vol. IV., p. 10).

Eliminating $\zeta_{11}, \zeta_{22}, \zeta_{33}, \zeta_{23}, \zeta_{31}, \zeta_{12}$ from these six equations, we obtain

$$\begin{vmatrix} 0, & z_{33}, & z_{22}, & 2z_{23}, & 0, & 0 \\ z_{33}, & 0, & z_{11}, & 0, & 2z_{13}, & 0 \\ z_{22}, & z_{11}, & 0, & 0, & 0, & 2z_{12} \\ z_{23}, & 0, & 0, & -z_{11}, & z_{12}, & z_{13} \\ 0, & z_{13}, & 0, & z_{12}, & -z_{22}, & z_{23} \\ 0, & 0, & z_{12}, & z_{13}, & z_{23}, & -z_{33} \end{vmatrix} = 0.$$

It may be verified that the above determinant is the square of the determinant

$$\begin{vmatrix} z_{11}, & z_{12}, & z_{13} \\ z_{21}, & z_{22}, & z_{23} \\ z_{31}, & z_{32}, & z_{33} \end{vmatrix},$$

so that, unless this latter determinant vanishes for every set of three suffixes, $\zeta_{rs} = 0$ for all values of $r$ and $s$ is the only solution of the system of equations (A).

3. Let us now suppose that the surface $z = f(x_1, \ldots x_n)$ is not of such special form that all of these determinants vanish. We deduce from $\zeta_{rs} = 0$ that

$$\zeta = c_1 x_1 + \ldots + c_n x_n + c,$$

where $c_1, \ldots c_n$ and $c$ are arbitrary constants. From

$$\frac{d^2 \xi_\kappa}{dx_r dx_s} + z_{rs}\zeta_\kappa + z_\kappa \zeta_{rs} = 0,$$

we now deduce $\quad \xi_\kappa = a_{1\kappa} x_1 + \ldots + a_{n\kappa} x_n - c_\kappa z + a_{0\kappa}$;

and, since $\quad \dfrac{d}{dx}\xi_\kappa + \dfrac{d}{dx_\kappa}\xi_i + z_i \zeta_\kappa + z_\kappa \zeta_i = 0,$

we see that the arbitrary constants $a_{rs}$ are connected by the relations

$$a_{rr} = 0 \quad \text{and} \quad a_{r\kappa} + a_{\kappa r} = 0.$$

We thus see that, when $n > 2$ and the surface is not a special one, the most general infinitesimal transformation which leaves the length of arcs on the surface unaltered involves $\dfrac{(n+1)(n+2)}{2}$ arbitrary constants, and is, so far as it affects the surface, at once identified with the most general infinitesimal motion of $n+1$ dimensional flat space.

It is not necessary that the constants $a$ and $c$ should be absolute constants, but only that they should be such functions of $z, x_1, \ldots x_n$ as are constant when $z = f(x_1 \ldots x_n)$; that is, the expression for the infinitesimal transformation is a mere motion so far as it affects the given surface, but not when applied to other points in space. If the single relation which connects $z, x_1, \ldots x_n$ had taken that particular form which involved $z$ only, then $z$ would have been a constant; and the discussion would have reduced to a consideration of those infinitesimal transformations which leave $dx_1^2 + \ldots + dx_n^2$ unaltered; *e.g.*, in case $n = 3$, we should have had the formulæ which give the infinitesimal displacements of a rigid body in ordinary space.

Lie (*Transformationsgruppen*, III., p. 314) discusses by a different method the more general problem of this type, namely, the transformations which leave the equation

$$a_{11}dx_1^2 + \ldots + a_{nn}dx_n^2 + 2a_{12}dx_1dx_2 + \ldots = 0$$

unaltered, where $a_{11}, \ldots a_{12}, \ldots$ are functions of the $n$ independent variables $x_1, \ldots x_n$. Some of his conclusions closely resemble results at which I have arrived in the, in some respects, more general problem where the variables are connected by a relation.

We can now write down the extended form of the transformation

$$\pi_\kappa = c_\kappa - \sum_{i=1}^{i=n} z_i (a_{\kappa i} - c_i z_\kappa),$$

$$\pi_{i\kappa} = z_{i\kappa} \sum_{r=1}^{r=n} c_r z_r - \sum_{r=1}^{r=n} (a_{ir} - c_r z_i) z_{\kappa r} - \sum_{r=1}^{r=n} (a_{\kappa r} - c_r z_\kappa) z_{ir}.$$

4. Now it is easily seen, conversely, that every such transformation does leave the length of arcs on the surface unaltered; but it must also be proved that, corresponding to every solution of the system (A), even when the surface is a special one, there will be an infinitesimal transformation with the required property.

Let $z_{rs}\zeta_\kappa + z_\kappa \zeta_{rs}$, where $\zeta$ satisfies the system (A), be written $-P_{rs\kappa}$; then we must prove that $P_{rs\kappa}$ is of the form $\dfrac{d^2V_\kappa}{dx_r dx_s}$.

We see at once that

$$\frac{d}{dx_h} P_{rs\kappa} = \frac{d}{dx_r} P_{hs\kappa},$$

and therefore

$$P_{rs\kappa} = \frac{dV}{dx_r}, \quad P_{hs\kappa} = \frac{dV}{dx_h},$$

where $V$ is some function of $x_1, \ldots x_n, z$; but

$$\frac{d}{dx_r} P_{ms\kappa} = \frac{d}{dx_m} P_{rs\kappa} = \frac{d}{dx_r} \frac{dV}{dx_m};$$

therefore
$$\frac{d}{dx_r}\left(P_{ms\kappa} - \frac{dV}{dx_m}\right) = 0.$$

Similarly, we obtain
$$\frac{d}{dx_h}\left(P_{ms\kappa} - \frac{dV}{dx_m}\right) = 0.$$

Thus we have
$$P_{ms\kappa} = \frac{dV}{dx_m} + Q,$$

where $Q$ is a function independent of $x_r$ and $x_h$; and it is easily seen that any such function may be also written in the form $\frac{dQ'}{dx_m}$, where $Q'$ is independent of $x_r$ and $x_h$, so that

$$P_{ms\kappa} = \frac{d}{dx_m}(V + Q').$$

Now
$$P_{rs\kappa} = \frac{dV}{dx_r} = \frac{d(V+Q')}{dx_r},$$

and
$$P_{hs\kappa} = \frac{dV}{dx_h} = \frac{d(V+Q')}{dx_h};$$

we therefore conclude that

$$P_{rs\kappa} = \frac{dV_{s\kappa}}{dx_r}$$

whatever value $r$ may have.

Similarly,
$$P_{sr\kappa} = \frac{dV_{r\kappa}}{dx_s},$$

whatever $s$ may be; but we have identically

$$P_{rs\kappa} = P_{sr\kappa};$$

therefore
$$\frac{d}{dx_s} V_{r\kappa} = \frac{d}{dx_r} V_{s\kappa},$$

and, just as before, we see that

$$V = \frac{dW_\kappa}{dx_s},$$

whatever value $s$ may have, and therefore

$$P_{rs\kappa} = \frac{d^2 W_\kappa}{dx_r\, dx_s},$$

whatever values $r$ and $s$ may have.

The conclusion is, if $\zeta$ is a function which satisfies the system of equations (A), then there are $n$ functions $W_1, \ldots W_n$, which satisfy the system of equations

$$\frac{d^2 W_\kappa}{dx_r dx_s} + z_{rs}\zeta_\kappa + z_\kappa \zeta_{rs} = 0;$$

and it is clear that, if $W_1, \ldots W_n$ are any $n$ particular solutions, the most general solution is of the form

$$\xi_\kappa = W_\kappa + a_{1\kappa}x_1 + \ldots + a_{n\kappa}x_n + b_\kappa,$$

where the letters $a$ and $b$ denote arbitrary constants.

Now we can easily verify that

$$\frac{d}{dx_r}\left(\frac{dW_\kappa}{dx_h} + \frac{dW_h}{dx_\kappa} + z_\kappa \zeta_h + z_h \zeta_\kappa\right) = 0$$

for all values of $h$, $r$, and $\kappa$; therefore

$$\frac{dW_\kappa}{dx_h} + \frac{dW_h}{dx_\kappa} + z_\kappa \zeta_h + z_h \zeta_\kappa = e_{\kappa h},$$

a constant.

If, then, we choose the constants $a$ and $b$ in the most general way to satisfy the system of equations

$$a_{h\kappa} + a_{\kappa h} + e_{\kappa h} = 0,$$

we see that

$$\frac{d\xi_\kappa}{dx} + \frac{d\xi_h}{dx_\kappa} + z_\kappa \zeta_h + z_h \zeta_\kappa = 0;$$

and that, corresponding to any particular solution $\zeta$ of the equation system (A), there is an infinitesimal transformation involving $\frac{n(n+1)}{2}$ arbitrary constants which has the property of leaving $ds$ unaltered. This may be otherwise expressed by saying that there are $\frac{n(n+1)}{2}$ of such independent transformations.

5. We have seen that, except for a special kind of surface,

$$z = f(x_1, \ldots x_n),$$

the only solution of the system of equations (A) is $\zeta_{rs} = 0$ for all values of $r$ and $s$. This result may also be arrived at geometrically and some further information as to the solution of the system obtained.

Since $z$ is a function of $x_1, \ldots x_n$, we may take $z = 0$ to represent a

surface in $n$ dimensions. The quadric which is an emanant of $z'$ with respect to $x_1', \ldots x_n'$ is symbolically written

$$U \equiv \left(x_1 \frac{d}{dx_1'} + \ldots + x_n \frac{d}{dx_n'}\right)^2 z' = 0.$$

Similarly, the quadric which is the emanant of $\zeta'$ with respect to $x_1', \ldots x_n'$ is

$$V \equiv \left(x_1 \frac{d}{dx_1'} + \ldots + x_n \frac{d}{dx_n'}\right)^2 \zeta' = 0.$$

Now the condition that the join of two points $x_1 \ldots x_n$ and $y_1 \ldots y_n$, should be divided harmonically by these quadrics is

$$U_x V_y + U_y V_x = \tfrac{1}{2} \sum_{i=1}^{i=n} y_i \frac{dU}{dx_i} \sum_{i=1}^{i=n} y_i \frac{dV}{dx_i},$$

where $U_y$ denotes the result of substituting $y$ for $x$ in $U$; for we have only to substitute for $x_1 \ldots x_n, \lambda x_1 + \mu y_1 \ldots \lambda x_n + \mu y_n$ in $U = 0$, and we obtain, to determine the ratio in which the join of $x_1 \ldots x_n$ and $y_1 \ldots y_n$ is divided by $U = 0$, the equation

$$\lambda^2 U_x + \lambda\mu \sum_{i=1}^{i=n} y_i \frac{dU}{dx_i} + \mu^2 U_y = 0.$$

Now we can see at once that the relation between $z$ and $\zeta$ involved in the equation system (A) may be thus stated: Every line is divided harmonically by the quadratic emanants of $z$ and $\zeta$ with respect to any point. Now, what special kind of quadrics must $U = 0$ and $V = 0$ be in order to have this property?

There is, of course, the obvious solution that the coefficients in one quadric should vanish identically. Rejecting this for the present, let us also assume that both the quadrics do not degenerate into planes. Suppose that $U = 0$ does not degenerate; take any tangent line at any point $P$ on $U = 0$; then it cannot be divided harmonically by $U = 0$ and $V = 0$ unless $V = 0$ also passes through $P$; that is, $U = 0$ and $V = 0$ must coincide. If, then, they coincide, take any two non-coincident points on $U = 0$, $P$ and $Q$; we have now two coincident points at $P$ (one on $U = 0$ and one on $V = 0$) and two at $Q$, so that $PQ$ cannot be divided harmonically. The conclusion is that both quadrics must degenerate into planes (for then any tangent line coincides with the surface along its whole length). Take now the case that both quadrics degenerate, and let $U$ degenerate into planes $X = 0$, $Y = 0$, and $V$ into planes $Z = 0$, $W = 0$. It is now easily seen that the join of two points common to $X = 0$ and $Y = 0$

cannot be divided harmonically by $U=0$ and $V=0$ unless those points are also common to $Z=0$ or $W=0$; thus, $Z=0$, say, must pass through the points common to $X=0$ and $Y=0$. Similarly, the join of two points common to $X=0$ and $W=0$ cannot be divided harmonically unless those points lie on $Y=0$ or $Z=0$. So that, finally, each of the four planes passes through the points common to any two. By transformation of axes, then, we can write the quadrics in these forms—

$$U=ax_1^2+2hx_1x_2+bx_2^2 \quad \text{and} \quad V=a'x_1^2+2h'x_1x_2+b'x_2^2,$$

and the only other necessary relation between them is

$$ab'+ba'=2hh'.$$

Now, the Hessian of $z+\kappa\zeta$ is a joint covariant, and, by considering that the quadratic emanants can be thrown into the above forms, we see that, if $H$ is the Hessian of $z$,

$$\Sigma\zeta_{rs}\frac{dH}{dz_{rs}}\equiv 0.$$

This result, which will be required later on, is true when $n=2$ as well as when $n>2$. When $n>2$, we can prove that the equation system (A) can only be satisfied by equating to zero all such expressions as $\zeta_{rs}$, unless the surface $z=f(x_1,\ldots x_n)$ is the envelope of a plane which only contains two arbitrary parameters.

$H$ denoting the Hessian of $z$, we see that the quadratic emanant of $z$ can then, and only then, degenerate into planes $X$, $Y$, when every $(n-3)^{\text{rd}}$ minor of $H$ vanishes; and, remembering that the Hessian of $z$ is the Jacobian of $z_1,\ \ldots\ z_n$, we conclude that there must be $n-2$ functional relations between $z_1,\ \ldots\ z_n$, but the quadratic emanant of $\zeta$ also degenerates into planes $\lambda X+\mu Y$ and $\lambda' X+\mu' Y$, which form, with $XY$, a harmonic pencil. We can conclude, then, that every determinant of the third order must vanish which can be derived from the matrix

$$\begin{vmatrix} z_{11}, & z_{12}, & \ldots, & z_{1n} \\ z_{21}, & z_{22}, & \ldots, & z_2{}^n \\ \ldots & \ldots & \ldots & \ldots \\ \zeta_{11}, & \zeta_{12}, & \ldots, & \zeta_{1n} \\ \zeta_{21}, & \zeta_{22}, & \ldots, & \zeta_{2n} \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

This means that of the sets $\zeta_1 \dots \zeta_n$ and $z_1 \dots z_n$ only two are independent, say, $z_1$ and $z_2$.

6. We have now proved that, for a transformation which is not a mere motion,

$$z = f(x_1 \dots x_n)$$

must be such a surface that

$$z_3 = {}_3f(z_1, z_2) \dots z_n = {}_nf(z_1, z_2).$$

The question, however, arises: Given such a surface, does the required transformation exist? In other words, can we always obtain functions

$$\zeta_1 = {}_1\phi(z_1, z_2) \dots \zeta_n = {}_n\phi(z_1, z_2)$$

which satisfy the necessary conditions

$$\frac{d}{dx_r}{}_s\phi = \frac{d}{dx_s}{}_r\phi$$

for all values of $r$ and $s$?

It will first be proved that, if ${}_1\phi \dots {}_n\phi$ satisfy these conditions for all values of $r$, and for $s = 1$ and $s = 2$, they will satisfy the conditions for all other values of $s$.

By our data,
$$\frac{d_1\phi}{dx_2} = \frac{d_2\phi}{dx_1};$$

therefore
$${}_2\phi_1 z_{11} + {}_2\phi_2 z_{12} = {}_1\phi_1 z_{12} + {}_1\phi_2 z_{22}, \tag{1}$$

where ${}_1\phi_1$ denotes $\frac{d}{dz_1}{}_1\phi$ and ${}_1\phi_2$ denotes $\frac{d}{dz_2}{}_1\phi$; and, because

$$\frac{d}{dx_1}{}_r\phi = \frac{d}{dx_r}{}_1\phi \quad \text{and} \quad \frac{d}{dx_2}{}_r\phi = \frac{d}{dx_r}{}_2\phi,$$

we have
$$\left.\begin{aligned} {}_r\phi_1 z_{11} + {}_r\phi_2 z_{12} &= {}_1\phi_1 z_{r1} + {}_1\phi_2 z_{r2} \\ {}_r\phi_1 z_{12} + {}_r\phi_2 z_{22} &= {}_2\phi_1 z_{r1} + {}_2\phi_2 z_{r2} \end{aligned}\right\}. \tag{2}$$

Using (1), we can write these equations in the form

$$\left.\begin{aligned} {}_r\phi_1 (z_{11}z_{22} - z_{12}) &= {}_1\phi_1 (z_{r1}z_{22} - z_{12}z_{r2}) + {}_2\phi_1 (z_{r2}z_{11} - z_{r1}z_{12}) \\ {}_r\phi_2 (z_{11}z_{22} - z_1^2) &= {}_1\phi_2 (z_{r1}z_{22} - z_{12}z_{r2}) + {}_2\phi_2 (z_{r2}z_{11} - z_{r1}z_{12}) \end{aligned}\right\}. \tag{3}$$

But
$$\left.\begin{aligned} z_{r1} &= {}_rf_1 z_{11} + {}_rf_2 z_{12} \\ z_{r2} &= {}_rf_1 z_{12} + {}_rf_2 z_{22} \end{aligned}\right\}, \tag{4}$$

so that we have
$$\left.\begin{aligned} {}_r\phi_1 &= {}_1\phi_1\, {}_rf_1 + {}_2\phi_1\, {}_rf_2 \\ {}_r\phi_2 &= {}_1\phi_2\, {}_rf_1 + {}_2\phi_2\, {}_rf_2 \end{aligned}\right\}. \tag{5}$$

Now $$\frac{d_r\phi}{dx_s} = {}_r\phi_1 z_{1s} + {}_r\phi_2 z_{2s}$$

$$= A\,{}_rf_1\,{}_sf_1 + B\,{}_rf_2\,{}_sf_2 + H\,({}_rf_1\,{}_sf_2 + {}_rf_2\,{}_sf_1), \text{ by (4)},$$

where $$A = {}_1\phi_1 z_{11} + {}_1\phi_2 z_{12},$$

$$B = {}_2\phi_1 z_{12} + {}_2\phi_2 z_{22},$$

$$H = {}_1\phi_1 z_{12} + {}_1\phi_2 z_{22} = {}_2\phi_1 z_{11} + {}_2\phi_2 z_{12}, \text{ by (1)}.$$

We thus see that, since $r$ and $s$ are involved symmetrically,

$$\frac{d_r\phi}{dx_s} = \frac{d_s\phi}{dx_r}.$$

If the equations (5) are consistent, they define uniquely (save as to an additive constant) the function ${}_r\phi$. They are consistent if, and only if,

$$\frac{d}{dz_2}({}_1\phi_1\,{}_rf_1 + {}_2\phi_1\,{}_rf_2) = \frac{d}{dz_1}({}_1\phi_2\,{}_rf_1 + {}_2\phi_1\,{}_rf_2),$$

that is, if $${}_1\phi_1\,{}_rf_{12} + {}_2\phi_1\,{}_rf_{22} = {}_1\phi_2\,{}_rf_{11} + {}_2\phi_2\,{}_rf_{21}.$$

The necessary and sufficient conditions, then, that we may be able to determine the required kind of functions ${}_1\phi, \ldots {}_n\phi$ is that we can determine functions ${}_1\phi$ and ${}_2\phi$ to satisfy the equations

$$\left.\begin{aligned} {}_1\phi_1 z_{12} + {}_1\phi_2 z_{22} &= {}_2\phi_1 z_{11} + {}_2\phi_2 z_{12} \\ {}_1\phi_1\,{}_rf_{12} + {}_2\phi_1\,{}_rf_{22} &= {}_1\phi_2\,{}_rf_{11} + {}_2\phi_2\,{}_rf_{21} \end{aligned}\right\} \qquad (6)$$

for all values of $r$ from 3 to $n$. These are the conditions necessary in order that the two quadric emanants may degenerate into plane pairs with a common line of intersection; one and only one further condition is supplied by the fact that any join of two points is divided harmonically by them.

This condition $$z_{22}\frac{d}{dx_1}{}_1\phi + z_{11}\frac{d}{dx_2}{}_2\phi = 2z_{12}\frac{d}{dx_2}{}_1\phi$$

may be written

$$z_{22}({}_1\phi_1 z_{11} + {}_1\phi_2 z_{12}) + z_{11}({}_2\phi_1 z_{12} + {}_2\phi_2 z_{22}) = z_{12}({}_1\phi_1 z_{12} + {}_1\phi_2 z_{22}) + z_{12}({}_2\phi_1 z_{11} + {}_2\phi_2 z_{12}),$$

that is, $$({}_1\phi_1 + {}_2\phi_2)(z_{11}z_{22} - z_{12}^2) = 0.$$

We have supposed $z_1$ and $z_2$ to be independent throughout, that is, we have assumed that the surface is not a mere developable; we cannot therefore take the factor

$$z_{11}z_{22} - z_{12}^2 = 0,$$

so that we have $${}_1\phi_1 + {}_2\phi_2 \equiv 0;$$

we can then write

$${}_1\phi \equiv \frac{d\psi}{dz_2} \equiv \psi_2 \quad \text{and} \quad {}_2\phi \equiv -\frac{d\psi}{dz_1} \equiv -\psi_1,$$

where $\psi$ is some function of $z_1$ and $z_2$. We can now write the equations (6) in the simpler form

$$\left.\begin{aligned} \psi_{11}\ z_{11} + \psi_{22}\ z_{22} + 2\psi_{12}\ z_{12} &= 0 \\ \psi_{11}\ {}_rf_{22} + \psi_{22}\ {}_rf_{11} - 2\psi_{12}\ {}_rf_{12} &= 0 \end{aligned}\right\} \tag{7}$$

for all values of $r$ from 3 to $n$ inclusive; and the necessary and sufficient conditions that the given surface should admit transformations (other than motions) which leave the lengths of arcs unaltered is that this system of equations may be self-consistent.

In order to see if these equations are consistent it will be necessary to obtain a different form of the first equation of (7).

The surface which satisfies the equations

$$z_3 = {}_3f(z_1, z_2) \ldots z_n = {}_nf(z_1, z_2)$$

is the envelope of the plane

$$z = ax_1 + bx_2 + {}_3f(a, b)\, x_3 + \ldots + {}_nf(a, b)\, x_n + {}_{n+1}f(a, b).$$

Differentiating with respect to $a$ and $b$, we obtain

$$\left.\begin{aligned} x_1 + \sum_{r=3}^{r=n} {}_rf_1 x_r + {}_{n+1}f_1 &= 0 \\ x_2 + \sum_{r=3}^{r=n} {}_rf_2 x_r + {}_{n+1}f_2 &= 0 \end{aligned}\right\}, \tag{8}$$

where ${}_rf_1$ denotes, as usual, $\dfrac{d}{da}\,{}_rf(a, b)$;

but $a = z_1$ and $b = z_2$; these last two equations will therefore determine for us $z_1$ and $z_2$ in terms of $x_1 \ldots x_n$.

Looking on the equations (8) in this light, we differentiate the first of them with respect to $x_2$, and the second with respect to $x_1$, and we obtain

$$\sum_{r=3}^{r=n} x_r\,({}_rf_{11}z_{12} + {}_rf_{12}z_{22}) + {}_{n+1}f_{11}z_{12} + {}_{n+1}f_{12}z_{22} = 0,$$

$$\sum_{r=3}^{r=n} x_r\,({}_rf_{12}z_{11} + {}_rf_{22}z_{12}) + {}_{n+1}f_{12}z_{11} + {}_{n+1}f_{22}z_{12} = 0.$$

Eliminating $z_{11}$, $z_{12}$, and $z_{22}$ between these equations and

$$\psi_{11}z_{12}+\psi_{22}z_{22}+2\psi_{12}z_{12}=0,$$

and remembering that

$$\psi_{11}\ {}_rf_{22}+\psi_{22}\ {}_rf_{11}-2\psi_{12}\ {}_rf_{12}=0,$$

we obtain

$$\psi_{11\,n+1}f_{22}+\psi_{22\,n+1}f_{11}-2\psi_{12\,n+1}f_{12}=0\,;$$

the equations (7) will be self-consistent then if, and only if, the system in two independent variables $z_1$, $z_2$ is consistent.

$$\psi_{11\ r}f_{22}+\psi_{22\,r}f_{11}-2\psi_{12\,r}f_{12}=0 \quad (r=3,\ldots n+1). \tag{9}$$

The result at which we have arrived might be thus stated. In order to find a surface which will admit a transformation of the required type (that is, one which leaves the length of arcs unaltered, and is yet not a motion), take any arbitrary function of $x$ and $y$, $\psi(x, y)$; find any $n-2$ solutions ${}_rf(x, y)$, $(r=3,\ldots n+1)$ of the equation

$$\frac{d^2\psi}{dx^2}\frac{d^2f}{dy^2}+\frac{d^2\psi}{dy^2}\frac{d^2f}{dx^2}-2\frac{d^2\psi}{dx\,dy}\frac{d^2f}{dx\,dy}=0$$

[that is, the equation which determines the surfaces applicable on $z=\psi(x, y)$ in ordinary space]; then the envelope of the plane

$$z=ax_1+bx_2+{}_3f(a, b)\,x_3+\ldots+{}_nf(a, b)\,x_n+{}_{n+1}f(a, b)$$

will be a surface of the required type.

The function $\psi(x, y)$ which implicitly determines this surface will also determine the transformation which the surface admits.

It has been assumed throughout that the $(n-2)^{\text{nd}}$ minors of $H$ did not vanish; if they do, we see that $z_2, \ldots z_n$ are all functions of $z_1$ only, and easily deduce that $\zeta_1, \ldots \zeta_n$ must also each be functions of $z_1$ only. We have then

$$z_r={}_rf(z_1);$$

therefore

$$z_{rs}={}_rf'(z_1)\times z_{s1}={}_rf'\times{}_sf';$$

similarly,

$$\zeta_r={}_r\phi'\times{}_s\phi',$$

and, in order that the equation system (A) may hold for all values of $r$ and $s$, we see that we must have

$$\frac{{}_rf'}{{}_r\phi'}\equiv\frac{{}_sf'}{{}_s\phi'}$$

for all values of $r$ and $s$.

7. The object of the remainder of this paper is to prove that infinitesimal transformations which leave arcs unaltered in length also leave unaltered the expression

$$H \div (1+z_1+\ldots+z_n^2)^{\frac{1}{2}(n+2)},$$

where $H$ is the Hessian of $z$. For the case $n=2$, this is the well-known result that surfaces applicable on one another have the same measure of curvature at corresponding points.

Using only the system of equations

$$\frac{d^2\xi_\kappa}{dx_r\,dx_s}+z_{rs}\zeta_\kappa+z_\kappa\zeta_{rs}=0,$$

and not the deduced consequences of those equations (so as not t exclude the exceptional cases, $n=2$ and special surfaces), we see from the above that

$$\pi_{i\kappa}=z_{i\kappa}\sum_{r=1}^{r=n}z_r\zeta_r-\sum_{r=1}^{r=n}z_{\kappa r}\frac{d\xi_r}{dx_i}-\sum_{r=1}^{r=n}z_{ir}\frac{d\xi_r}{dx_\kappa}+\zeta_{i\kappa}(1+z_1^2+\ldots+z_n^2).$$

Let $Z_{i\kappa}$ denote the first minor of $z_{i\kappa}$ in the Hessian; then

$$\sum_{\kappa=1}^{\kappa=n}\sum_{i=1}^{i=n}\pi_{i\kappa}Z_{i\kappa}-(1+z_1^2+\ldots+z_n^2)\sum_{\kappa=1}^{\kappa=n}\sum_{i=1}^{i=n}\zeta_{i\kappa}Z_{i\kappa}-\sum_{r=1}^{r=n}z_r\zeta_r\sum_{\kappa=1}^{\kappa=n}\sum_{i=1}^{i=n}z_{i\kappa}Z_{i\kappa}$$

$$=-\sum_{i=1}^{i=n}\sum_{r=1}^{r=n}\frac{d\xi_r}{dx_i}\sum_{\kappa=1}^{\kappa=n}z_{\kappa r}Z_{i\kappa}-\sum_{\kappa=1}^{\kappa=n}\sum_{r=1}^{r=n}\frac{d\xi_r}{dx_\kappa}\sum_{i=1}^{i=n}z_{ir}Z_{i\kappa}.$$

Now,
$$\sum_{\kappa=1}^{\kappa=n}z_{\kappa r}Z_{i\kappa}=0,$$

unless $r=i$, and its value is then $H$; and

$$\sum_{i=1}^{i=n}z_{ir}Z_{i\kappa}=0,$$

unless $r=\kappa$, and its value is then $H$.

Now, remembering that

$$\pi_{i\kappa}=\pi_{\kappa i} \quad \text{and} \quad Z_{i\kappa}=Z_{\kappa i},$$

and that
$$2Z_{i\kappa}=\frac{dH}{dz_{i\kappa}},$$

if
$$i\neq\kappa \quad \text{and} \quad Z_{ii}=\frac{dH}{dz_{ii}},$$

we see that
$$\sum_{\kappa=1}^{\kappa=n}\sum_{i=1}^{i=n}\pi_{i\kappa}Z_{i\kappa}=\Sigma\pi_{i\kappa}\frac{dH}{dz_{i\kappa}}=\frac{\partial}{\partial t}H,$$

and $$\sum_{i=1}^{i=n}\sum_{\kappa=1}^{\kappa=n}\zeta_{i\kappa}Z_{i\kappa}=\Sigma\zeta_{i\kappa}\frac{dH}{dz_{i\kappa}}=0,\text{ by (B)};$$

finally, $$\sum_{i=1}^{i=n}\sum_{\kappa=1}^{\kappa=n}z_{i\kappa}Z_{i\kappa}=nH,$$

so that we have

$$\frac{\partial H}{\partial t}=nH\sum_{r=1}^{r=n}z_r\zeta_r-2H\sum_{r=1}^{r=n}\frac{d\xi_r}{dx_r}=(n+2)\,H\sum_{r=1}^{r=n}z_r\zeta_r,$$

since $$\frac{d\xi_r}{dx_r}+z_r\zeta_r=0.$$

[The sign of summation $\Sigma$ in $\Sigma\pi_{i\kappa}\frac{dH}{dz_{i\kappa}}$, for instance, means that the summation is to be taken for every value $z_{i\kappa}$ which occurs in $H$.]

Remembering that $$\pi_i=\zeta_i-\sum_{r=1}^{r=n}z_r\frac{d\xi_r}{dx_r},$$

and that $$\frac{d\xi_\kappa}{dx_r}+\frac{d\xi_r}{dx_\kappa}+z_\kappa\zeta_r+z_r\zeta_\kappa=0,$$

we have $$\sum_{i=1}^{i=n}\pi_iz_i=\sum_{r=1}^{r=n}z_r\zeta_r(1+z_1^2+\ldots+z_n);$$

therefore $$\frac{\partial}{\partial t}(1+z_1^2+\ldots+z_n^2)=2\,(1+z_1^2+\ldots+z_n^2)\sum_{r=1}^{r=n}z_r\zeta_r,$$

so that $$\frac{\partial H}{\partial t}\div H=\frac{n+2}{2}\,\frac{\partial}{\partial t}(1+z_1^2+\ldots+z_n^2)\div(1+z_1^2+\ldots+z_n^2),$$

or $$\frac{\partial}{\partial t}\log\frac{H}{(1+z_1^2+\ldots+z_n^2)^{\frac{1}{2}(n+2)}}=0,$$

that is, $$H\div(1+z_1^2+\ldots+z_n^2)^{\frac{1}{2}(n+2)}$$

is unaltered by the infinitesimal transformation, and therefore by any transformation which leaves the length of arcs on the surface unaltered.

*On the Intersections of Two Cubics.* *By* H. M. Taylor. Received January 6th, 1898. Read January 13th, 1898. Received, in amended form, May 5th, 1898.

It has long been a well-known theorem that every cubic drawn through eight given points passes through a ninth fixed point. Geometrical methods have been given for finding this ninth point by means of conics, and also by means of straight lines only.*

In this paper expressions will be found for the coordinates of the ninth point in terms of the other eight. It will appear that (I.) the coordinates of the ninth point are functions of the eighth degree in the coordinates of each of the other points; (II.) that when seven of the points are given, if the eighth point lies upon a straight line, the locus of the ninth point is an octavic curve having a triple point at each of the seven given points: and that, if the straight line on which the eighth point lies passes through one or two of the seven points, the locus of the ninth point degenerates and contains respectively one or two nodal cubics, having nodes at those points through which the straight line passes; and also, if the eighth point lies upon a conic, the locus of the ninth point is a curve of the sixteenth degree having a sextuple point at each of the seven given points: and that, if the conic on which the eighth point lies passes through one, two, three, four, or five of the seven given points, the locus of the ninth point degenerates and contains respectively one, two, three, four, or five nodal cubics having nodes at the points through which the conic passes.

The general equation of a cubic may be written

$$\phi(x, y, z) \equiv ax^3 + by^3 + cz^3 + fx^2y + gy^2z + hz^2x + lx^2z + my^2x + nz^2y + pxyz = 0.$$

If this passes through nine points $(x_r, y_r, z_r)$, then

$$\phi(x_r, y_r, z_r) = 0,$$

for $$r = 1, 2, \ldots, 9.$$

* Weddle, *Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 83, and Hart, *Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 181; and Cayley, *Quarterly Journal*, Vol. V., p. 222, *Collected Works*, Vol. IV., p. 495.

When these nine equations are independent, they determine the ratios of the constant coefficients in $\phi$; but, if

$$x_q^\rho y_q^\sigma z_q^\tau = \sum_{i=1}^{4} \lambda_i x_i^\rho y_i^\sigma z_i^\tau$$

for all combinations of $\rho$, $\sigma$, $\tau$ such that

$$\rho + \sigma + \tau = 3,$$

then the nine equations $\phi(\ \ ) = 0$ are equivalent to only eight, and any cubic curve through eight of the points will pass through the ninth. In this case we see that, by elimination of the constants $\lambda$ from nine of the last ten equations, we obtain ten relations between the coordinates of the nine points, each relation being symmetrical with respect to the points, and only two of them being independent.

If we represent the determinants formed by leaving out the first, second, ..., tenth columns in the array

$$\| x_i^3,\ y_i^3,\ z_i^3,\ x_i^2 y_i,\ y_i^2 z_i,\ z_i^2 x_i,\ x_i^2 z_i,\ y_i^2 x_i,\ z_i^2 y_i,\ x_i y_i z_i \|,$$

for $i = 0, 1, 2, \ldots, 8$, where $x_0, y_0, z_0 = x, y, z$, by $A'$, $B'$, $C'$, $F'$, $G'$, $H'$, $L'$, $M'$, $N'$, $P'$, then each of the equations $A' = 0$, $B' = 0$, ... represents the fact that the point 9 lies on a cubic curve passing through the points 1, 2, ..., 8.

Every pair of these ten cubics intersect in the eight given points and a fixed ninth point. We shall proceed to find the coordinates of this ninth point in terms of the coordinates of the eight given points.

If these equations were written in the form

$$\begin{aligned}
A' &\equiv A_b y^3 + A_c z^3 + A_f x^2 y + A_g y^2 z + A_h z^2 x + A_l x^2 z + A_m y^2 x + A_n z^2 y + A_p xyz \\
B' &\equiv B_a x^3 + B_c z^3 + B_f x^2 y + B_g y^2 z + B_h z^2 x + B_l x^2 z + B_m y^2 x + B_n z^2 y + B_p xyz \\
C' &\equiv C_a x^3 + C_b y^3 + C_f x^2 y + C_g y^2 z + C_h z^2 x + C\ x^2 z + C_m y^2 x + C_n z^2 y + C_p xyz \\
F' &\equiv F_a x^3 + F_b y^3 + F_c z^3 + F_g y^2 z + F_h z^2 x + F_l x^2 z + F_m y^2 x + F_n z^2 y + F_p xyz \\
G' &\equiv G_a x^3 + G_b y^3 + G_c z^3 + G_f x^2 y + G_h z^2 x + G_l x^2 z + G_m y^2 x + G_n z^2 y + G_p xyz \\
H' &\equiv H_a x^3 + H_b y^3 + H_c z^3 + H_f x^2 y + H_g y^2 z + H_l x^2 z + H_m y^2 x + H_n z^2 y + H_p xyz \\
L' &\equiv L_a x^3 + L_b y^3 + L_c z^3 + L_f x^2 y + L_g y^2 z + L_h z^2 x + L_m y^2 x + L_n z^2 y + L_p xyz \\
M' &\equiv M_a x^3 + M_b y^3 + M_c z^3 + M_f x^2 y + M_g y^2 z + M_h z^2 x + M_l x^2 z + M_n z^2 y + M_p xyz \\
N' &\equiv N_a x^3 + N_b y^3 + N_c z^3 + N_f x^2 y + N_g y^2 z + N_h z^2 x + N_l x^2 z + N_m y^2 x + N_p xyz \\
P' &\equiv P_a x^3 + P_b y^3 + P_c z^3 + P_f x^2 y + P_g y^2 z + P_h z^2 x + P_l x^2 z + P_m y^2 x + P_n z^2 y
\end{aligned}$$

then $A_b = B_a$, $A_c = -C_a$, $A_f = F_a$, $A_g = -G_a$, ...,

and, since from any pair of these equations we can obtain each of the other eight equations, we deduce relations among the coefficients of the form

$$\frac{A_a B_f - A_f B_a}{F_a} \equiv -B_a$$

$$= \frac{A_b B_f - A_f B_b}{F_b} = \frac{A_c B_f - A_f B_c}{F_c} = \&c.$$

Consider two cubics $\phi(x, y, z) = 0$

and $\Phi(x, y, z) \equiv Ax^3 + By^3 + Cz^3 + Fx^2y + Gy^2z + Hz^2x$

$$+ Lx^2z + My^2x + Nz^2y + Pxyz = 0;$$

we have, at once, for their nine points of intersection

$$\frac{x_1 x_2 \dots x_9}{U} = \frac{y_1 y_2 \dots y_9}{V} = \frac{z_1 z_2 \dots z_9}{W},$$

where $U$ denotes the eliminant of

$$by^3 + cz^3 + gy^2z + nz^2y \quad \text{and} \quad By^3 + Cz^3 + Gy^2z + Nz^2y,$$

$V$ denotes the eliminant of

$$ax^3 + cz^3 + hz^2x + lx^2z \quad \text{and} \quad Ax^3 + Cz^3 + Hz^2x + Lx^2z,$$

and $W$ denotes the eliminant of

$$ax^3 + by^3 + fx^2y + my^2x \quad \text{and} \quad Ax^3 + By^3 + Fx^2y + My^2x;$$

for, if $x$ be eliminated between $\phi = 0$ and $\Phi = 0$, the result is

$$Wy^9 + \dots + Vz^9 = 0,$$

and likewise for the elimination of $y$ and of $z$.

Now, suppose that the two cubics $\phi = 0$, $\Phi = 0$ are identical with the two cubics $A' = 0$, $B' = 0$. Then, after some reductions in the determinantal forms of the eliminants, we find

$$U = A_b^3 \{ C_n B_g N_g + C_g^2 B_g - C_b C_g B_n + C_n B_n^2 - 2 C_b C_n B_g + C_b^2 B_c \},$$

$$V = A_b^3 \{ C_h L_h A_l + \dots \},$$

$$W = A_b^3 \{ A_f M_f B_n + \dots \}.$$

As the relation $x_1 = 0$ does not necessitate either of the relations

$$y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 = 0, \quad z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8 z_9 = 0,$$

it follows that $x_1$ is a factor of the expression

$$C_n B_g N_g + \&c.$$

Similarly, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$, $x_8$ must be factors of the expression.

Similarly the products $y_1y_2y_3y_4y_5y_6y_7y_8$ and $z_1z_2z_3z_4z_5z_6z_7z_8$ must be factors of the expressions

$$C_kL_hA_l + \&c.,$$
$$A_fM_fB_m + \&c.$$

Hence we may write $$\frac{x_\theta}{O_a} = \frac{y_\theta}{O_b} = \frac{z_\theta}{O_c},$$

where $O_a$, $O_b$, $O_c$ are homogeneous functions of the eighth degree in the coordinates of each of the points 1, 2, 3, 4, 5, 6, 7, 8.

We might infer from these equations that, if seven of the points were given, and an eighth lay upon a given straight line, then the ninth must lie on an octavic curve having a triple point at each of the seven given points; but, for convenience, we will choose points 1, 2, 3 as coincident with the angular points $A$, $B$, $C$ of the triangle of reference.

In this case the equation of the cubic takes the form

$$fx^2y + gy^2z + hz^2x + lx^2z + my^2x + nz^2y + pxyz = 0,$$

and, since $$x_1 = y_2 = z_3 = 1$$

and $$y_1 = z_1 = z_2 = x_2 = x_3 = y_3 = 0,$$

each of the factors $B_g$, $C_n$, $C_h$, $A_l$, &c., is of the first degree in the coordinates which vanish, but $N_g$, $L_h$, $M_f$ do not vanish with these coordinates; hence, if we denote by $f$, $g$, $h$, $l$, $m$, $n$, $p$ the determinants formed by leaving out the first, second, ..., seventh columns in the scheme

$$\| x_i^2y_i,\ y_i^2z_i,\ z_i^2x_i,\ x_i^2z_i,\ y^2x_i,\ z_i^2y_i,\ x_iy_iz_i \|,$$

for $i = 0, 4, 5, 6, 7, 8$, where

$$x_0, y_0, z_0 = x, y, z,$$

and if further we denote the coefficients in the equations $f = 0$, $g = 0$, ..., $p = 0$ by a notation similar to one used already and explained by the following scheme:—

$$f \equiv f_g\ y^2z + f_h\ z^2x + f_l\ x^2z + f_m\ y^2x + f_n\ z^2y + f_p\ xyz = 0,$$
$$g \equiv g_f\ x^2y + g_h\ z^2x + g_l\ x^2z + g_m\ y^2x + g_n\ z^2y + g_p\ xyz = 0,$$
$$h \equiv h_f\ x^2y + h_g\ y^2z + h_l\ x^2z + h_m\ y^2x + h_n\ z^2y + h_p\ xyz = 0,$$
$$l \equiv l_f\ x^2y + l_g\ y^2z + l_h\ z^2x + l_m\ y^2x + l_n\ z^2y + l_p\ xyz = 0,$$
$$m \equiv m_fx^2y + m_gy^2z + m_hz^2x + m_i\ x^2z + m_n\ z^2y + m_pxyz = 0,$$
$$n \equiv n_f\ x^2y + n_g\ y^2z + n_h\ z^2x + n_l\ x^2z + n_m\ y^2x + n_p\ xyz = 0,$$
$$p \equiv p_f\ x^2y + p_g\ y^2z + p_h\ z^2x + p_l\ x^2z + p_m\ y^2x + p_n\ z^2y = 0,$$

we may write the equations in the form

$$\frac{x_1x_2x_3x_4x_5x_6x_7x_8x_9}{x_2m_gx_3h_hn_g+\&c.}=\frac{y_1y_2y_3y_4y_5y_6y_7y_8y_9}{y_3n_hy_1f_il_h+\&c.}=\frac{z_1z_2z_3z_4z_5z_6z_7z_8z_9}{z_1l_fz_2g_mm_f+\&c.}.$$

Hence
$$\frac{x_4\dots x_9}{m_gh_hn_g}=\frac{y_4\dots y_9}{n_hf_il_h}=\frac{z_4\dots z_9}{l_fg_mm_f},$$

or
$$\frac{x_4\dots x_9}{\frac{n_g}{l_f}}=\frac{y_4\dots y_9}{\frac{l_h}{m_g}}=\frac{z_4\dots z_9}{\frac{m_f}{n_h}}.$$

Now,
$$n_g=x_4x_5x_6x_7x_8S_1,$$

where $S_1$ is the determinant

$$|\ x_ry_r,\ x_rz_r,\ y_r^2,\ z_r^2,\ y_rz_r\ |,$$

for $r=4, 5, 6, 7, 8$.

Similarly,
$$l_h=y_4y_5y_6y_7y_8S_2,$$

and
$$m_f=z_4z_5z_6z_7z_8S_3,$$

with corresponding values for $S_2$ and $S_3$. Therefore we may write

$$\frac{x_9}{\frac{S_1}{l_f}}=\frac{y_9}{\frac{S_2}{m_g}}=\frac{z_9}{\frac{S_3}{n_h}}.$$

If the points 4, 5, 6, 7 be assumed given, the equation $S_1=0$ implies that the point 8 must lie on a conic through the points $A$, 4, 5, 6, 7. Similarly, $S_2=0$ is the conic through $B$, 4, 5, 6, 7; $S_3=0$ is the conic through $C$, 4, 5, 6, 7.

Similarly, the equation

$$l_f\equiv|\ y_r^2z_r,\ y_r^2x_r,\ z_r^2x_r,\ z_r^2y_r,\ x_ry_rz_r\ |=0,$$

where $r=4, 5, 6, 7, 8$, implies that the point 8 lies on a cubic which has a double point at $A$, and passes through the points $B, C$, 4, 5, 6, 7.

Similarly, $m_g=0$ is a cubic which has a double point at $B$, and passes through the points $A$, $C$, 4, 5, 6, 7; and $n_h=0$ is a cubic which has a double point at $C$ and passes through the points $A$, $B$, 4, 5, 6, 7.

We conclude that, if the point 9 lies on a given straight line

$$\lambda x+\mu y+\nu z=0,$$

then the point 8 must lie on an octavic curve

$$\lambda \frac{S_1}{l_f} + \mu \frac{S_2}{m_g} + \nu \frac{S_3}{n_h} = 0,$$

having a triple point at each of the points $A$, $B$, $C$, 4, 5, 6, 7.

Every such octavic contains only two arbitrary constants corresponding to the two constants determining the straight line on which the point 9 lies.

Two such octavics intersect nine times at each of the points $A$, $B$, $C$, 4, 5, 6, 7; and therefore have but one other intersection.

If the point 9 lies on a straight line through the point 1,

$$\mu y + \nu z = 0,$$

the octavic locus of the point 8 degenerates into the cubic $l_f = 0$ and a quintic curve

$$u \frac{S_2}{l_f} + \nu \frac{S_3}{n_h} = 0,$$

passing through $A$ and having double points at $B$, $C$, 4, 5, 6, 7.

Again, if the point 9 lies on the straight line $BC$ joining the points 2, 3, then the octavic degenerates into the two cubics $m_g = 0$, $n_h = 0$ and the conic $S_1 = 0$.

If the point 9 were upon the straight line joining two of the other points, 6, 7, then the octavic locus of the point 8 would degenerate into two cubics $C_6 = 0$, $C_7 = 0$ and a conic $S' = 0$. $C_6 = 0$ has a double point at 6 and passes through $A$, $B$, $C$, 4, 5, 7, and $C_7 = 0$ has a double point at 7 and passes through $A$, $B$, $C$, 4, 5, 6, and $S' = 0$ passes through $A$, $B$, $C$, 4, 5. Hence we see that two of the cubics, $m_g = 0$ and $C_6 = 0$, intersect twice at each of the points $B$, 6 and once at each of the points $A$, $C$, 4, 5, 7; and therefore at no other point.

Similarly, one of the cubics and one of the conics, say $n_h = 0$ and $S' = 0$, intersect twice at $C$ and once at each of the points $A$, $B$, 4, 5, 6; and therefore at no other point.

It appears therefore that in this case the point 8 is the fourth intersection of the two conics $S_1 = 0$ and $S' = 0$, both being conics circumscribing the triangle $ABC$—a result which might have been anticipated.

If the point 9 lies upon a conic

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \lambda yz + \mu zx + \nu xy = 0,$$

the point 8 must lie on a curve of the sixteenth degree

$$\alpha\frac{S_1^2}{l_f^2}+\beta\frac{S_2^2}{m_g^2}+\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having sextuple points at the points $A$, $B$, $C$, 4, 5, 6, 7. Two such sedecimics will intersect thirty-six times at each of the seven given points, and will have four other intersections corresponding to the four intersections of the two conics from which the sedecimics were obtained.

If the point 9 lies upon a conic

$$\beta y^2+\gamma z^2+\lambda yz+\mu zx+\nu xy=0,$$

passing through $A$, the sedecimic locus of 8 degenerates into the cubic $l_f=0$ and a tredecimic

$$\beta\frac{S_2^2}{m_g^2}+\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having a quadruple point at $A$ and quintuple points at $B$, $C$, 4, 5, 6, 7.

If the point 9 lies on a conic

$$\gamma z^2+\lambda yz+\mu zx+\nu xy=0,$$

passing through two of the points $A$, $B$, the locus of the point 8 degenerates into the two cubics $l_f=0$ and $m_g=0$ and a decimic curve

$$\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having triple points at $A$, $B$ and quadruple points at $C$, 4, 5, 6, 7.

If the point 9 lies on a conic

$$\lambda yz+\mu zx+\nu xy=0,$$

passing through three of the points $A$, $B$, $C$, then the locus of the point 8 degenerates into the cubics $l_f=0$, $m_g=0$, $n_h=0$ and a septimic

$$\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having double points at each of the points $A$, $B$, $C$ and triple points at each of the points 4, 5, 6, 7.

It has been seen three times in succession that, if the conic locus of the point 9 pass through one of the given points, the locus of the

point 8 degenerated and contained a cubic curve. It can be shown that this degeneration still continues when the conic locus of 9 passes through other of the fixed points.

If the conic locus of the point 9 pass through the point 4, then

$$\frac{\lambda}{x_4} + \frac{\mu}{y_4} + \frac{\nu}{z_4} = 0,$$

and the septimic may be written

$$\frac{\lambda}{z_4} l_f S_2 S_3 + \frac{\mu}{z_4} m_g S_3 S_1 - \left(\frac{\lambda}{x_4} + \frac{\mu}{y_4}\right) n_h S_1 S_2 = 0,$$

or

$$\lambda S_2 \left(\frac{l_f}{z_4} S_3 - \frac{n_h}{x_4} S_1\right) + \mu S_1 \left(\frac{m_g}{z_4} - \frac{n_n}{y_4} S_2\right) = 0.$$

Now, after reductions, we find

$$\frac{S_3}{z_4} l_f - \frac{S_1}{x_4} n_p = \frac{Y y_4}{P_5 \Delta_5} \Sigma_4,$$

$$\frac{S_3}{z_4} m_g - \frac{S_2}{y^4} n_h = \frac{X x_4}{P_5 \Delta_5} \Sigma_4,$$

where $P_5$ is the determinant $(y_5 z_5,\ z_6 x_6,\ x_7 y_7)$,

$\Delta_5$ ,, ,, $(x_5,\ y_6,\ z_7)$,

$X$ ,, ,, $(x^2,\ y_5 z_5,\ z_6 x_6,\ x_7 y_7)$,

$Y$ ,, ,, $(y^2,\ y_5 z_5,\ z_6 x_6,\ x_7 y_7)$,

and $Z$ ,, ,, $(z^2,\ y_5 z_5,\ z_6 x_6,\ x_7 y_7)$.

Hence the septimic may be written

$$\lambda S_2 \frac{Y y_4}{P_5 \Delta_5} \Sigma_4 + \mu S_1 \frac{X x_4}{P_5 \Delta_5} \Sigma_4 = 0,$$

and it therefore degenerates into the cubic $\Sigma_4 = 0$ and the quartic

$$\lambda S_2 Y y_4 + \mu S_1 X x_4 = 0.$$

Since $X = 0$, $Y = 0$, $Z = 0$ represent the conics $BC567$, $CA567$, $AB567$ respectively, the cubic $\Sigma_4 = 0$ passes through the six points $A$, $B$, $C$, 5, 6, 7 and has a double point at the point 4, and the quartic passes through the points $A$, $B$, $C$, 4 and has double points at 5, 6, 7.

If the conic locus of the point 9 pass also through the point 5, then

$$\frac{\lambda}{x_5}+\frac{\mu}{y_5}+\frac{\nu}{z_5}=0,$$

and therefore $\lambda : \mu = x_4x_5(y_4z_5-y_5z_4) : y_4y_5(z_4x_5-z_5x_4)$.

Using these values, we find that the quartic $\lambda S_2 Yy_4+\mu S_1 Xx_4=0$ degenerates into a line $D=(x, y_6, z_7)=0$ and into a cubic $\Sigma_5=0$, where we may write

$$\Sigma_5 = D_5PL+P_5DQ,$$

$P$ denoting the determinant $(yz, z_6x_6, x_7y_7)$, $Q$ denoting

$$x_5yz\begin{vmatrix} y_6, & z_6 \\ y_2, & z_7 \end{vmatrix}\begin{vmatrix} P_4, & P_5 \\ x_4D_4, & x_5D_5 \end{vmatrix} + \text{corresponding terms in } zx,\ xy,$$

and $L=0$ being a tangent to $Q=0$ at the point 5. Now this cubic $\Sigma_5$ passes through all the points $A$, $B$, $C$, 4, 5, 6, 7; manifestly it has a double point at the point 5, because $L=0$ is a tangent to the conic $Q=0$.

This proves that, in the case when the conic locus of 9 passes through the points $A$, $B$, $C$, 4, 5, the locus of the point 8 degenerates into the five cubics $l_f=0$, $m_g=0$, $n_h=0$, $\Sigma_4=0$, $\Sigma_5=0$, having double points at $A$, $B$, $C$, 4, 5 respectively, and the straight line $D=0$ joining the remaining pair of the seven fixed points 6, 7.

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*Thursday, March 10th,* 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Present, seventeen members and three visitors.

Mr. A. N. Whitehead was admitted into the Society, and then read a paper entitled "The Geodesic Geometry of Surfaces in non-Euclidean Space." Messrs. Berry, Macaulay, and Burnside joined in a discussion on the paper.

Prof. Burnside read a paper on "Linear Homogeneous Continuous Groups whose Operations are Permutable."

Mr. T. I. Dewar, in the absence of Prof. Greenhill, exhibited, with the aid of stereoscopes, some stereoscopic diagrams of Pseudo-Elliptic Catenaries and Geodesics. A unanimous vote of thanks was passed to Mr. Dewar.

Lt.-Col. Cunningham gave a short supplement to his paper "On Aurifeuillians."

The President briefly communicated a paper by Mr. W. F. Sheppard, "On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Division of a Scale";* and then (Lt.-Col. Cunningham in the Chair) read a paper "On the Transformation of Linear Partial Differential Operators by extended Linear Continuous Groups."

The following presents were made to the Library :—

"Queen's College, Galway, Calendar for 1897-98," 8vo; Dublin, 1898.

"Proceedings of the Royal Society," Vol. LXII., Nos. 384–5.

"Year-Book of the Royal Society," 1897–8.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 2; Leipzig, 1898.

"Nyt Tidsskrift for Matematik," Aargang 8, A, Nr. 6, 1898; B, Nr. 4, 1897; Copenhagen.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 42, 1897, Heft 3–4.

"Zeitschrift für Mathematik und Physik," Bd. XLIII., Heft 1; Leipzig, 1898.

"The Nautical Almanac for 1901," 8vo; London, 1898. From the Lords Commissioners of the Admiralty.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. IV., Fasc. 2; Napoli, 1898.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. IV., No. 5; New York, 1898.

Lorenz, L.—"Œuvres Scientifiques," revues et annotées, par H. Valentiner, Tome I., Fasc. 2, 8vo; Copenhague, 1898.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXVI., Heft 3; Berlin, 1898.

"Bulletin des Sciences Mathématiques," Tome XXII., Fév., 1898; Paris.

Bruno, Peter.—"Beobachtungen am sechszölligen Repsoldschen Heliometer der Leipziger Sternwarte," Abh. II., No. 3, 8vo; Leipzig, 1898.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLII., Pt. 1; 1897–98.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," Nos. XL.–LIII.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 2, 3; Roma, 1898.

Ball, Sir R.—"The Twelfth and Concluding Memoir on the Theory of Screws," 4to; Dublin, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XI.; Paris, 1897.

"Educational Times," March, 1898.

"Annals of Mathematics," Vol. XI., No. 6; Virginia, 1898.

Cayley, A.—"The Collected Mathematical Papers, Supplementary Volume containing Titles of Papers and Index," 4to; Cambridge, 1898.

"Indian Engineering," Vol. XXIII., Nos. 4–7, Jan. 22–Feb. 5, 1898.

* The original title of the paper has been changed in accordance with the suggestion of one of the referees.

# *The Geodesic Geometry of Surfaces in non-Euclidean Space.* By A. N. Whitehead. Received and read March 10th, 1898.

## *Introduction.*

The present paper is concerned with the properties of geodesic lines drawn on surfaces in elliptic and in hyperbolic space. There are two forms of elliptic space, which have been named by Burnside the "single" and the "double" forms. They have also been named the "polar" and "antipodal" forms. The antipodal (or double) form has also been called "spherical." In this paper the polar form of elliptic space will be the only one considered. Thus, both in the elliptic and the hyperbolic geometry of this paper, two straight lines in a plane intersect in one point only; and, in hyperbolic geometry, this point may lie in antispace (also called "ideal space").

Now the relations between the properties of geodesics on surfaces and non-Euclidean geometry, as far as they have hitherto been investigated, to my knowledge, are as follows:

It has been proved by Beltrami that the "geodesic geometry" of surfaces of constant curvature in *Euclidean* space is the same as the geometry of straight lines in planes in elliptic or in hyperbolic space, according as the curvature of the surface is positive or negative.

The geometry of great circles on a sphere of radius $\rho$ in elliptic space of "space-constant" $\gamma$ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sin \frac{\rho}{\gamma}$.

The geometry of great circles on a sphere of radius $\rho$ in hyperbolic space of "space-constant" $\gamma$ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sinh \frac{\rho}{\gamma}$

The geometry of geodesics (that is, lines of equal distance), on a surface of equal distance, $\sigma$, from a plane in hyperbolic space of space-constant $\gamma$, is the same as that of straight lines in planes in hyperbolic space of space-constant $\gamma \cosh \frac{\sigma}{\gamma}$.

Finally, the geometry of geodesics (that is, limit-lines), on a limit-surface in hyperbolic space—which may be conceived either as a sphere of infinite radius or as a surface of equal, but infinite, distance

from a plane—is the same as that of straight lines in planes in Euclidean space.

The preceding four propositions are due directly, or almost directly, to John Bolyai, though, of course, he only directly treats of hyperbolic space. I have given demonstrations of them elsewhere.*

From the popularization of Beltrami's results by Helmholtz, and from the unfortunate adoption of the name "radius of space curvature" for $\gamma$ (here called the space-constant), many philosophers, and, it may be suspected from their language, many mathematicians, have been misled into the belief that some peculiar property of flatness is to be ascribed to Euclidean space, in that planes of other sorts of space can be represented as surfaces in it. This idea is sufficiently refuted, at least, as regards hyperbolic space, by Bolyai's theorem respecting the geodesic geometry of limit-surfaces. For a Euclidean plane can thereby be represented by a surface in hyperbolic space.

[*Added, May*, 1898. It is to be noticed that the comparison of the lengths of the space-constants of different "spaces" is nonsense. A space-constant can only be said to be of length $\gamma$ in comparison with the length of some arbitrary unit straight line *in that space*. No comparison exists between lines in different spaces. But we may compare space-constants when we are really, as above, discussing the geodesic geometry of surfaces in a space of three dimensions; for, in this case, the test of congruence can be applied so as to compare the length of a geodesic on one surface with the length of any other line in the complete space.

All non-Euclidean geometry can be interpreted as geodesic geometry (allowing space of four dimensions). The preceding argument of this preface is directed against the assumption that this is the necessary interpretation, the refutation being based on the fact that Euclidean geometry can also be interpreted as geodesic geometry in a non-Euclidean space.

In the application of geometry to the interpretation of phenomena the idea of a fourth dimension is useless, as we have no such intuition. But the comparison of the space-constant $\gamma$ to our empirically given units, such as a mile or the earth's diameter, is the fundamental problem of applied geometry. In pure geometry a disembodied mind is contemplating a possible idea of space. To such a mind there are only three kinds of geometry: namely, hyperbolic, Euclidean, elliptic (polar and antipodal). But in applied geometry there is the mini-

* *Cf. A Treatise on Universal Algebra*, § 262.

mum length which can be perceived—some small fraction of an inch, —and a maximum length—namely, the distance of the furthest fixed star. These are empirically given lengths, and the properties of space in relation to our experience will be different according to the ratio of the space-constant to some intermediate empirically given standard length, such as the Earth's diameter.

It has been stated that any value of the space-constant may be assumed, since the laws of nature can be altered to suit. This is untrue if the possibility of measuring lines apart from any assumption of a special geometry is allowed. Thus the ratio of the circumference of a circle of radius $r$ to its diameter is $\pi\gamma \sin r/\gamma$, $\pi$, and $\pi\gamma \sinh r/\gamma$ in the three geometries respectively: accordingly, it is possible to determine $\gamma$ by actual measurement without any dependence on the more hypothetical laws of nature. The practical accuracy which can be thus attained is not to the point; the possibility of the experiment proves that $\gamma$ is not arbitrary.]

It is the object of this paper to extend and complete Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane. Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution. In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The surfaces fall into two main types, of which figures of meridian sections are given; the limit-surfaces form a transition case between these types.

In elliptic space there is only one type of such a surface of revolution; but it may have a finite or an infinite number of sheets according as a certain number is rational or irrational. The simplest surface of this type is one-sheeted, and a figure of it is given. Its equation, referred to the most convenient rectangular axes, is

$$3\,(\xi^2+\xi_3^2)^2\,\xi_3^2 = 3\,(3\xi^2+\xi_3^2)(\xi^2+\xi_3^2)(\xi_1^2+\xi_2^2)+(2\xi^2-\xi_3^2)(\xi_1^2+\xi_2^2)^2.$$

The same principles, only more laborious in their application, would enable the problem to be solved of the discovery in any kind of space of surfaces with their "geodesic" geometry identical with that of planes in any other kind of space. The distinguishing characteristic of surfaces is found, but I have not worked out the problem in detail.

As a preliminary to these investigations some general theorems are proved. If a family of geodesics and its orthogonal curves be traced on a surface in Euclidean geometry, Gauss has proved that the formula for an element of arc becomes

$$(\delta\sigma)^2 = (\delta\rho)^2 + \Pi^2 (\delta\phi)^2,$$

where the curves $\phi =$ constant are the geodesics, and the curves $\rho =$ constant are the orthogonal family. Such curvilinear co-ordinates are called in this paper "semi-geodesic orthogonal co-ordinates." Polar geodesic coordinates form a particular case of such coordinates. It is shown in this paper that all the well-known general properties of semi-geodesic orthogonal coordinates also hold for non-Euclidean space. In particular, the important equation

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \frac{\Pi}{\rho_1 \rho_2} = 0,$$

where $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are the principal measures of curvature of the surface at the point, becomes in elliptic space

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \left( \frac{1}{\gamma^2} + \frac{1}{\rho_1 \rho_2} \right) \Pi = 0;$$

and in hyperbolic space

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \left( \frac{1}{\rho_1 \rho_2} - \frac{1}{\gamma^2} \right) \Pi = 0.$$

In this connexion it is necessary to note that in non-Euclidean geometry a distinction has to be drawn between the inverse of the measure of curvature of a curve, which may be called the radius of curvature, and the radius of the circle through three consecutive points on the curve, which may be called the radius of the circle of curvature. If $\delta\sigma$ be a small arc, and $\delta\epsilon$ the angle between the tangents at its extremities, the radius of curvature is the limit of $\delta\sigma/\delta\epsilon$. Also, if this be called $\kappa$, and $\mu$ be the radius of the circle of curvature, then it is easy to prove * that

$$\kappa = \gamma \frac{\tan}{\tanh} \frac{\mu}{\gamma},$$

according as the space is elliptic or hyperbolic.

* *Cf. Universal Algebra*, § 288 (4).

The condition that a curve drawn on a surface in non-Euclidean space may be a geodesic is investigated, and this condition is put into different forms. Thus, if any point on the surface be denoted by curvilinear coordinates $\theta$ and $\phi$, the condition, that the curve

$$\theta = f(\tau), \quad \phi = F(\tau)$$

is a geodesic, is found in the form of a differential equation involving, $\theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta}, \ddot{\phi}$ : this equation is the same equation as that which holds for Euclidean space. In particular, if $\rho$ and $\phi$ be semi-geodesic orthogonal coordinates, and $\rho$ be taken as the independent variable, the simplified form of the differential equation of a geodesic, involving $\rho$, $\phi$, $\frac{d\phi}{d\rho}$, $\frac{d^2\phi}{d\rho^2}$, is found. This is also the same as the corresponding equation for Euclidean space.

Hence the proof given by Darboux for Beltrami's theorem on the geodesic representation of surfaces on planes is found to hold for non-Euclidean space; thus the theorem is extended to non-Euclidean space. The theorem is as follows:

Surfaces of constant curvature are the only surfaces for which geodesics can be transformed into straight lines when the surface is represented point for point on a plane. Surfaces of constant curvature are here named elliptic, parabolic, or hyperbolic, according as in elliptic geometry $\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2}$ is positive, zero, or negative, and according as in hyperbolic geometry $\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2}$ is positive, zero, or negative.

The "geodesic" geometry of an elliptic surface of constant curvature would be that of a plane in elliptic geometry; of a hyperbolic surface of constant curvature would be that of a plane in hyperbolic geometry. But these cases are not here considered in detail. The case of parabolic surfaces of constant curvature is considered in detail, with the results already mentioned.

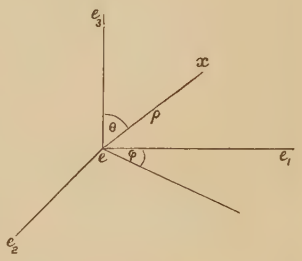
The elementary theorems respecting the curvature of surfaces in non-Euclidean geometry, and the application of Gauss' curvilinear coordinates to such surfaces, are given by Darboux, *Théorie générale des Surfaces*, Livre VII., chapter xiv. I have also developed them by the same methods and notation as are used in this paper, in *A Treatise on Universal Algebra*, Book VI., chapter vii.

The theorems in this paper are proved by the use of Grassmann's "Calculus of Extension." A sketch of the method of this Calculus,

especially in relation to non-Euclidean space, has been given by Buchheim in the *Proceedings* of this Society; *cf.* Vol. xv. The Calculus is also investigated in detail in the previously mentioned volume (*Universal Algebra*), and I have ventured to shorten my paper by reference to it for results assumed.

The brevity and facility of reasoning gained by the use of Grassmann's methods are very great. They can be immediately transformed into ordinary equations; but, by their use, the theorems of non-Euclidean geometry can be proved analytically even more easily than those of Euclidean geometry by ordinary methods.

1. Let the small italic letters represent points. Thus, let $e$ be the origin of coordinates, and $ee_1$, $ee_2$, $ee_3$ be three mutually rectangular axes. Also, let $e$, $e_1$, $e_2$, $e_3$ be a set of four quadrantal points; so that, if $\gamma$ be the "space-constant," the distance between any two of these points is $\frac{1}{2}\pi\gamma$, being half the complete length of a straight line. Let




$x$ be any point, let the length $ex$ be $\rho$, the angle $xee_3$ be $\theta$, and the angle between the planes $e_1ee_3$ and $xee_3$ be $\varphi$. Thus $\rho$, $\theta$, $\phi$ are analogous to the polar coordinates of Euclidean space. Also it is well known that, if we assume $\xi$, $\xi_1$, $\xi_2$, $\xi_3$, so that

$$\frac{\xi}{\cos\frac{\rho}{\gamma}} = \frac{\xi_1}{\sin\frac{\rho}{\gamma}\sin\theta\cos\phi} = \frac{\xi_2}{\sin\frac{\rho}{\gamma}\sin\theta\sin\phi} = \frac{\xi_3}{\sin\frac{\rho}{\gamma}\cos\theta},$$

then $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ have analogous properties partly to rectangular Cartesian coordinates, and partly to quadriplanar coordinates in Euclidean space. All the equations can be made homogeneous, and equations of the first degree represent planes. When the above proportions are turned into equations, so that

$$\xi = \cos\rho/\gamma, \ \&c.,$$

then $\xi, \xi_1, \xi_2, \xi_3$ will be called *the* rectangular coordinates of the point $x$. In this case they satisfy the equation

$$\xi^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1.$$

The symbols which represent points are combined according to the laws of Grassmann's "Calculus of Extension." Thus, if $x$ and $y$ be any two points,

$$xy = -yx \quad \text{and} \quad xx = 0.$$

Also $xy$ is called a "linear element" or a "force." Again, if $z$ and $u$ be two other points,

$$xyz = yzx = zxy = -yxz = \&c.$$

Also,
$$xxy = 0.$$

A product of three points, such as $xyz$, is called a "planar element." Again,

$$(xyzu) = -(uxyz) = (zuxy) = \&c.;$$

such a product is treated as an ordinary algebraic quantity, a mere number.

The relation of the point $x$ to the points $e$, $e_1$, $e_2$, $e_3$ is expressed by the equation

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$

where $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ are the quantities already defined by reference to the geometrical relations of $x$ and the axes $ee_1$, $ee_2$, $ee_3$. Then the products $xy$, $xyz$, $(xyzu)$ can be expressed in terms of the products of $e$, $e_1$, $e_2$, $e_3$, taken respectively two together, three together, and four together. But the mention of $(ee_1e_2e_3)$ is avoided by the convention that in elliptic geometry

$$(ee_1e_2e_3) = 1,$$

and in hyperbolic geometry

$$(ee_1e_2e_3) = \sqrt{-1}.$$

Thus, if

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$
$$y = \eta e + \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3,$$
$$z = \zeta e + \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3,$$
$$u = \nu e + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3,$$

then

$$xy = (\xi\eta_1 - \xi_1\eta)\, ee_1 + (\xi\eta_2 - \xi_2\eta)\, ee_2 + \&c.,$$

$$xyz = \begin{vmatrix} \xi, & \xi_1, & \xi_2 \\ \eta, & \eta_1, & \eta_2 \\ \zeta, & \zeta_1, & \zeta_2 \end{vmatrix} ee_1e_2 + \&c.,$$

and
$$(xyzu) = \begin{vmatrix} \xi, & \xi_1, & \xi_2, & \xi_3 \\ \eta, & \eta_1, & \eta_2, & \eta_3 \\ \zeta, & \zeta_1, & \zeta_2, & \zeta_3 \\ \upsilon, & \upsilon_1, & \upsilon_2, & \upsilon_3 \end{vmatrix}.$$

Accordingly $xyz = 0$ denotes that $x$, $y$, and $z$ are collinear, and $(xyzu) = 0$ denotes that $x$, $y$, $z$, $u$ are coplanar.

Also in elliptic geometry
$$(x \mid y) = \xi\eta + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$
and in hyperbolic geometry
$$(x \mid y) = \xi\eta - \xi_1\eta_1 - \xi_2\eta_2 - \xi_3\eta_3.$$
Furthermore, the following formulæ always hold:—
$$(xy \mid zu) = (x \mid z)(y \mid u) - (x \mid u)(y \mid z),$$
$$(xyz \mid uvw) = \begin{vmatrix} (x \mid u), & (x \mid v), & (x \mid w) \\ (y \mid u), & (y \mid v), & (y \mid w) \\ (z \mid u), & (z \mid v), & (z \mid w) \end{vmatrix}.$$

The most important special cases of these formulæ arise in the evaluation of the forms $(xy \mid xy)$ and $(xyz \mid xyz)$.

If $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ are the coordinates of the point $x$, then $(x \mid x) = 1$. The point $x$ is said to be at unit intensity. But, if $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ are only proportionals to these coordinates, then $+\sqrt{(x \mid x)}$ is called the intensity of the point $x$. It will usually be convenient to work with points at unit intensity.

The polar plane of a point $x$ with respect to the absolute is the planar element $\mid x$, the pole of a planar element $P$ is $\mid P$, the polar line of the linear element $xy$ is $\mid xy$. Also $\mid\mid x = -x$, $\mid\mid P = -P$, $\mid\mid xy = xy$, and $\mid xy = \mid x \,.\, \mid y$.

These formulæ, and others, are more particularly explained in the work (*Universal Algebra*, Books IV. and VI.) already referred to. But the preceding explanations will enable any reader unacquainted with the calculus of extension to translate the reasoning into the language of ordinary algebra.

Hence, with the notation of the calculus of extension as applied to this subject (*cf. Universal Algebra*, Book IV., chapter iii., also §§ 204, 210), the distance $\overline{xy}$ between two points $x$ and $y$ is given by
$$\cos\frac{\overline{xy}}{\gamma} = \frac{(x \mid y)}{\sqrt{\{(x \mid x)(y \mid y)\}}}, \quad \sin\frac{\overline{xy}}{\gamma} = \sqrt{\frac{(xy \mid xy)}{\{(x \mid x)(y \mid y)\}}}.$$

2. Now (*cf. Universal Algebra*, Book VI., chapter vii.) let $x$ be any unit point on a surface, so that throughout the subsequent reasoning we assume $(x \mid x) = 1$. Also let the position of $x$ on the surface be defined, according to Gauss' method, by two curvilinear coordinates $\theta$ and $\phi$, which are not to be identified with the $\theta$ and $\phi$ of the previous explanations respecting rectangular axes. Thus the equations $\theta =$ constant and $\phi =$ constant represent two families of curves traced on the surface, and the ordinary algebraic quantities which define $x$ are functions of $\theta$ and $\phi$.

Let $x_1$ stand for $\dfrac{\partial x}{\partial \theta}$, $x_{11}$ for $\dfrac{\partial^2 x}{\partial \theta^2}$, $x_2$ for $\dfrac{\partial x}{\partial \phi}$, $x_{12}$ for $\dfrac{\partial^2 x}{\partial \theta \partial \phi}$, $x_{22}$ for $\dfrac{\partial^2 x}{\partial \phi^2}$, and so on for partial differential coefficients of a higher order. Thus, if $x$ be written in the form

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$

$$x_1 = \frac{\partial \xi}{\partial \theta} e + \frac{\partial \xi_1}{\partial \theta} e_1 + \frac{\partial \xi_2}{\partial \theta} e_2 + \frac{\partial \xi_3}{\partial \theta} e_3,$$

and similarly for $x_{11}$, and so on. Then, by differentiating the equation

$$(x \mid x) = 1,$$

which is to hold for all positions of $x$ on the surface, we find

$$(x \mid x_1) = 0 = (x \mid x_2),$$

also

$$(x_1 \mid x_1) + (x \mid x_{11}) = 0,$$

$$(x_1 \mid x_2) + (x \mid x_{12}) = 0,$$

and so on.

The tangent plane at $x$ is represented by the planar element $xx_1x_2$ [*cf. Universal Algebra*, Book IV., chapters i. and ii., and § 293 (5)]; the normal at $x$ is represented by the linear element $\mid x_1 x_2$; the tangent lines to the curves $\phi =$ constant and $\theta =$ constant through $x$ are $xx_1$ and $xx_2$ respectively.

The angle between these tangent lines [*cf. Universal Algebra*, § 295 (2)] is

$$\cos^{-1} \frac{(x_1 \mid x_2)}{\sqrt{\{(x_1 \mid x_1)(x_2 \mid x_2)\}}}.$$

The length $(\delta\sigma)$ of the element of arc joining the points $x$ and $x + \delta x$ on the surface, where $x + \delta x$ corresponds to the values $\theta + \delta\theta$ and

$\phi+\delta\phi$ of the coordinates, is given (*cf. Universal Algebra*, § 293) by

$$\frac{(\delta\sigma)^2}{\gamma^2} = (x_1 \mid x_1)(\delta\theta)^2 + 2\,(x_1 \mid x_2)\,\delta\theta\,\delta\phi + (x_2 \mid x_2)(\delta\phi)^2.$$

Thus $(x_1 \mid x_1)$, $(x_1 \mid x_2)$, and $(x_2 \mid x_2)$ are the quantities which in Gauss' theory of surfaces are usually denominated by $E$, $F$, and $G$, respectively. Also

$$H^2 = EG - F^2 = (x_1 \mid x_1)(x_2 \mid x_2) - (x_1 \mid x_2)^2 = (x_1 x_2 \mid x_1 x_2).$$

The advantages of the forms $(x_1 \mid x_1)$, &c., are that they are nearly as short as the single letters $E$, $F$, $G$, and at the same time they express completely the rules for their formation and transformation.

Also, if $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ be the principal *measures of curvature* of the surface at $x$, then [*cf. Universal Algebra*, § 294 (3)]

$$\frac{\gamma^2\{(x_1 \mid x_1)(x_2 \mid x_2) - (x_1 \mid x_2)^2\}^2}{\rho_1\rho_2} = \Delta_1 - \Delta_2 + \{(x_1 \mid x_1)(x_2 \mid x_2) - (x_1 \mid x_2)^2\}$$

$$\times \{(x_1 \mid x_2)^2 - (x_1 \mid x_1)(x_2 \mid x_2) + (x_1 \mid x_2)_{12} - \tfrac{1}{2}(x_2 \mid x_2)_{11} - \tfrac{1}{2}(x_1 \mid x_1)_{22}\}, \quad (1)$$

where $$(x_1 \mid x_2)_{12} = \frac{\partial^2}{\partial\theta\partial\phi}(x_1 \mid x_2),$$

with similar meaning for the other terms of similar form; and $\Delta_1$ and $\Delta_2$ stand for the determinants

$$\Delta_1 = \begin{vmatrix} (x_1 \mid x_1), & (x_1 \mid x_2), & (x_1 \mid x_2)_2 - \tfrac{1}{2}(x_2 \mid x_2)_1 \\ (x_1 \mid x_2), & (x_2 \mid x_2), & \tfrac{1}{2}(x_2 \mid x_2)_2 \\ \tfrac{1}{2}(x_1 \mid x_1)_1, & (x_1 \mid x_2)_1 - \tfrac{1}{2}(x_1 \mid x_1)_2, & 0 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} (x_1 \mid x_1), & (x_1 \mid x_2), & \tfrac{1}{2}(x_1 \mid x_1)_2 \\ (x_1 \mid x_2), & (x_2 \mid x_2), & \tfrac{1}{2}(x_2 \mid x_2)_1 \\ \tfrac{1}{2}(x_1 \mid x_1)_2, & \tfrac{1}{2}(x_2 \mid x_2)_1, & 0 \end{vmatrix}.$$

3. The equations $\theta = f(\tau), \quad \phi = F(\tau),$

where $\tau$ is some independent variable, and $f$ and $F$ are functional signs, define a curve on the surface.

Let the point (at unit intensity) corresponding to the value $\tau+\delta\tau$

of the variable be denoted by

$$x+\dot{x}\,\delta\tau+\tfrac{1}{2}\ddot{x}\,\delta\tau^2+\dots;$$

then
$$\dot{x}=\frac{dx}{d\tau},\quad \ddot{x}=\frac{d^2x}{d\tau^2},$$

and so on. Hence

$$\dot{x}=x_1\dot{\theta}+x_2\dot{\phi},$$

$$\ddot{x}=x_1\ddot{\theta}+x_2\ddot{\phi}+x_{11}\dot{\theta}^2+2x_{12}\dot{\theta}\dot{\phi}+x_{22}\dot{\phi}^2.$$

Also the length of an element of arc ($\delta\sigma$) on the curve is given by

$$\left(\frac{\delta\sigma}{\gamma}\right)^2=(\dot{x}\mid\dot{x})\,\delta\tau^2.$$

Thus, if $\tau$ denote the length of the arc of the curve measured from a fixed point, then in this special case

$$(\dot{x}\mid\dot{x})=\frac{1}{\gamma^2},$$

since
$$\delta\tau=\delta\sigma.$$

The osculating plane of the curve is [*cf. Universal Algebra*, § 287 (6)] $x\dot{x}\ddot{x}$; the tangent line is $x\dot{x}$.

Let $\omega$ be the angle between the normal section at $x$ containing the tangent line $x\dot{x}$ and the osculating plane of the curve. Then the angle between the tangent plane at $x$ and the osculating plane at $x$ is $\frac{1}{2}\pi-\omega$. Thus (*cf. Universal Algebra*, § 221)

$$\sin\omega=\frac{(x\dot{x}\ddot{x}\mid xx_1x_2)}{\sqrt{\{(x\dot{x}\ddot{x}\mid x\dot{x}\ddot{x})(xx_1x_2\mid xx_1x_2)\}}}.$$

Now, if $\frac{1}{\kappa}$ be the measure of curvature of the curve, that is $\frac{\delta\epsilon}{\delta\sigma}$, where $\delta\epsilon$ is the angle of contingence, it is easily proved [*cf. Universal Algebra*, § 288 (1)] that

$$\frac{\gamma^2}{\kappa^2}=\frac{(x\dot{x}\ddot{x}\mid x\dot{x}\ddot{x})}{(\dot{x}\mid\dot{x})^3}.$$

Also
$$(xx_1x_2\mid xx_1x_2)=(x_1x_2\mid x_1x_2),$$

since
$$(x\mid x_1)=0=(x\mid x_2);$$

and similarly $(x\dot{x}\ddot{x} \mid xx_1x_2) = (\dot{x}\ddot{x} \mid x_1x_2)$.

Thus $$\sin\omega = \frac{\kappa\,(\dot{x}\ddot{x} \mid x_1x_2)}{\gamma\sqrt{\{(\dot{x} \mid \dot{x})^3\,(x_1x_2 \mid x_1x_2)\}}}.$$

Let $\frac{\sin\omega}{\kappa}$ be defined as the geodesic curvature of the curve, and let it be written $\frac{1}{\mu_g}$. Then

$$\frac{1}{\mu_g} = \frac{(\dot{x}\ddot{x} \mid x, x_2)}{\gamma\sqrt{\{(\dot{x} \mid \dot{x})^3\,(x_1x_2 \mid x_1x_2)\}}}.$$

The condition for a geodesic is that

$$(\dot{x}\ddot{x} \mid x_1x_2) = 0 \qquad (2)$$

may hold at every point of the curve. For this condition secures that the osculating planes of the curve are normal planes of the surface; also, since the geometry of infinitely small figures becomes ultimately that of Euclidean space, the condition is thereby secured that the length of each element of arc of the curve is a minimum distance (on the surface) between its extremities.

This equation can be put into another form, which explicitly relates $\theta$ and $\phi$ and their differential coefficients with respect to $\tau$. For

$$\dot{x} = x_1\dot{\theta} + x_2\dot{\phi},$$

$$\ddot{x} = x_1\ddot{\theta} + x_2\ddot{\varphi} + x_{11}\dot{\theta}^2 + 2x_{12}\dot{\theta}\dot{\phi} + x_{22}\dot{\phi}^2;$$

hence $$\dot{x}\ddot{x} = (\dot{\theta}\ddot{\varphi} - \ddot{\theta}\dot{\phi})\,x_1x_2 + x_1x_{11}\dot{\theta}^3 + (2x_1x_{12} + x_2x_{11})\,\dot{\theta}^2\dot{\varphi} + (2x_2x_{12} + x_1x_{22})\,\dot{\theta}\dot{\phi}^2 + x_2x_{22}\dot{\phi}^3.$$

Hence the equation $(\dot{x}\ddot{x} \mid x_1x_2) = 0$

becomes

$$(\dot{\theta}\ddot{\varphi} - \ddot{\theta}\dot{\phi})(x_1x_2 \mid x_1x_2) + \dot{\theta}^3\,(x_1x_{11} \mid x_1x_2) + \{2\,(x_1x_{12} \mid x_1x_2) + (x_2x_{11} \mid x_1x_2)\}\,\dot{\theta}^2\dot{\varphi} + \{2\,(x_2x_{12} \mid x_1x_2) + (x_1x_{22} \mid x_1x_2)\}\,\dot{\theta}\dot{\phi}^2 + (x_2x_{22} \mid x_1x_2)\,\dot{\varphi}^3 = 0.$$

The coefficients can easily be expressed in terms of the differential coefficients of $(x_1 \mid x_1)$, $(x_1 \mid x_2)$, $(x_2 \mid x_2)$. For

$$(x_1x_{11} \mid x_1x_2) = (x_1 \mid x_1)(x_{11} \mid x_2) - (x_1 \mid x_2)(x_1 \mid x_{11})$$

$$= (x_1 \mid x_1)(x_1 \mid x_2)_1 - \tfrac{1}{2}\,(x_1 \mid x_1)(x_1 \mid x_1)_2 - \tfrac{1}{2}\,(x_1 \mid x_2)(x_1 \mid x_1)_1,$$

and $2(x_1x_{12} \mid x_1x_2) + (x_2x_{11} \mid x_1x_2)$

$$= \tfrac{1}{2}(x_1x_2 \mid x_1x_2)_1 + \tfrac{1}{2}(x_1 \mid x_1)(x_2 \mid x_2)_1 - \tfrac{1}{2}(x_1 \mid x_2)(x_1 \mid x_1)_2$$

with similar equations for the coefficients of $\dot\theta\dot\phi^2$ and of $\dot\phi^3$.

Hence equation (2) can be written in the form

$$\begin{aligned} 2(\dot\theta\ddot\phi - \ddot\theta\dot\phi)(x_1x_2 \mid x_1x_2) \\ = \quad & \{(x_1 \mid x_1)(x_1 \mid x_1)_2 + (x_1 \mid x_2)(x_1 \mid x_1)_1 - 2(x_1 \mid x_1)(x_1 \mid x_2)_1\}\dot\theta^3 \\ & + \{3(x_1 \mid x_2)(x_1 \mid x_1)_2 - 3(x_1 \mid x_1)(x_2 \mid x_2)_1 + (x_1x_2 \mid x_1x_2)_1\}\dot\theta^2\dot\phi \\ & - \{3(x_1 \mid x_2)(x_2 \mid x_2)_1 - 3(x_2 \mid x_2)(x_1 \mid x_1)_2 + (x_1x_2 \mid x_1x_2)_2\}\dot\theta\dot\phi^2 \\ & - \{(x_2 \mid x_2)(x_2 \mid x_2)_1 + (x_1 \mid x_2)(x_2 \mid x_2)_2 - 2(x_2 \mid x_2)(x_1 \mid x_2)_2\}\dot\phi^3. \quad (3) \end{aligned}$$

It is easy to verify that this equation for geodesics on surfaces in elliptic space is the same as that for Euclidean space given by Darboux in his *Leçons sur la Théorie générale des Surfaces*, Livre v., chapter iv., § 314. It is obvious that equation (3) must be the same in Euclidean and in non-Euclidean geometry; for, in each case, it is the immediate expression of the fact that $E\delta\theta^2 + 2F\delta\theta\delta\phi + G\delta\phi^2$ is a minimum. Equation (3) is practically an interpretation of equation (2) in terms of ordinary algebra, and is, therefore, useful when it is required to integrate it or to discuss it in any way in connexion with the theory of differential equations.

The equation of a curve on a surface can be put into the form

$$\psi(\theta, \phi) = \text{constant},$$

thus avoiding the use of an auxiliary variable $\tau$. The preceding formula can be transformed into this notation. For

$$\dot x = x_1\dot\theta + x_2\dot\phi,$$

where

$$\frac{\partial\psi}{\partial\theta}\dot\theta + \frac{\partial\psi}{\partial\phi}\dot\phi = 0.$$

Now let $\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\psi}{\partial\phi}$ be denoted by $\psi_1$ and $\psi_2$ respectively. Then

$$\frac{\dot\theta}{\psi_2} = \frac{\dot\phi}{-\psi_1} = \frac{\dot\sigma}{\gamma\surd\{(x_1 \mid x_1)\psi_2^2 - 2(x_1 \mid x_2)\psi_1\psi_2 + (x_1 \mid x_1)\psi_1^2\}},$$

where $\dot\sigma$ stands for $\dfrac{d\sigma}{d\tau}$.

Now, following Beltrami's notation for Euclidean space, put

$$\{(x_1 \mid x_1)\,\psi_2^2 - 2\,(x_1 \mid x_2)\,\psi_1\psi_2 + (x_2 \mid x_2)\,\psi_1^2\} / (x_1x_2 \mid x_1x_2) = \Delta\psi.$$

Hence
$$\frac{\dot\theta}{\psi_2} = \frac{\dot\phi}{-\psi_1} = \frac{\dot\sigma}{\gamma\{(x_1x_2 \mid x_1x_2)\,\Delta\psi\}^{\frac{1}{2}}} = \frac{\dot\sigma}{\lambda},\ \text{say},$$

where
$$\lambda = \gamma\{(x_1x_2 \mid x_1x_2)\,\Delta\psi\}^{\frac{1}{2}}.$$

Hence, choosing $\tau$ so that $\dot\sigma = 1$,

$$\dot x = \frac{1}{\lambda}(x_1\psi_2 - x_2\psi_1) = \frac{1}{\lambda}\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right)x.$$

Now let $\delta$ stand for the symbolic operator

$$\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right);$$

also let
$$\delta^2 = \left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right)\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right).$$

Then
$$\dot x = \frac{1}{\lambda}\delta x;$$

also
$$(\dot x \mid \dot x) = \frac{1}{\gamma^2} = \frac{1}{\lambda^2}(\delta x \mid \delta x).$$

Similarly,
$$\ddot x = \frac{1}{\lambda}\delta\frac{1}{\lambda}\delta x + \frac{1}{\lambda^2}\delta^2 x.$$

Hence
$$\dot x\ddot x = \frac{1}{\lambda^3}\delta x\delta^2 x.$$

Thus
$$\frac{\gamma}{\kappa} = \frac{(x\,\delta x\,\delta^2 x \mid x\,\delta x\,\delta^2 x)^{\frac{1}{2}}}{\{(x_1x_2 \mid x_1x_2)\,\Delta\psi\}^{\frac{3}{2}}};$$

also
$$\frac{\gamma}{\mu_g} = \frac{(\delta x\,\delta^2 x \mid x_1x_2)}{(x_1x_2 \mid x_1x_2)(\Delta\psi)^{\frac{3}{2}}}.$$

The condition for a geodesic is given by

$$(\delta x\,\delta^2 x \mid x_1x_2) = 0. \qquad (4)$$

Thus, in the special case in which $\psi = \phi$, then $\psi_2 = 1$ and $\psi_1 = 0$; hence $\delta$ becomes $\dfrac{\partial}{\partial\theta}$. Thus the condition for $\phi =$ constant being a geodesic is that $(x_1x_{11} \mid x_1x_2) = 0$ at all points of it.

4. Now assume that the curves $\phi =$ constant are a family of geodesics, and that the curves $\theta =$ constant are the family of curves cutting the geodesics at right angles.

Then the condition that the two families are everywhere orthogonal is

$$(x_1 \mid x_2) = 0.$$

The condition that the family $\phi =$ constant is geodesic is

$$(x_1 x_{11} \mid x_1 x_2) = 0;$$

that is,
$$(x_1 \mid x_1)(x_{11} \mid x_2) - (x_1 \mid x_2)(x_1 \mid x_{11}) = 0.$$

Hence, using the condition of orthogonality, we find

$$(x_{11} \mid x_2) = 0.$$

But, by differentiating $(x_1 \mid x_2) = 0$ partially with respect to $\phi$, we find

$$(x_{11} \mid x_2) + (x_1 \mid x_{12}) = 0.$$

Hence
$$(x_1 \mid x_{12}) = 0.$$

But
$$(x_2 \mid x_{12}) = \tfrac{1}{2}(x_1 \mid x_1)_2.$$

Hence
$$(x_1 \mid x_1)_2 = 0;$$

that is to say, $(x_1 \mid x_1)$ is independent of $\phi$. Now

$$\frac{\delta\sigma^2}{\gamma} = (x_1 \mid x_1)\,\delta\theta^2 + (x_2 \mid x_2)\,\delta\phi^2.$$

But, since $(x_1 \mid x_1)$ is independent of $\phi$, we may write

$$\int (x_1 \mid x_1)\, d\theta = \frac{\rho}{\gamma},$$

and use $\rho$ and $\phi$ as coordinates instead of $\theta$ and $\phi$. The curves $\rho =$ constant and $\phi =$ constant are the same as the curves $\theta =$ constant and $\phi =$ constant. Also let the subscript 1 denote for the future partial differentiation with respect to $\rho$. Then

$$\delta\sigma^2 = \delta\rho^2 + \gamma^2 (x_2 \mid x_2)\,\delta\phi^2.$$

Thus, if a family of geodesics be cut orthogonally, the distance between any two of the orthogonal curves measured along an orthogonal geodesic is the same for that pair of curves whatever geodesic be chosen. For consider the pair of curves $\rho = \rho_1$ and $\rho = \rho_2$; and let

the geodesic be $\phi = a$. Then an element of arc along the geodesic is

$$\delta\sigma = \delta\rho.$$

Then $$\sigma = \rho_2 \sim \rho_1.$$

Let $\rho_2 \sim \rho_1$ be called the "geodesic distance between the curves."

5. Conversely, assume that the curves $\phi =$ constant are a family of geodesics, and that the length of the arc of any geodesic $\phi = a$, between any given pair of curves $\theta = \theta_1$ and $\theta = \theta_2$, is the same for all values of $a$. It is required to prove that the two families are orthogonal, if any particular curve $\theta = \theta_0$ is orthogonal to all the curves $\phi =$ constant. The conditions are

$$(x_1 x_{11} \mid x_1 x_2) = 0,$$

and that $(x_1 \mid x_1)$ is independent of $\phi$. For then the element of the arc of $\phi = a$ is given by

$$\delta\sigma = \gamma\sqrt{(x_1 \mid x_1)}\,\delta\theta,$$

and is independent of $a$.

It follows from the second condition, by differentiating with respect to $\phi$, that

$$(x_1 \mid x_{12}) = 0.$$

Hence $$(x_1 \mid x_2)_1 = (x_{11} \mid x_2) + (x_1 \mid x_{12}) = (x_{11} \mid x_2).$$

Again, the first condition becomes, after performing the multiplications indicated,

$$(x_1 \mid x_1)(x_2 \mid x_{11}) - (x_1 \mid x_2)(x_1 \mid x_{11}) = 0;$$

that is, $$(x_1 \mid x_1)(x_1 \mid x_2)_1 - \tfrac{1}{2}(x_1 \mid x_2)(x_1 \mid x_1)_1 = 0.$$

Hence, $$2\frac{(x_1 \mid x_2)_1}{(x_1 \mid x_2)} = \frac{(x_1 \mid x_1)_1}{(x_1 \mid x_1)}.$$

Hence, by integrating, we find

$$(x_1 \mid x_2)^2 = \Phi(x_1 \mid x_1);$$

where $\Phi$ is a function of $\varphi$ only. But, by hypothesis, when $\theta = \theta_0$,

$$(x_1 \mid x_2) = 0,$$

for all values of $\phi$. Hence

$$\Phi(x_1 \mid x_1)_{\theta=\theta_0} = 0,$$

for all values of $\phi$. But $(x_1 \mid x_1)$ cannot vanish, for an element of the arc of any geodesic $\phi = a$ at the point $(\theta_0, a)$ is

$$\delta\sigma = \gamma(x_1 \mid x_1)\,\delta\theta.$$

Thus $\Phi = 0$ for all values of $\phi$. Hence

$$(x_1 \mid x_2) = 0$$

at all points of the surface.

6. Collecting results, we see that, if the coordinate curves are geodesics and their orthogonals, and if the length $\rho$ of the geodesic arc from a given orthogonal be taken as one coordinate, then

$$(x \mid x_2) = \frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0.$$

To these must be added the general condition that $x$ is a unit point, namely,

$$(x \mid x) = 1.$$

Also all equations hold which can be deduced from these by partial differentiations with respect to $\rho$ or $\phi$.

Any element of arc $\delta\sigma$ takes the form

$$(\delta\sigma)^2 = (\delta\rho)^2 + \gamma^2 (x_2 \mid x_2)(\delta\phi)^2 = \delta\rho^2 + \gamma^2 \Pi^2 (\delta\phi)^2, \qquad (5)$$

where $\Pi^2$ is written for $(x_2 \mid x_2)$. Let this type of coordinates be called "semi-geodesic orthogonal coordinates."

7. A special form of such coordinates can always be found as follows:—Let the family of geodesics be the geodesics issuing from any point $c$ on the surface; let $\rho$ be the arc of the geodesic through the point $x$, measured from $c$, and let $\phi$ be the angle which that geodesic makes at $c$ with a given geodesic of the family.

Let such coordinates be called geodesic polar coordinates; and let the curves $\rho =$ constant be called "geodesic circles centre $c$," and the curves $\phi =$ constant be called the "geodesic radii from $c$."

It is easily seen that the orthogonal family to the geodesic radii is the family of geodesic circles. For, by § 5, we have only to prove that one of the geodesic circles is orthogonal to the radii. But it is obvious that a geodesic circle with an infinitely small radius, $\rho$, is orthogonal, since the figure becomes ultimately a plane figure.

Since geodesic circles with small radii are ultimately plane, and since the properties of small figures are ultimately the same as those of figures in Euclidean space, it follows that an element of arc of the circle $\rho = a$, when $a$ is small, takes the form $\rho\,\delta\phi$. Hence the limit of $\Pi$, when $\rho$ is small, is $\rho/\gamma$. This result is only necessarily true for polar geodesic coordinates.

8. The formula (1) in § 2 for $\frac{1}{\rho_1\rho_2}$ becomes, using the semi-geodesic orthogonal coordinates of § 6 and simplifying,

$$\frac{(x_2 \mid x_2)^2}{\gamma^2\rho_1\rho_2} = \Delta_1 - \Delta_2 + \frac{(x_2 \mid x_2)}{\gamma^2}\left\{-\frac{(x_2 \mid x_2)}{\gamma^2} - \tfrac{1}{2}(x_2 \mid x_2)_{11}\right\},$$

where $$\Delta_1 = 0, \quad \Delta_2 = -\tfrac{1}{4}\frac{(x_2 \mid x_2)_1^2}{\gamma^2}.$$

Hence $$\frac{(x_2 \mid x_2)^2}{\rho_1\rho_2} = \tfrac{1}{4}(x_2 \mid x_2)_1 - \frac{(x_2 \mid x_2)^2}{\gamma^2} - \tfrac{1}{2}(x_2 \mid x_2)(x_2 \mid x_2)_{11}.$$

Now $$\Pi^2 = (x_2 \mid x_2).$$

Therefore, substituting and reducing, we find

$$\frac{\partial^2\Pi}{\partial\rho^2} + \left(\frac{1}{\gamma^2} + \frac{1}{\rho_1\rho_2}\right)\Pi = 0. \tag{6}$$

This is the analogue in elliptic geometry of Gauss's well-known equation in Euclidean geometry for the measure of curvature at any point of a surface in terms of semi-geodesic orthogonal coordinates (*cf.* Salmon's *Solid Geometry*, 3rd ed., § 329, and Darboux, § 524. Darboux also considers for Euclidean space semi-geodesic orthogonal coordinates, Livre v., chapter v.). If $\gamma$ be made infinite, the equation reduces to the Euclidean form, as it ought to do.

9. Now, transforming equation (3) into the special form which it assumes when the semi-geodesic orthogonal coordinates $(\rho, \phi)$ are substituted, we obtain as the equation satisfied by geodesics, after division by $2\Pi^2/\gamma^4$,

$$(\dot\rho\,\ddot\phi - \ddot\rho\,\dot\phi) = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}\dot\rho^2\dot\phi - \frac{1}{\Pi}\frac{\partial\Pi}{\partial\phi}\dot\rho\,\dot\phi^2 - \Pi\frac{\partial\Pi}{\partial\rho}\dot\phi^3. \tag{7}$$

Also $\rho$ can be taken instead of $\tau$ as the independent variable. Hence

$$\dot\rho = 1, \quad \ddot\rho = 0.$$

Thus the equation (7) becomes

$$\ddot\phi = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}\dot\phi - \frac{1}{\Pi}\frac{\partial\Pi}{\partial\phi}\dot\phi^2 - \Pi\frac{\partial\Pi}{\partial\rho}\dot\phi^3. \tag{8}$$

This is the same as the equation for Euclidean geometry given by Darboux, § 598.

10. Beltrami's theorem, that the surfaces of constant curvature are the only surfaces for which geodesics are transformed into straight lines when the surface is represented on a plane, can now be proved by exactly the same method as is employed by Darboux (*cf.* § 598) to prove it for Euclidean space.

The following proof is substantially taken from Darboux :—

Let the point $e_0$ on the plane correspond to the point $e$ on the surface. Then the straight lines through $e_0$ correspond to the geodesics on the surface through $e$. Also, if $\rho_0$ and $\theta$ be the polar coordinates of any point on the plane, and $\rho$ and $\phi$ the polar geodesic coordinates of the corresponding point on the surface, we must have

$$\rho_0 = f(\rho, \phi), \quad \theta = F(\phi),$$

where $\theta$ and $\varphi$ are measured respectively from a corresponding pair formed by a straight line through $e_0$ and a geodesic through $e$. But the general equation of a straight line is

$$\alpha \sin \frac{\rho_0}{\gamma} \cos\theta + \beta \sin \frac{\rho_0}{\gamma} \sin\theta + \delta = 0.$$

Hence
$$\alpha + \beta\mu + \delta\sigma = 0,$$

where $\mu$ is a function of $\phi$ only and $\sigma$ is a function of $\rho$ and $\phi$, is the general integral for geodesics on the surface.

But the differential equation (8) can be written, substituting $\mu$ for $\phi$ as the coordinate,

$$\ddot{\mu} = -\frac{2}{\Pi}\frac{\partial \Pi}{\partial \rho}\dot{\mu} - \frac{1}{\Pi}\frac{\partial \Pi}{\partial \mu}\dot{\mu}^2 - \Pi\frac{\partial \Pi}{\partial \rho}\dot{\mu}^3.$$

But, differentiating the integral form, we have

$$\beta\dot{\mu} - \delta(\sigma_1 + \sigma_2\dot{\mu}) = 0,$$
$$\beta\ddot{\mu} + \delta(\sigma_{11} + 2\sigma_{12}\dot{\mu} + \sigma_{22}\dot{\mu}^2 + \sigma_2\ddot{\mu}) = 0.$$

Eliminating $\beta$ and $\delta$,

$$\ddot{\mu}(\sigma_1 + \sigma_2\dot{\mu}) = \dot{\mu}(\sigma_{11} + 2\sigma_{12}\dot{\mu} + \sigma_{22}\dot{\mu}^2 + \sigma_2\ddot{\mu}).$$

Hence
$$\ddot{\mu} = \frac{\sigma_{11}}{\sigma_1}\dot{\mu} + \frac{2\sigma_{12}}{\sigma_1}\dot{\mu}^2 + \frac{\sigma_{22}}{\sigma_1}\dot{\mu}^3.$$

Accordingly, by comparison,

$$\frac{\sigma_{11}}{\sigma_1} = -\frac{2}{\Pi}\frac{\partial \Pi}{\partial \rho}, \quad \frac{2\sigma_{12}}{\sigma_1} = -\frac{1}{\Pi}\frac{\partial \Pi}{\partial \mu}, \quad \frac{\sigma_{22}}{\sigma_1} = -\Pi\frac{\partial \Pi}{\partial \rho}. \tag{9}$$

From the first two of equations (9) we deduce

$$\Pi^2\sigma_1 = \frac{1}{\Phi^3}, \quad \Pi\sigma_1^2 = \frac{1}{P^3},$$

where $\Phi$ and P are arbitrary functions of $\mu$ and $\rho$ respectively. Hence

$$\Pi = \frac{P}{\Phi^2}, \quad \sigma_1 = \frac{\Phi}{P^2}.$$

Thus

$$\sigma = \Phi \int \frac{d\rho}{P^2} + M,$$

where M is another arbitrary function of $\mu$ only. Hence, from the third of equations (9),

$$\Phi_{22} \int \frac{d\rho}{P^2} + \mu_{22} = -\frac{\Phi}{P^2}\frac{P}{\Phi^2}\frac{P_1}{\Phi^2} = -\frac{P_1}{P\Phi^3}.$$

Differentiating both sides with respect to $\rho$,

$$\frac{\Phi_{22}}{P^2} = -\frac{1}{\Phi^3}\frac{d}{d\rho}\left(\frac{P_1}{P}\right).$$

Hence

$$\Phi^3\Phi_{22} = -P^2\frac{d}{d\rho}\left(\frac{P_1}{P}\right).$$

But the left-hand side is a function of $\mu$ only, and the right-hand side is a function of $\rho$ only. Hence both sides are constant. Thus

$$\Phi^3\Phi_{22} = -\epsilon, \quad P^2\frac{d}{d\rho}\left(\frac{P_1}{P}\right) = \epsilon.$$

The second equation can be written

$$\frac{P_1}{P}\frac{d}{d\rho}\left(\frac{P_1}{P}\right) = \frac{\epsilon P_1}{P^3}.$$

Hence, by integration,

$$\tfrac{1}{2}\frac{P_1^2}{P^2} = -\frac{\epsilon}{2P^2} + \tfrac{1}{2}\eta,$$

where $\eta$ is constant. Hence

$$P_1^2 = -\epsilon + \eta P^2.$$

Therefore

$$P_1 P_{11} = \eta P P_1, \quad P_{11} = \eta P.$$

Now, from equation (6),

$$\frac{1}{\Pi}\frac{\partial^2\Pi}{\partial\rho^2} = -\left(\frac{1}{\gamma^2} + \frac{1}{\rho_1\rho_2}\right).$$

But

$$\Pi = \frac{P}{\Phi^2};$$

therefore $$\frac{1}{\Pi}\frac{\partial^2\Pi}{\partial\rho^2} = \frac{P_{11}}{P} = \eta.$$

Hence $$\frac{1}{\rho_1\rho_2} = -\eta - \frac{1}{\gamma^2}.$$

Thus the only surfaces for which the geodesics correspond to straight lines are surfaces of constant curvature.

11. There are three types of surfaces of constant curvature, according as $\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2}$ is positive, negative, or zero.

Let $$\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2} = \frac{1}{\epsilon^2}, \quad -\frac{1}{\epsilon^2}, \text{ or } 0,$$

in the three cases.

$$\left.\begin{aligned} &\text{In case (1)} && \Pi = \sigma \sin\left(\frac{\rho}{\epsilon} - \tau\right) \\ &\text{in case (2)} && \Pi = \sigma \sinh\left(\frac{\rho}{\epsilon} - \tau\right) \\ &\text{in case (3)} && \Pi = \sigma\rho + \tau \end{aligned}\right\}, \tag{10}$$

where, if semi-geodesic orthogonal coordinates be used, $\sigma$ and $\tau$ are functions of $\phi$ only.

Let the surfaces of type (1) be called "elliptic surfaces of constant curvature," of type (2) "hyperbolic surfaces of constant curvature," of type (3) "parabolic surfaces of constant curvature."

12. If geodesic polar coordinates be used, it follows from § 7, by making $\rho$ small, that $\tau = 0$ and $\sigma = \epsilon/\gamma$ for the elliptic and hyperbolic surfaces, and $\tau = 0$, $\sigma = \frac{1}{\gamma}$ for the parabolic surfaces.

Hence, with these coordinates,

$$\Pi = \frac{\epsilon}{\gamma} \sin\frac{\rho}{\epsilon} \text{ for elliptic surfaces,}$$

$$\Pi = \frac{\epsilon}{\gamma} \sinh\frac{\rho}{\epsilon} \text{ for hyperbolic surfaces,}$$

$$\Pi = \frac{\rho}{\gamma} \text{ for parabolic surfaces.}$$

Conversely, if the limit of $\Pi$ be $\frac{\sigma\rho}{\epsilon}$, when $\rho$ is small, then the curves $\rho =$ constant are geodesic circles.

For, if $\delta\sigma_2$ be an element of the arc of one of these curves, for which $\rho$ is a small constant,

$$\delta\sigma_2 = \frac{\gamma\sigma}{\epsilon}\rho\delta\phi.$$

Now write $\psi = \frac{\gamma}{\epsilon}\int \sigma d\phi$, remembering that $\sigma$ is a function of $\phi$ only. Then

$$\left[\sigma_2\right]_{\psi_1}^{\psi_2} = \rho\,(\psi_2 - \psi_1).$$

Hence the length of arc $\left[\sigma_2\right]_{\psi_1}^{\psi_2}$ is very small, when $\rho$ is small. Thus the limit of the curve $\rho =$ constant must be an evanescent curve round a point. Hence $\rho$ and $\psi$ are polar geodesic coordinates.

13. The only surfaces in elliptic space on which two orthogonal families of geodesic curves can be drawn are parabolic surfaces of constant curvature. For, by the collected results of § 6, if we take these orthogonal geodesics as curvilinear coordinate curves, it follows that we may write

$$(x_1 \mid x_1) = \frac{1}{\gamma^2} = (x_2 \mid x_2), \quad \text{and} \quad (x_1 \mid x_2) = 0. \tag{11}$$

Here the orthogonal curves are the families $\xi =$ constant and $\eta =$ constant, and the element of arc $\delta\sigma$ takes the form

$$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2;$$

also the suffix 1 denotes partial differentiation with respect to $\xi$, and the suffix 2 with respect to $\eta$.

It follows from equations (11) that

$$(x_1 \mid x_1)_1 = 0 = (x_1 \mid x_1)_2 = (x_2 \mid x_2)_1 = (x_2 \mid x_2)_2 = (x_1 \mid x_2)_1 = (x_1 \mid x_2)_2.$$

Hence the equation (1) in § 2 for $\frac{1}{\rho_1\rho_2}$ becomes

$$\frac{1}{\rho_1\rho_2} = -\frac{1}{\gamma^2}.$$

This proves the proposition.

14. Conversely, two orthogonal families of geodesics can always be drawn on any parabolic surface of constant curvature so as to include any given geodesic as a member of one of the families. For, from each point of the given geodesic, draw the geodesics cutting the given geodesic at right angles. Let the length of the geodesic arc drawn in this manner to any point $x$ be $\rho$. Let the family of geodesics so

drawn be denoted by the equation $\phi =$ constant. Then (by § 5 above) $\rho$ and $\phi$ are semi-geodesic orthogonal coordinates; and, of the curves $\rho =$ constant, one curve, namely, $\rho = 0$, is a geodesic. We will now prove that all the curves $\rho =$ constant are geodesics.

For, from §§ 6 and 11, we have

$$(x_1 \mid x_1) = \frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0, \quad (x_2 \mid x_2) = (\sigma\rho + \tau)^2,$$

where $\sigma$ and $\tau$ are functions of $\phi$ only. Also, by §§ 3, equation (4), the condition that the curves $\rho =$ constant may be geodesics is that $(x_2 x_{22} \mid x_1 x_2)$ should vanish at each point of the surface. Now

$$\begin{aligned}(x_2 x_{22} \mid x_1 x_2) = &\quad (x_1 \mid x_2)(x_2 \mid x_{22}) - (x_2 \mid x_2)(x_1 \mid x_{22}) \\ = &- (x_2 \mid x_2)(x_1 \mid x_{22}).\end{aligned}$$

But $$(x_{12} \mid x_2) + (x_1 \mid x_{22}) = (x_1 \mid x_2)_2 = 0;$$

and $$(x_{12} \mid x_2) = \tfrac{1}{2}(x_2 \mid x_2)_1.$$

Hence $$(x_1 \mid x_{22}) = -\tfrac{1}{2}(x_2 \mid x_2)_1.$$

Therefore $$\begin{aligned}(x_2 x_{22} \mid x_1 x_2) = &\quad \tfrac{1}{2}(x_2 \mid x_2)(x_2 \mid x_2)_1 \\ = &\ \sigma(\sigma\rho + \tau)^3.\end{aligned}$$

But, when $\rho = 0$, by hypothesis,

$$(x_2 x_{22} \mid x_1 x_2) = 0.$$

Hence $$\sigma\tau^3 = 0.$$

Now, if $\tau = 0$, the curve $\rho = 0$ is, by § 12, an evanescent circle, and is, therefore, not a geodesic, contrary to hypothesis.

Hence, $\sigma = 0$; and, therefore, at all points of the surface

$$(x_2 x_{22} \mid x_1 x_2) = 0.$$

Thus, the family of curves $\rho =$ constant are geodesics.

Thus, on parabolic surfaces of constant curvature, and only on such surfaces, geodesic orthogonal coordinates are possible; and the coordinate curves can be chosen so as to include any assigned geodesic.

14. On any parabolic surface of constant curvature let geodesic orthogonal coordinates be chosen. Let the curves $\xi =$ constant denote one family of geodesics, and the curves $\eta =$ constant denote the orthogonal family of geodesics. Also choose $\xi$ and $\eta$ so that $\delta\xi$ denotes an element of arc of a curve $\eta =$ constant, and $\delta\eta$ denotes an

element of arc of a curve $\xi =$ constant. Then, by § 6,

$$(x_1 \mid x_1) = \frac{1}{\gamma^2} = (x_2 \mid x_2),$$

and $$(x_1 \mid x_2) = 0.$$

Hence $$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2. \qquad (11)$$

The identity of the geodesic geometry of parabolic surfaces of constant curvature with the geometry of straight lines in a Euclidean plane follows immediately from this equation. It may be convenient to some readers to illustrate this identity by the following investigation, occupying the remainder of this article (14).

Let any geodesic be given by

$$\xi = f(\tau), \quad \eta = \phi(\tau);$$

and let differentiation with respect to $\tau$ be denoted by $\dot{\xi}, \dot{\eta}$, and so on. The condition for a geodesic has been given by equation (3) of § 2. This now takes the special form

$$\dot{\xi}\ddot{\eta} - \dot{\xi}\ddot{\eta} = 0.$$

Integrating the form $$\frac{\ddot{\xi}}{\dot{\xi}} - \frac{\ddot{\eta}}{\dot{\eta}} = 0,$$

we have $$\dot{\eta} = a\dot{\xi}. \qquad (12)$$

Integrating again, $$\eta = a\xi + \beta. \qquad (13)$$

This is the general equation of a geodesic in terms of orthogonal curvilinear coordinates. The tangent line at any point $x$ of the geodesic (13) is $x\dot{x}$. But

$$x\dot{x} = x(x_1\dot{\xi} + x_2\dot{\eta}) = x(x_1 + ax_2)\dot{\xi}.$$

Hence $x(x_1 + ax_2)$ is the tangent line. The tangent line to any other geodesic

$$\eta = a'\xi + \beta',$$

through the same point $x$, is $x(x_1 + a'x_2)$. The angle $\theta$ between these lines is given by

$$\cos\theta = \frac{\{x(x_1+ax_2) \mid x(x_1+a'x_2)\}}{\sqrt{[\{x(x_1+ax_2) \mid x(x_1+ax_2)\}\{x(x_1+a'x_2) \mid x(x_1+a'x_2)\}]}}$$

$$= \frac{1+aa'}{\sqrt{\{(1+a^2)(1+a'^2)\}}};$$

that is, $$\tan\theta = \frac{a \sim a'}{1 + aa'}.$$

Thus the angles that the geodesics severally make with $\eta = 0$ at their points of intersection with it are $\tan^{-1} a$ and $\tan^{-1} a'$.

15. We will now find the equation of the general type of those parabolic surfaces of constant curvature which are also surfaces of revolution. By so doing we also prove the important proposition that parabolic surfaces of constant curvature in elliptic space are real surfaces.

Let $ee_3$ be the axis of revolution, and let $e$, $e_1$, $e_2$, $e_3$ be a set of unit-normal (or quadrantal) points $e_3$ [*cf. Universal Algebra*, § 223 (1) and § 245]. Then the three lines $ee_1$, $ee_2$, $ee_3$ form a system of three axes mutually at right angles, origin $e$.

Any unit point $$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$$

satisfies the condition $$(x \mid x) = 1,$$

that is, $$\xi^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1;$$

and $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ are the ordinary rectangular coordinates of elliptic geometry.

The meridian curves of the surface of revolution lie in planes through $ee_3$, and are geodesics of the surface. The parallels of latitude are curves (not geodesics) which cut the meridians orthogonally. Take these two families of curves to define a system of semi-geodesic orthogonal coordinates. In accordance with the notation of § 6, let a point $x$ on the surface be defined by $\rho$, which is the geodesic arc measured from some fixed parallel of latitude, and by $\phi$, which is the angle between the plane $ee_3x$ and the plane $ee_3e_1$.

Then, from § 11, equation (10), we have

$$\Pi = \alpha\rho + \beta, \tag{14}$$

where $\alpha$ and $\beta$ are constants, since the surface is one of revolution.

Also, since the surface is one of revolution, we may assume

$$x = \xi e + \mu\,(e_1 \cos\phi + e_2 \sin\phi) + \xi_3 e_3, \tag{15}$$

where $\xi$, $\mu$, and $\xi_3$ are functions of $\rho$ only.

Also, since $$(x \mid x) = 1,$$

we have $$\xi^2 + \mu^2 + \xi_3^2 = 1. \tag{16}$$

Since $$(x_1 \mid x_1) = \frac{1}{\gamma^2},$$

we have $$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\mu}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \frac{1}{\gamma^2}. \tag{17}$$

Since $$(x_2 \mid x_2) = \Pi^2 = (a\rho+\beta)^2,$$

we have $$\mu^2 = (a\rho+\beta)^2;$$

that is, $$\mu = a\rho+\beta. \tag{18}$$

The condition $(x_1 \mid x_2) = 0$ is satisfied identically.

From (17) and (18) we find

$$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \frac{1}{\gamma^2} - a^2.$$

Hence for a real surface we must have $a^2 < \frac{1}{\gamma^2}$. Put

$$a = \frac{\cos\delta}{\gamma}.$$

Then $$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \left(\frac{\sin^2\delta}{\gamma^2}\right). \tag{19}$$

Also equations (16) and (18) give

$$\xi^2 + \xi_3^2 = 1-(a\rho+\beta)^2 = \sigma^2,$$

say. Assume $$\xi = \sigma\cos\varpi, \quad \xi_3 = \sigma\sin\varpi, \tag{20}$$

where $\varpi$ is a function of $\rho$ only. Hence equation (19) becomes

$$\sigma^2\left(\frac{d\varpi}{d\rho}\right)^2 + \left(\frac{d\sigma}{d\rho}\right)^2 = \frac{\sin^2\delta}{\gamma^2}.$$

Thus $$\frac{d\varpi}{d\rho} = \frac{\left\{\frac{\sin^2\delta}{\gamma^2} - \left(\frac{d\sigma}{d\rho}\right)^2\right\}}{\sigma}.$$

Now, put $$a\rho+\beta = \tau,$$

so that $$\sigma^2 = 1-\tau^2.$$

Then $$\varpi = \sec\delta\int \frac{(\sin^2\delta-\tau^2)^{\frac{1}{2}}}{1-\tau^2}\,d\tau.$$

The argument of the integral is rationalized by the assumption

$$\sin^2\delta-\tau^2 = \tau^2\zeta^2.$$

Finally, we find

$$\varpi-\epsilon=\tan^{-1}\frac{\{\gamma^2\sin^2\delta-(\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{(\rho\cos\delta+\kappa)\cos\delta}$$
$$-\sec\delta\tan^{-1}\frac{\{\gamma^2\sin^2\delta-(\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{\rho\cos\delta+\kappa},\qquad(20)$$

where we have written $\gamma\beta=\kappa$,

and $\epsilon$ is an arbitrary constant.

Thus the required surface is given by

$$\left.\begin{aligned} x&=\xi e+\frac{\rho\cos\delta+\kappa}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3 e_3,\\ \text{where}\quad \xi&=\left\{1-\frac{(\rho\cos\delta+\kappa)^2}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi,\\ \xi_3&=\left\{1-\frac{(\rho\cos\delta+\kappa)^2}{\gamma^2}\right\}^{\frac{1}{2}}\sin\varpi,\end{aligned}\right\}\qquad(21)$$

and $\varpi$ is given by equation (20).

If we measure $\rho$ from the parallel of latitude which in equations (21) is denoted by

$$\rho=-\kappa\sec\delta,$$

the equations become

$$\left.\begin{aligned} x&=\xi e+\frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3 e_3,\\ \xi&=\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi,\\ \xi_3&=\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\varpi,\\ \varpi-\epsilon&=\tan^{-1}\frac{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos^2\delta}-\sec\delta\tan^{-1}\frac{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta}.\end{aligned}\right\}\qquad(22)$$

The equation of the surface can also be put in the form

$$\tan^{-1}\frac{\xi_3}{\xi}-\epsilon=\tan^{-1}\frac{\{\sin^2\delta-\xi_1^2-\xi_2^2\}}{(\xi_1^2+\xi_2^2)^{\frac{1}{2}}\cos\delta}-\sec\delta\tan^{-1}\frac{\{\sin^2\delta-\xi_1^2-\xi_2^2\}^{\frac{1}{2}}}{(\xi_1^2+\xi_2^2)^{\frac{1}{2}}}.\qquad(23)$$

The equations (22) and (23) can be conceived in two different ways, which are not quite identical in their results. Equation (23) may be

taken as the fundamental equation; and $\rho$ must then be conceived as standing merely for the analytical expression

$$\frac{\sec\delta}{\gamma}\left\{\xi_1^2+\xi_2^2\right\}^{\frac{1}{2}}.$$

Then the equations (22), together with the equations

$$\xi_1=\frac{\rho\cos\delta}{\gamma}\cos\phi,\quad \xi_2=\frac{\rho\cos\delta}{\gamma}\sin\phi,$$

define the coordinates $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ in terms of the auxiliary variables $\rho$ and $\phi$. The surface will be found to consist of many sheets, and the geometrical meaning of $\rho$ must be varied to make it appropriate to each particular sheet.

The second method of conceiving equations (22) and (23) is the one by which they have been attained; namely, $\rho$ is conceived as the arc of the meridian geodesic measured from a given parallel of latitude. This method is appropriate for studying the properties of any one particular sheet of the surface; and, since the different sheets are merely repetitions of each other, which arise owing to the multiple values of the inverse tangents in equation (23), this will be the convenient method for us to continue to adhere to for the present. It will save confusion if, during our adherence to this geometrical definition of $\rho$, the surface is called "the sheet." The change of name is appropriate, for we are considering one sheet of the surface obtained by the first point of view.

16. The parallel of latitude $\rho=0$ is an evanescent circle surrounding and coinciding with some point on the axis $ee_3$. Let this point be called "the conical point of the corresponding sheet," and let the sheet be considered as consisting of all that part of the complete surface denoted by equation (23) which is represented by equations (22) when the ambiguities of the inverse tangents are so determined that $\rho$ represents the length of the meridian geodesic drawn from their conical point.

Firstly, let $\rho$ be assumed to be positive, and let $\lambda$ and $\mu$ be positive acute angles such that

$$\cot\lambda=\frac{\left\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}{\rho\cos^2\delta},\qquad \cot\mu=\frac{\left\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}{\rho\cos\delta}.$$

Then the most general expression of the last of equations (22), without any determination of the ambiguities, is

$$\varpi-\epsilon=(n\pi+\tfrac{1}{2}\pi-\lambda)-\sec\delta\,(m\pi+\tfrac{1}{2}\pi-\mu),$$

where $m$ and $n$ are any two integers, positive or negative. Now let

$$\epsilon = \tfrac{1}{2}\pi\,(\sec\delta - 1). \tag{24}$$

Then the equation for $\varpi$ becomes

$$\varpi = (n-m)\,\pi - 2m\epsilon - \mu\sec\delta - \lambda. \tag{25}$$

The only equations in which $\varpi$ is used are the following:—

$$\left.\begin{aligned}\xi &= \left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n-m}\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\left\{-2m\epsilon+\mu\sec\delta-\lambda\right\}\\ \xi_3 &= \left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\varpi = (-1)^{n-m}\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\left\{-2m\epsilon+\mu\sec\delta-\lambda\right\}\end{aligned}\right\}. \tag{26}$$

Secondly, let $\rho$ be assumed to be negative and equal to $-\rho'$, and let $\lambda'$ and $\mu'$ be positive acute angles such that

$$\cot\lambda' = \frac{\left\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\right\}^{\frac{1}{2}}}{\rho'\cos^2\delta}, \quad \cot\mu' = \frac{\left\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\right\}^{\frac{1}{2}}}{\rho'\cos\delta}.$$

Then $\lambda'$ and $\mu'$ are respectively the same functions of $\rho'$ as $\lambda$ and $\mu$ are of $\rho$.

Hence the last of equations (22) becomes, when $\rho$ is negative,

$$\varpi - \epsilon = (n'\pi + \tfrac{1}{2}\pi + \lambda') - \sec\delta\,(m'\pi + \tfrac{1}{2}\pi + \mu'),$$

where $m'$ and $n'$ are any integers, positive or negative. Hence, using equation (24), this equation becomes

$$\varpi = (n'-m')\,\pi - 2m'\epsilon - \mu'\sec\delta + \lambda'. \tag{27}$$

Substituting in the other equations (22), we find

$$\left.\begin{aligned}\xi &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n'-m'}\left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\left\{2m'\epsilon+\mu'\sec\delta-\lambda'\right\}\\ \xi_3 &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n'-m'+1}\left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\left\{2m'\epsilon+\mu'\sec\delta-\lambda'\right\}\end{aligned}\right\}. \tag{28}$$

The factor $(-1)^{n-m}$ in equations (26) and $(-1)^{n'-m'}$ in equations (28) may be omitted. For the points $(\pm\xi_1 \pm \xi_3, \xi_1, \xi_2)$, that is to say, the points $(\xi, \xi_3, \pm\xi_1, \pm\xi_2)$, are both on the surface if one of them is.

17. Now, when $\rho$ is small, so that $\rho^2$ is neglected,

$$\lambda = \frac{\rho\cos^2\delta}{\gamma\sin\delta}, \quad \mu = \frac{\rho\cos\delta}{\sin\delta}, \quad \mu\sec\delta - \lambda = \frac{\rho\sin\delta}{\gamma}.$$

Hence equations (26) become

$$\left.\begin{aligned} \xi &= \cos 2m\epsilon + \frac{\rho \sin \delta}{\gamma} \sin 2m\epsilon \\ \xi_3 &= -\sin 2m\epsilon + \frac{\rho \sin \delta}{\gamma} \cos 2m\epsilon \end{aligned}\right\}. \tag{29}$$

Similarly, when $\rho'$ is small, equations (28) become

$$\left.\begin{aligned} \xi &= \cos 2m'\epsilon - \frac{\rho' \sin \delta}{\gamma} \sin 2m'\epsilon \\ \xi_3 &= -\sin 2m'\epsilon - \frac{\rho' \sin \delta}{\gamma} \cos 2m'\epsilon \end{aligned}\right\}. \tag{30}$$

Hence the series of conical points on $ee_3$ are given by

$$\xi = \cos 2m\epsilon, \quad \xi_3 = -\sin 2m\epsilon \ (m = 0, \pm 1, \pm 2, \ldots).$$

Let these points be denoted by $d_0$, $d_1$, $d_{-1}$, $d_2$, $d_{-2}$, ..., so that

$$d_m = e \cos 2m\epsilon - e_3 \sin 2m\epsilon, \quad d_{-m} = e \cos 2m\epsilon + e_3 \sin 2m\epsilon;$$

also $d_0$ is the point $e$. The distance between any two consecutive conical points, such as $d_m$ and $d_{m+1}$ is $\kappa$, where

$$\cos \frac{\kappa}{\gamma} = \frac{(d_m \mid d_{m+1})}{\sqrt{\{(d_m \mid d_m)(d_{m+1} \mid d_{m+1})\}}} = (d_m \mid d_{m+1})$$

$$= \cos 2m\epsilon \cos \overline{2m+2}\,\epsilon + \sin 2m\epsilon \sin \overline{2m+2}\,\epsilon = \cos 2\epsilon.$$

Hence $\kappa = 2\gamma\epsilon$, where, since the whole line is of length $\pi\gamma$, a number of complete circuits of the line may be included. If $\epsilon/\pi$ be a rational number, that is, if $\cos \delta$ be a rational number, there are only a finite number of conical points to the surface. But, if $\cos \delta$ be an irrational number, there are an infinite number of conical points to the surface.

18. Let us now study the sheet of which $e$, that is $d_0$, is the conical point.

Putting $m = 0$, equations (26) become

$$\left.\begin{aligned} \xi &= \left\{1 - \frac{\rho^2 \cos \delta}{\gamma^2}\right\}^{\frac{1}{2}} \cos\{\mu \sec \delta - \lambda\} \\ \xi_3 &= \left\{1 - \frac{\rho^2 \cos \delta}{\gamma^2}\right\}^{\frac{1}{2}} \sin\{\mu \sec \delta - \lambda\} \end{aligned}\right\}, \tag{31}$$

where $\lambda$ and $\mu$ are positive acute angles given by

$$\tan\lambda = \frac{\rho\cos^2\delta}{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}, \quad \tan\mu = \frac{\rho\cos\delta}{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}.$$

Also, when $\rho$ is small and $\rho^2$ is neglected, equations (29) become

$$\xi = 1, \quad \xi_3 = \frac{\rho\sin\delta}{\gamma}. \tag{32}$$

Also putting $m' = 0$, equations (28) become

$$\left.\begin{aligned} \xi &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\{\mu'\sec\delta-\lambda'\} \\ \xi_3 &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\{\mu'\sec\delta-\lambda'\} \end{aligned}\right\}, \tag{33}$$

where $\lambda'$ and $\mu'$ are positive acute angles given by

$$\tan\lambda' = \frac{\rho'\cos^2\delta}{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}}, \quad \tan\mu' = \frac{\rho'\cos\delta}{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}}.$$

Also, when $\rho'$ is small and $\rho'^2$ is neglected, equations (30) become

$$\xi = 1, \quad \xi_3 = -\frac{\rho'\sin\delta}{\gamma^2}. \tag{34}$$

Hence, from equations (32), (34), and the first of equations (22), a point $x$ on the sheet in the neighbourhood of $e$ is

$$x = e + \frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi) + \frac{\rho\sin\delta}{\gamma}e_3; \tag{35}$$

and this equation holds whether $\rho$ be positive or negative. Hence, in the neighbourhood of $e$,

$$x_1 = \frac{\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi) + \frac{\sin\delta}{\gamma}e_3. \tag{36}$$

Hence (*cf. Universal Algebra*, § 293) the tangent at $e$ to the meridian geodesic in the plane $\phi$ is the line $ex_1$, that is,

$$\frac{\cos\delta}{\gamma}(ee_1\cos\phi+ee_2\sin\phi) + \frac{\sin\delta}{\gamma}ee_3.$$

Now the angle $\chi$ between $ex_1$ and $ee_3$ is given by

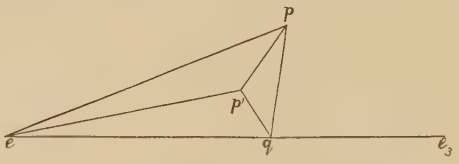
$$\cos\chi = \frac{(ex_1 \mid ee_3)}{\sqrt{\{(ex_1 \mid ex_1)(ee_3 \mid ee_3)\}}} = \sin\delta,$$

since $(ee_1 \mid ee_3) = 0 = (ee_2 \mid ee_3)$, and $(ee_3 \mid ee_3) = 1$.

Hence $$\chi = \tfrac{1}{2}\pi - \delta.$$

Hence the tangents at $e$ to the meridian geodesics form a cone of revolution of semi-vertical angle $\frac{1}{2}\pi - \delta$.

It is now easy to state the relation of the semi-geodesic coordinates $\rho$ and $\phi$ to polar geodesic coordinates $\rho$ and $\theta$. For the two geodesics $\phi$ and $\phi + \delta\phi$ make an angle $\delta\theta$ with each other at their point of intersection $e$. Now let $p$ and $p'$ be two points respectively on these




geodesics, and at the same small geodesic distance $\rho$ from $e$. Draw the perpendiculars $pq$ and $p'q$ to $ee_3$. Then, remembering that the properties of small figures are ultimately those of Euclidean space,

$$\overline{pp'} = \overline{ep} \,.\, \delta\theta = \rho\,\delta\theta.$$

Also
$$\overline{pp'} = \overline{pq} \,.\, \delta\phi = \rho \cos\delta\, \delta\phi.$$

Hence
$$\delta\theta = \cos\delta\, \delta\phi, \quad \theta = \phi \cos\delta.$$

Hence, if $\delta\sigma$ be an element of arc at any point of the surface,

$$(\delta\sigma)^2 = (\delta\rho)^2 + \rho^2 \cos^2\delta\, (\delta\phi)^2$$
$$= (\delta\rho)^2 + \rho^2\, (\delta\theta)^2;$$

and this agrees with § 12 and with the formula of Euclidean plane geometry which holds for the geodesic geometry of this sheet.

19. Recurring to equations (31), note that, as $\rho$ increases from 0 to $\gamma \tan\delta$, $\lambda$ and $\mu$ increase from 0 to $\frac{1}{2}\pi$; and that, when $\rho > \gamma \tan\delta$, $\lambda$ and $\mu$ are imaginary. Hence, for $\rho$ positive, the sheet is real only when $\rho < \gamma \tan\delta$.

Thus, as $\rho$ increases from 0 to $\gamma \tan\delta$, $\xi$ decreases from 1 to $\cos\delta \cos\epsilon$, $\xi_3$ increases from 0 to $\cos\delta \sin\epsilon$.

The plane of the parallel of latitude through $x$ is perpendicular to $ee_3$, and is, therefore, the plane $xe_1e_2$.

Hence the centre of the circular parallel of latitude is the point $xe_1e_2 \,.\, ee_3$, that is, $(xe_1e_2e_3)\, e - (xe_1e_2e)\, e_3$, that is, $\xi e + \xi_3 e_3$.

Hence, as $\rho$ increases from 0 to $\gamma \tan\delta$, the centres of the successive parallels of latitude of the sheet occupy successive positions on the line $ee_3$ between the point $e$ and the point $e\cos\epsilon + e_3 \sin\epsilon$, which is the centre of the extreme parallel of latitude on the sheet. Let $e\cos\epsilon + e_3 \sin\epsilon$ be called the point $c$.

The distance between $e$ and $c$ is $\gamma\epsilon$. But, in § 17, it is proved that the distance between $e$ and $d_{-1}$ is $2\gamma\epsilon$. Hence $c$ bisects the intercept between $e$ and $d_{-1}$, of which the length is $2\gamma\epsilon$.

Similarly, when $\rho$ is negative, and equal to $-\rho'$, equations (33) and (34) show that the remainder of the sheet is, in all respects, a reflexion in the plane $ee_1e_2$ of the part of the sheet for $\rho$ positive. Thus there is an extreme parallel of latitude, of which the centre is the point $e\cos\epsilon - e_3\sin\epsilon$ $(=c'$, say), and the point $c'$ bisects the intercept between $e$ and $d_1$ which is of length $2\gamma\epsilon$.

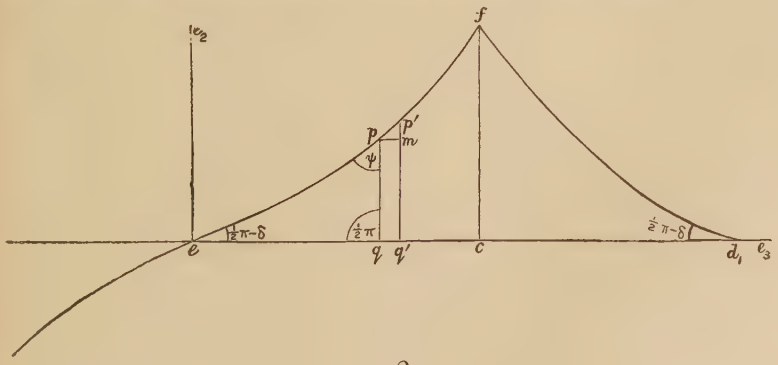
The perpendicular distance, $\zeta$, of any point $x$ on the sheet from the axis $ee_3$ is given (*cf. Universal Algebra*, § 226) by

$$\sin\frac{\zeta}{\gamma} = \sqrt{\left\{\frac{(xee_3 \mid xee_3)}{(x \mid x)(ee_3 \mid ee_3)}\right\}} = \surd\{(xee_3 \mid xee_3)\} = \frac{\rho\cos\delta}{\gamma}. \quad (38)$$

Hence, as $\rho$ increases from 0 to $\gamma\tan\delta$, the radii of the corresponding parallels of latitude increase from 0 to $\delta\gamma$.

Though the real part of each sheet is bounded by the two extreme parallels of latitude corresponding to $\rho^2 = \gamma^2\tan^2\delta$, the complete surface denoted by equation (23) is not discontinuous. For, since $c_1$ bisects the intercept between $e$ and $d_{-1}$, the point $c$ is also the centre of one of the extreme parallels of latitude of the sheet corresponding to the conical point $d_{-1}$, and the two parallels of latitude centre $c$ coincide. Thus, at the termination of one sheet, another sheet commences, corresponding to the next conical point given by the series $\ldots\ d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots$. It will be proved that the junction of two sheets forms a cuspidal line on the surface.

20. The simpler properties of the meridian curves can now be easily stated. Consider the meridian curve in the plane $ee_2e_3$. Take any point $p$ on the curve and draw $pq$ perpendicular to $ee_3$.



Let $ep$ denote the distance between $e$ and $p$, and $\rho$ the length of the arc $ep$ of the curve. At $e$ the curve cuts the line $ee_3$ at an angle $\frac{1}{2}\pi-\delta$. From equation (38),

$$\sin\frac{\overline{pq}}{\gamma} = \frac{\rho\cos\delta}{\gamma} = \frac{\text{arc } ep\,.\cos\delta}{\gamma}.$$

This may be looked on as the geometrical definition of the curve.

The angle $\psi$, which the tangent at $p$ makes with $pq$, can be found from this equation. For, let $p'$ be a neighbouring point to $p$ on the curve; draw $p'q'$ perpendicular to $ee_3$. Let $pm$ be an arc of the circle of equal distance from $ee_3$. Then the arc $pm$ is ultimately a straight line perpendicular to $pq$. Also

$$p'm = p'q'-pq.$$

Hence, by differentiating equation (38),

$$\cos\frac{\overline{pq}}{\gamma}\,\frac{p'm}{\gamma} = \frac{\delta\rho\cos\delta}{\gamma} = \frac{pp'\cos\delta}{\gamma}.$$

Hence
$$\cos\psi = \text{Lt.}\,\frac{p'm}{pp'} = \frac{\cos\delta}{\cos\frac{\overline{pq}}{\gamma}} = \frac{\gamma\cos\delta}{\{\gamma^2-\rho^2\cos^2\delta\}^{\frac{1}{2}}}. \qquad (39)$$

The angle $\psi$ increases with $\rho$; hence the curve is convex to the axis $ee_3$.

The value of $\psi$ is imaginary if

$$\frac{\gamma\cos\delta}{\{\gamma^4-\rho^2\cos^2\delta\}^{\frac{1}{2}}} > 1,$$

that is, if
$$\rho > \gamma\tan\delta.$$

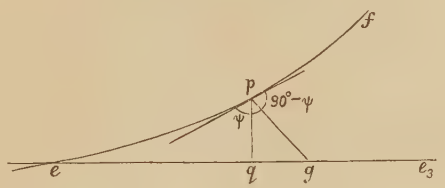
This agrees with § 19.

When $\rho = \gamma\tan\delta$, $\psi = 0$. Hence the tangent at $f$, the extreme point of the curve, is perpendicular to the axis $ee_3$. Hence, as previously stated, the junction of two sheets is a cuspidal line.

The inverse measure of curvature ($\rho_1$) of the curve at $p$ is most easily found from the formula

$$\frac{1}{\rho_1\rho_2} = -\frac{1}{\gamma^2},$$

which expresses the condition that the curve is a meridian geodesic of a parabolic surface of constant curvature.




For draw the normal $pg$ cutting $ee_3$ in $g$. Then, from Napier's formulæ for right-angled triangles,

$$\tan\frac{\overline{pg}}{\gamma} = \tan\frac{\overline{pq}}{\gamma}\sec(90°-\psi) = \tan\frac{\overline{pq}}{\gamma}\operatorname{cosec}\psi.$$

Hence, from equations (38) and (39),

$$\tan\frac{\overline{pg}}{\gamma} = \frac{\rho\cos\delta}{\left\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}.$$

But $\overline{pg}$ is the radius of circular curvature at $p$ of the line of curvature perpendicular to the meridian; this must be distinguished [*cf. Universal Algebra*, § 288 (4)] from the inverse measure of curvature $\sigma_2$. The relation between the two is

$$\rho_2 = \gamma\tan\frac{\overline{pg}}{\gamma}.$$

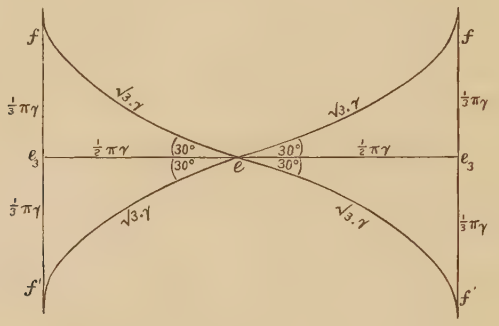
Hence
$$\rho_2 = \frac{\gamma\rho\cos\delta}{\left\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}.$$

Hence
$$\rho_1 = -\frac{\gamma^2}{\rho_2} = -\frac{\gamma\left\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}{\rho\cos\delta}. \qquad (40)$$

Therefore $\rho$ decreases numerically from $\infty$ at $e$ to 0 at $f$.

21. The distance between successive conical points of the complete surface is (*cf.* § 17) $2\gamma\epsilon$, that is, $\pi\gamma$ (sec $\delta-1$). But the length of a complete straight line is $\pi\gamma$. Hence, if $\cos\delta = \frac{1}{2}$, that is, if $\delta = \frac{1}{3}\pi$, the surface has only one conical point. In this case the surface has only one sheet, and the two extremities of the sheet, for $\rho$ positive and $\rho$ negative respectively, coincide along a parallel of latitude of radius

$\frac{1}{3}\pi\gamma$ [*cf.* equation (38)]. The total length of a meridian arc, measured from $e$, is $\gamma \tan \frac{1}{3}\pi$, that is, $\sqrt{3} \cdot \gamma$.




The accompanying figure, of course, cannot, in Euclidean space, represent the form of the meridian section of the surface in elliptic space; but it may help the imagination to realize the geometrical relations of the parts. The two lines $fe_3f'$, at the extremities of the figure, must be conceived as coinciding—thus the figure may be wrapped on a cylinder so that these lines become the same generator. The lines are tangents, so that the surface in elliptic space has a cuspidal line.

With the notation of the end of § 18, $\theta = \phi \cos \delta = \frac{1}{2}\varphi$. Now $\phi$ varies from 0 to $2\pi$. Hence, each half of the sheet of a single-sheeted parabolic surface of revolution of constant curvature corresponds to a semi-circle of radius $\sqrt{3} \cdot \gamma$ on a Euclidean plane in Euclidean space.

The equation (23) can be rationalized in this case, since $\delta = \frac{1}{3}\pi$, $\epsilon = \frac{1}{2}\pi$. Thus, the equation of the one-sheeted parabolic surface of revolution of constant curvature is

$$3\,(\xi^2+\xi_3^2)^2\,\xi_3^2 = 3\,(3\xi^2+\xi_3^2)(\xi^2+\xi_3^2)\,(\xi_1^2+\xi_2^2)+(2\xi^2-\xi_3^2)(\xi_1^2+\xi_2^2)^2.$$

[*Added May*, 1898.—Professor Burnside, who has kindly read over this paper in manuscript, has pointed out that the detailed mention of the simplest case of all has been omitted. For in equation (16) put $a = 0$, so that $\pi = \beta$; this is equivalent to putting $\delta = \frac{1}{2}\pi$. Hence the equation for $\varpi$ becomes

$$\frac{d\varpi}{d\rho} = \frac{1}{\gamma\,(1-\beta^2)^{\frac{1}{2}}}, \quad \varpi = \frac{\rho+\epsilon}{\gamma\,(1-\beta^2)^{\frac{1}{2}}}.$$

Thus, instead of equations (21), we find

$$x = (1-\beta^2)^{\frac{1}{2}} \cos \frac{\rho+\epsilon}{\gamma(1-\beta^2)^{\frac{1}{2}}}\, e + \beta\,(e_1 \cos\phi + e_2 \sin\phi) + (1-\beta^2)^{\frac{1}{2}} \cos \frac{\rho+\epsilon}{\gamma(1-\beta^2)^{\frac{1}{2}}}\, e_3.$$

Thus the surface is a surface of equal distance from $ee_3$, and the constant distance from $ee_3$ is $\gamma \sin^{-1}\beta$. Professor Burnside makes this surface the basis of a synthetic treatment of the subject of this paper which is appended to this paper.]

*Geodesics on Surfaces in Hyperbolic Space.*

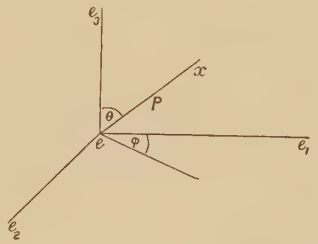
22. The preceding results can now easily be adapted to hyperbolic space; it will not be necessary to repeat the proofs in detail.

Let $e$ be the origin of three mutually rectangular axes $ee_1$, $ee_2$, $ee_3$; and let $e$, $e_1$, $e_2$, $e_3$ be a system of mutually normal unit points. Then (*cf. Universal Algebra*, Book VI., chapter iv.) $e_1$, $e_2$, $e_3$ are points in antispace, and

$$(e \mid e) = 1, \quad (e_1 \mid e_1) = -1 = (e_2 \mid e_2) = (e_3 \mid e_3).$$

Also, if $x$ be any point, it can be expressed in the form $\xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$. If it be a unit point in (hyperbolic) space, as distinct from antispace,

$$\xi^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 1.$$




Also, if $\rho$ be the distance $\overline{ex}$, $\theta$ the angle $xee_3$, $\phi$ the angle between the planes $xee_3$ and $e_1ee_3$, then, $x$ being a unit point,

$$\xi = \cosh\frac{\rho}{\gamma}, \quad \xi_1 = \sinh\frac{\rho}{\gamma}\sin\theta\cos\phi, \quad \xi_2 = \sinh\frac{\rho}{\gamma}\sin\theta\sin\phi,$$

$$\xi_3 = \sinh\frac{\rho}{\gamma}\cos\theta.$$

Hence $\gamma$ is the "space constant" of the hyperbolic space.

Also let $x$, as before, be a unit point on a surface, defined by curvilinear coordinates $\theta$ and $\phi$, distinct from the $\theta$ and $\phi$ of the previous explanation. Then $(x \mid x) = 1$. The formulæ remain substantially unchanged from those of elliptic space; they can be

derived from the "elliptic" formulæ by writing $\gamma\sqrt{(-1)}$ instead of $\gamma$. Hence formulæ not involving $\gamma$ will be absolutely unchanged, others involving $\gamma^2$ will have their signs changed, and some trigonometrical functions will be turned into hyperbolic functions. Then

$$\frac{\delta\sigma^2}{\gamma^2} = -(x_1 \mid x_1)\,(\delta\theta)^2 - 2\,(x_1 \mid x_2)\,\delta\theta\,\delta\phi - (x_2 \mid x_2)\,(\delta\phi)^2.$$

Also
$$\frac{-\gamma^2\{(x_1 \mid x_1)\,(x_2 \mid x_2) - (x_3 \mid x_2)\}^2}{\rho_1\rho_3}$$

$$= \Delta_1 - \Delta_2 + \{(x_1 \mid x_1)(x_2 \mid x_2) - (x_1 \mid x_2)^2\}$$
$$\times \{(x_1 \mid x_2)^2 - (x_1 \mid x_1)(x_2 \mid x_2) + (x_1 \mid x_2)_{12} - \tfrac{1}{2}(x_2 \mid x_2)_{11} - \tfrac{1}{2}(x_1 \mid x_1)_{22}\}, \tag{41}$$

where $\Delta_1$ and $\Delta_2$ have the same meanings as in the corresponding formula (equation 1) for elliptic geometry.

The condition that the curve $\theta = f(\tau)$, $\phi = F(\tau)$ may be a geodesic is

$$(\dot{x}\,\ddot{x} \mid x_1 x_2) = 0. \tag{42}$$

This can be expressed in the same form as equation (3).

23. The investigations of §§ (4), (5), and (6) hold for hyperbolic space; that is to say, all the properties of semi-geodesic orthogonal coordinates. Thus, if the curves $\phi$ = constant be the geodesic family, and the curves $\rho$ = constant be the orthogonal family, we have

$$(x_1 \mid x_1) = -\frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0. \tag{43}$$

Also
$$(\delta\sigma)^2 = (\delta\rho)^2 - \gamma^2(x_2 \mid x_2)\,(\delta\phi)^2 = (\delta\rho)^2 + \gamma^2\Pi^2(\delta\phi)^2, \tag{44}$$

where
$$\Pi^2 = -(x_2 \mid x_2).$$

The special properties of polar geodesic coordinates also hold, and with these coordinates the limit of $\Pi$, when $\rho$ is small, is $\rho/\gamma$.

Equation (6) of § 8 becomes, with semi-geodesic orthogonal coordinates,

$$\frac{\delta^2\Pi}{\delta\rho^2} + \left(\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2}\right)\Pi = 0. \tag{45}$$

Also equations (7) and (8) of § 9 for geodesics hold without any change of form. Hence Beltrami's theorem respecting the geodesic representation on planes of surfaces of constant curvature can be proved as in § 10. Also, as in § 11, putting

$$\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2} = \frac{1}{\epsilon^2}, \text{ or } -\frac{1}{\epsilon^2}, \text{ or } 0,$$

equations (10) hold without modification. Let these surfaces, as in elliptic space, be called respectively "elliptic," "hyperbolic," and "parabolic" surfaces of constant curvature.

The theorems of §§ 13 and 14 also hold, namely, the only surfaces in hyperbolic space which admit of two families of geodesics being drawn on them mutually orthogonal are parabolic surfaces of constant curvature, and, conversely, on any such surface two such families can be drawn, of which one family includes any assigned geodesic.

Let $\xi =$ constant denote one such family of geodesics, and let $\eta =$ constant denote the orthogonal family. Let the subscript 1 denote differentiation with respect to $\xi$ and $a$ with respect to $\eta$.

Then we may assume

$$(x_1 \mid x_1) = -\frac{1}{\gamma^2} = (x_2 \mid x_2), \quad (x_1 \mid x_2) = 0;$$

and

$$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2.$$

Thus, as in elliptic space, it may be seen that the geodesic geometry of the surface is Euclidean.

24. To find the equation of the general type of surfaces of revolution which are also parabolic surfaces of constant curvature. It may be noticed that Bolyai's limit-surfaces must be included as a special case of such surfaces.

Let $ee_3$ be the axis of revolution, and let $ee_1$, $ee_2$, $ee_3$ be three mutually rectangular axes of origin $e$. Let $e$, $e_1$, $e_2$, $e_3$ be a set of mutually normal unit points; thus $e_1$, $e_2$, $e_3$ are antispatial. Any unit spatial point

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$$

satisfies the condition $(x \mid x) = 1$,

that is,

$$\xi^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 1,$$

and $\xi$, $\xi_1$, $\xi_2$, $\xi_3$ are the ordinary rectangular coordinates of hyperbolic geometry. Now, as in § 15, take meridians and parallels of latitude as a system of semi-geodesic orthogonal coordinates. Let the length of the arc of the meridian through $x$, measured from a fixed parallel of latitude, be $\rho$, and let $\phi$ be the angle between the planes $xee_3$ and $e_1ee_3$. Then, as in § 15, we have, since the last of equations (10) holds,

$$(\delta\sigma)^2 = (\delta\rho)^2 + \gamma^2\Pi^2(\delta\phi)^2;$$

and

$$\Pi = a\rho + \beta, \tag{46}$$

where $\alpha$ and $\beta$ are constants, since the surface is one of revolution. Also, since the surface is one of revolution, we may assume

$$x = \xi e + \mu (e_1 \cos\phi + e_2 \sin\phi) + \xi_3 e_3, \tag{47}$$

where $\xi$, $\mu$, $\xi_3$ are functions of $\rho$ only. Also, since

$$(x \mid x) = 1,$$

we have

$$\xi^2 - \mu^2 - \xi_3^2 = 1. \tag{48}$$

Since

$$(x_1 \mid x_1) = -\frac{1}{\gamma^2},$$

we have

$$\left(\frac{d\mu}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 - \left(\frac{d\xi}{d\rho}\right)^2 = \frac{1}{\gamma^2}. \tag{49}$$

Since

$$(x_2 \mid x_2) = -\Pi^2 = -(\alpha\rho+\beta)^2,$$

we have

$$\mu = \alpha\rho + \beta. \tag{50}$$

The condition $(x_1 \mid x_2) = 0$ is satisfied identically. From equations (48) and (50) we find

$$\xi^2 - \xi_3^2 = 1 + \mu^2 = 1 + (\alpha\rho+\beta)^2 = \sigma^2, \text{ say.} \tag{51}$$

Assume $\xi = \sigma\cosh\varpi$, $\xi_3 = \sigma\sinh\varpi$, where $\varpi$ is some function of $\rho$ to be determined. Substituting for $\mu$, $\xi$, $\xi_3$ in equation (48), we find

$$\sigma^2\left(\frac{d\varpi}{d\rho}\right)^2 - \left(\frac{d\sigma}{d\rho}\right)^2 = \frac{1}{\gamma^2} - \alpha^2,$$

that is,

$$\frac{d\varpi}{d\rho} = \frac{\left\{\frac{1}{\gamma^2} - \alpha^2 + \frac{\mu^2}{\gamma^2}\right\}^{\frac{1}{2}}}{(1+\mu^2)}. \tag{52}$$

Hence these cases arise for examination according as

$$\text{(Case I.) } \alpha = \frac{1}{\gamma}, \quad \text{(Case II.) } \alpha < \frac{1}{\gamma}, \quad \text{(Case III.) } \alpha > \frac{1}{\gamma}.$$

25. *Case I.*—Let

$$\alpha = \frac{1}{\gamma}.$$

Equation (52) becomes

$$\frac{d\varpi}{d\rho} = \frac{\mu}{\gamma(1+\mu^2)} = \frac{\rho+\kappa}{\gamma^2+(\rho+\kappa)^2},$$

where $\kappa/\gamma$ is written for $\beta$. Hence

$$\begin{aligned}\varpi &= \log\left\{\gamma^2+(\rho+\kappa)^2\right\}^{\frac{1}{2}} + \text{constant}\\ &= \log\frac{\left\{\gamma^2+(\rho+\kappa)^2\right\}^{\frac{1}{2}}}{\delta},\end{aligned}$$

where $\delta$ is the constant introduced by integration. Hence

$$\left.\begin{aligned}\xi &= \frac{\left\{\gamma^2+(\rho+\kappa)^2\right\}^{\frac{1}{2}}}{\gamma}\cosh\varpi = \frac{\delta^2+\gamma^2+(\rho+\kappa)^2}{2\gamma\delta}\\ \xi_3 &= \frac{\left\{\gamma^2+(\rho+\kappa)^2\right\}^{\frac{1}{2}}}{\gamma}\sinh\varpi = \frac{\gamma^2-\delta^2+(\rho+\kappa)^2}{2\gamma\delta}\\ \xi_1 &= \frac{\rho+\kappa}{\gamma}\cos\phi\\ \xi_2 &= \frac{\rho+\kappa}{\gamma}\sin\phi\end{aligned}\right\}. \qquad (53)$$

Now measure $\rho$ from the evanescent parallel of latitude, which has hitherto been denoted by $\rho = -\kappa$, and choose $e$ to be the point in which $ee_3$ cuts the axis $ee_3$, so that $\delta = \gamma$, and we find, without diminishing the generality of the type of surface,

$$\xi = 1+\tfrac{1}{2}\frac{\rho^2}{\gamma^2}, \quad \xi_3 = \tfrac{1}{2}\frac{\rho^2}{\gamma^2},$$

$$\xi_1 = \frac{\rho}{\gamma}\cos\phi, \quad \xi_2 = \frac{\rho}{\gamma}\sin\phi.$$

Hence $$(x \mid x) = \left\{x \mid (e+e_3)\right\}^2.$$

But (*cf. Universal Algebra*, § 299) this is the equation of Bolyai's limit-surface, which passes through the point $e$, and has its centre at the point $e+e_3$ on the absolute.

26. *Case II.*—Let $$a = \frac{\cos\delta}{\gamma}.$$

Equation (52) becomes

$$\frac{d\varpi}{d\rho} = \frac{\left\{\sin^2\delta+\mu^2\right\}^{\frac{1}{2}}}{\gamma\,(1+\mu^2)} = \frac{\left\{\gamma^2\sin^2\delta+(\rho\cos\delta+\kappa)^2\right\}^{\frac{1}{2}}}{\gamma^2\,(\rho\cos\delta+\kappa)^2}.$$

Integrating by means of the transformation

$$\nu^2\sin^2\delta+(\rho\cos\delta+\kappa)^2 = \zeta^2\,(\rho\cos\delta+\kappa)^2,$$

we find

$$\varpi+\epsilon = \sec\delta\,\coth^{-1}\frac{\{\gamma^2\sin\delta+(\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{\rho\cos\delta+\kappa} - \coth^{-1}\frac{\{\gamma^2\sin^2\delta+(\rho\cos\delta+\kappa)^2\}}{(\rho\cos\delta+\kappa)\cos\delta}.$$

Hence, measuring $\rho$ from the parallel of latitude hitherto denoted by

$$\rho = -\kappa\sec\delta,$$

we find the following set of equations to determine the surface,

$$\left.\begin{aligned} x &= \xi e+\frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3 e_3 \\ \xi &= \left\{1+\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cosh\varpi \\ \xi_3 &= \left\{1+\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sinh\varpi \\ \varpi+\epsilon &= \sec\delta\,\coth^{-1}\frac{\{\gamma^2\sin^2\delta+\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta} \\ &\qquad -\coth^{-1}\frac{\{\gamma^2\sin^2\delta+\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos^2\delta} \end{aligned}\right\}. \quad (54)$$

The parallel of latitude $\rho = 0$ is the evanescent parallel surrounding the point in which the surface cuts the axis $ee_3$. Let this point be the point $e$; then this assumption determines $\epsilon$ by the equation

$$\epsilon = \sec\delta\,\coth^{-1}\infty - \coth^{-1}\infty = 0.$$

Also, in the neighbourhood of the point $e$, when $\rho^2$ is neglected,

$$\varpi = \rho\left(\frac{d\varpi}{d\rho}\right)_{\rho=0} = \frac{\rho\sin\delta}{\gamma};$$

and hence

$$\xi = 1, \quad \xi_3 = \frac{\rho\sin\delta}{\gamma}.$$

Thus, in the neighbourhood of $e$,

$$x = e+\frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\rho\sin\delta}{\gamma}e_3.$$

Hence

$$x_1 = \frac{\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\sin\delta}{\gamma}e_3.$$

Thus the tangent line at $e$ to the meridian curve in the plane $\phi$ is denoted by the line element $ex_1$, that is, by

$$\frac{\cos\delta}{\gamma}(ee_1\cos\phi+ee_2\sin\phi)+\frac{\sin\delta}{\gamma}ee_3.$$

Hence (*cf. Universal Algebra*, §§ 244, 248) the angle between this line and the axis is given by

$$\cos\theta=\frac{(ee_3\mid ex_1)}{\sqrt{\{(ee_3\mid ee_3)(ex_1\mid ex_1)\}}}=\sin\delta.$$

Hence the tangents at $e$ form a cone of semi-vertical angle $\frac{1}{2}\pi-\delta$, and the point $e$ is a conical point on the surface. By putting $-\rho'$ instead of $\rho$ in the equations it is easily seen that the surface is symmetrical with respect to the point $e$.

The plane of the parallel of latitude through the point $x$ is perpendicular to $ee_3$, and is therefore represented by the planar element $xe_1e_2$, that is, by the planar element $(\xi ee_1e_2+\xi_3e_3e_1e_2)$. The centre of the parallel of latitude lies on $ee_3$, and is therefore the point $xe_1e_2\,.\,ee_3$

But
$$\begin{aligned}xe_1e_2\,.\,ee_3&=\xi ee_1e_2\,.\,ee_3+\xi_3e_3e_1e_2\,.\,ee_3\\&=(\xi e+\xi_3e_3)(ee_1e_2e_3)\equiv\xi e+\xi_3e_3,\end{aligned}$$

omitting the numerical factor.

The distance $d$ of this point from $e$ is given by

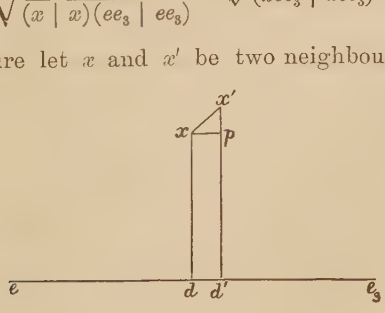
$$\cosh\frac{d}{\gamma}=\frac{\{e\mid(\xi e+\xi_3e_3)\}}{\sqrt{\lfloor(e\mid e)\{(\xi e-\xi_3e_3)\mid(\xi e+\xi_3e_3)\}\rfloor}}=\frac{\xi}{\sqrt{\{\xi^2-\xi_3^2\}}}=\cosh\varpi.$$

Hence
$$d=\varpi\gamma.$$

The radius ($\zeta$) of the parallel of latitude is the perpendicular distance from $x$ to the line $ee_3$. This is given by [*cf. Universal Algebra*, § 254 (5)]

$$\sinh\frac{\zeta}{\gamma}=\sqrt{\frac{-(xee_3\mid xee_3)}{(x\mid x)(ee_3\mid ee_3)}}=\sqrt{(xee_3\mid xee_3)}=\frac{\rho\cos\delta}{\gamma}. \qquad (55)$$

Now in the figure let $x$ and $x'$ be two neighbouring points on a



meridian, let $xd$ and $x'd'$ be the perpendiculars on to $ee_3$, and let $xp$ be the arc of the curve of equal distance from $ee_3$, which is ultimately a straight line perpendicular to $x'd'$. Then ultimately

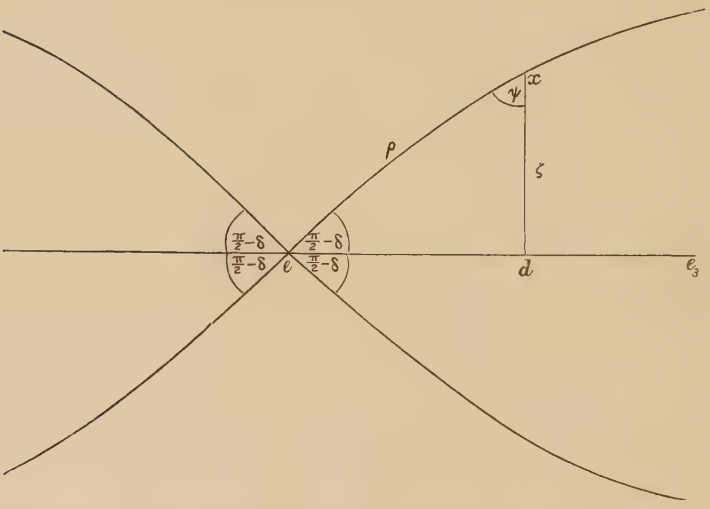
$$\overline{xx'} = \delta\rho, \quad \overline{x'p} = \delta\zeta.$$

Also, if $\psi$ be the angle which the tangent to the meridian at $x$ makes with $xd$, we have ultimately

$$\cos\psi = \frac{\overline{x'p}}{\overline{xx'}} = \frac{\delta\zeta}{\delta\rho} = \frac{\cos\delta}{\cosh\dfrac{\zeta}{\gamma}},$$

from equation (55). Thus

$$\cos\psi = \frac{\gamma\cos\delta}{\left\{\gamma^2+\rho^2\cos^2\delta\right\}^{\frac{1}{2}}}. \tag{56}$$




The form in hyperbolic space of the section of the surface by any meridian plane is suggested by the annexed figure. Both branches go to infinity at both ends. By putting $\delta = 0$, the surface becomes the limit surface investigated in Case I.

[*Added May*, 1898.—Here again, as in elliptic space, the simplest case has been omitted, namely, when $\delta = \frac{1}{2}\pi$. Then the equations become

$$\xi = (1+\beta^2)^{\frac{1}{2}}\cosh\frac{\rho+\epsilon}{\gamma(1+\beta^2)^{\frac{1}{2}}}, \quad \xi_3 = (1+\beta^2)^{\frac{1}{2}}\sinh\frac{\rho+\epsilon}{\gamma(1+\beta^2)^{\frac{1}{2}}},$$

$$\xi_1 = \beta\cos\phi, \quad \xi_2 = \beta\sin\phi.$$

Thus the equation of the surface is

$$\beta^2(\xi^2-\xi_3^2)-(1+\beta^2)(\xi_1^2+\xi_2^2)=0.$$

This is a surface of equal distance from the line $ee_3$.]

27. *Case III.*—Let $\alpha=\dfrac{\cosh\delta}{\gamma}$.

Equation (52) becomes

$$\frac{d\varpi}{d\rho}=\frac{\{\mu^2-\sinh^2\delta\}^{\frac{1}{2}}}{\gamma(1+\mu^2)}=\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac{1}{2}}}{\gamma^2+(\rho\cosh\delta+\kappa)^2}.$$

Integrating by means of the transformation

$$(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta=\xi^2(\rho\cosh\delta+\kappa)^2,$$

we find

$$\varpi+\epsilon=\operatorname{sech}\delta\tanh^{-1}\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac{1}{2}}}{\rho\cosh\delta+\kappa}$$
$$-\tanh^{-1}\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac{1}{2}}}{(\rho\cosh\delta+\kappa)\cosh\delta}.$$

The generality of the surface will not be impaired by assuming $\kappa=0=\epsilon$. Then the following set of equations determine the surface

$$\left.\begin{aligned}
x&=\xi e+\frac{\rho\cosh\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3 e_3\\
\xi&=\left\{1+\frac{\rho^2\cosh^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cosh\varpi\\
\xi_3&=\left\{1+\frac{\rho^2\cosh^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sinh\varpi\\
\varpi&=\operatorname{sech}\delta\tanh^{-1}\frac{\{\rho^2\cosh\delta-\gamma^2\sinh^2\delta\}^{\frac{1}{2}}}{\rho\cosh\delta}\\
&\qquad-\tanh^{-1}\frac{\{\rho^2\cosh^2\delta-\gamma^2\sinh^2\delta\}^{\frac{1}{2}}}{\rho\cosh^2\delta}
\end{aligned}\right\}. \qquad (57)$$

The value of $\varpi$ is imaginary if $\rho$ lie between $\pm\gamma\tanh\delta$; hence the surface does not cut the axis $ee_3$ in real points.

Also $\varpi=0$, when $\rho=\pm\gamma\tanh\delta$; and $\varpi$ is real as $\rho$ increases numerically from $\pm\gamma\tanh\delta$ to $\pm\infty$.

Also, as $\rho$ increases from $\gamma \tanh \delta$ to $+\infty$,

$$\frac{d\varpi}{d\rho} = \frac{\{\rho^2 \cosh^2 \delta - \gamma^2 \sinh^2 \delta\}^{\frac{1}{2}}}{\gamma^2 + \rho^2 \cosh^2 \delta},$$

and is therefore always positive. Hence $\varpi$ continually increases, and

$$\varpi_{(\rho=\infty)} = \operatorname{sech} \delta \tanh^{-1} 1 - \tanh^{-1} \{\operatorname{sech} \delta\} = \infty.$$

Hence when $\rho$ is infinite, the limit of $\xi/\xi_3 = 1$.

Negative values of $\rho$ give the same numerical values of $\varpi$, with the opposite sign, as the numerically equal positive values of $\rho$. Hence, for negative values of $\rho$, $\xi/\xi_3$ runs through the same numerical series of values only with the opposite sign, and the limit of $\xi/\xi_3$, when $\rho = -\infty$, is $-1$.

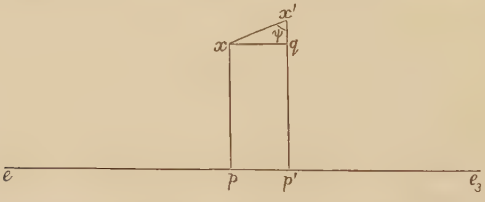
The perpendicular distance of $x$ from $ee_3$ is given by

$$\sinh \frac{\zeta}{\gamma} = \sqrt{\frac{-(xee_3 \mid xee_3)}{(x \mid x)(ee_3 \mid ee_3)}} = \sqrt{(xee_3 \mid xee_3)} = \frac{\rho \cosh \delta}{\gamma}.$$

As $\rho$ increases from $\gamma \tanh \delta$ to $\infty$, $\zeta$ increases from $\delta\gamma$ to $\infty$.

The centre of the parallel of latitude through $x$ is the point $xe_1e_2 . ee_3$, that is, the point $\xi e + \xi_3 e_3$. As $\varpi$ varies from 0 to $\infty$, this point moves from $e$ to $e + e_3$, a point on the absolute on $ee_3$; and, as $\varpi$ varies from 0 to $-\infty$, this point moves from $e$ to $e - e_3$, the other point on the absolute on $ee_3$.

Let $x$ and $x'$ be neighbouring points on the same meridian curve, and draw $xp$ and $x'p'$ perpendiculars to $ee_3$. Let $\psi$ be the angle



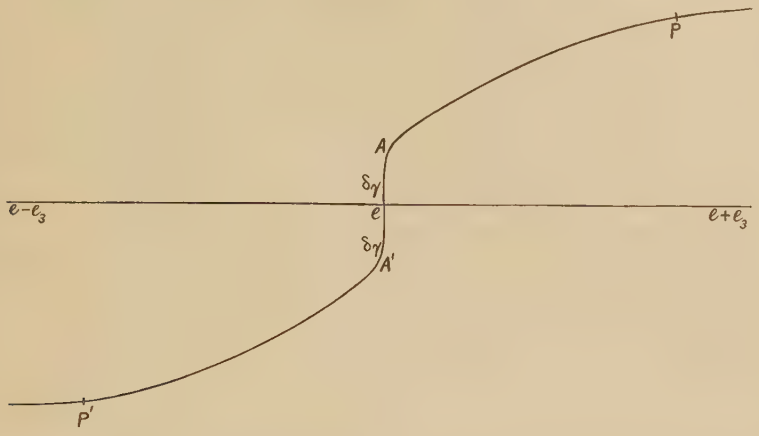

between the tangent at $x$ and $xp$. Then, by the same construction as before,

$$\cos \psi = \frac{d\zeta}{d\rho} = \frac{\cosh \delta}{\cosh \frac{\zeta}{\gamma}} = \frac{\gamma \cosh \delta}{\{\gamma^2 + \rho^2 \cosh^2 \delta\}^{\frac{1}{2}}}.$$

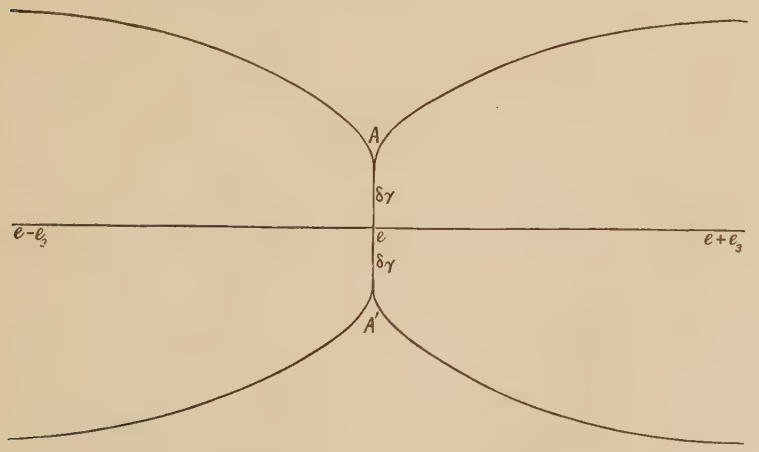
Hence, when $\rho = \gamma \tanh \delta$,

$\psi$ is zero, and as $\rho$ increases to $\infty$, $\psi$ increases to $\frac{1}{2}\pi$.

Thus the form in hyperbolic space of a meridian curve is suggested by the annexed figure.




There are two detached branches, $AP$ and $A'P'$, which both go to infinity; $AA'$ is the tangent to the curve both at $A$ and at $A'$. The arc $AP = \rho - \gamma \tanh \delta$, where the parameter $\rho$ defines the point $P$; also, if $\rho$ be negative and is written $-\rho'$, and if it defines a point $P'$ on the other branch, then the arc $A'P' = \rho' - \gamma \tanh \delta$. The surface is formed by revolving this figure round $ee_3$. A meridian section is shown in the annexed figure.




When $\delta = 0$, the surface becomes a Bolyai limit-surface.

## Appendix.

*Synthetic Proofs of some of the above results, communicated to me by Professor Burnside, F.R.S., April*, 1898.

At Mr. Whitehead's request I have written out for publication at the end of his paper a note which the latter part of the paper suggested to me when I read it in manuscript. The note is the outcome of an attempt to approach the subject dealt with in the last half of Mr. Whitehead's paper from a synthetical point of view.

The locus of a point whose perpendicular distance from a given straight line is constant is called an equidistant surface; the given line is the axis of the surface, and the constant distance the radius.

An equidistant surface is clearly self-congruent for all translations along and rotations round the axis, and also for rotations through two right angles about any radius of the surface.

If the section of the surface by a plane through the axis be called a meridian, and that by a plane perpendicular to the axis a circular section, then, by every displacement for which the surface is self-congruent, a meridian is changed into a meridian and a circular section into a circular section. The distance between any two given circular sections (or meridians) measured along a meridian (or circular section) is, therefore, the same at all points.

Consider now a curve on the surface which meets all the meridians at a constant angle. Through a given point $A$ on the surface, one, and only one, such curve can be drawn to meet the meridians at a given angle $\alpha$ $(0 \leqslant \alpha < \pi)$. If the surface be rotated through two right angles about the radius through $A$, the surface, and, therefore, also the curve, is changed into itself. Hence the osculating plane of the curve at $A$ must contain the radius through $A$; in other words, the curve is a geodesic.

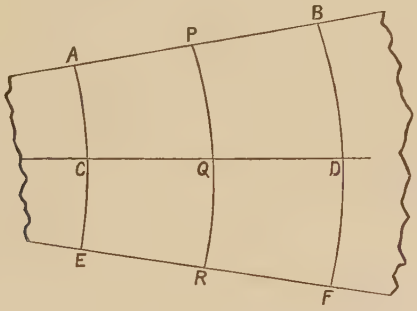
The geodesics on the surface are, therefore, the curves which meet the meridians (and circular sections) at a constant angle.

An infinite number of geodesics can clearly be drawn to join any two points on the surface. Among these, the shortest will be that one for which the (acute) angle $\alpha$ is least. Suppose, now, that any three points $A$, $B$, $C$ on the surface are joined by shortest geodesics $AB$, $BC$, $CA$. Since each of these makes equal angles with each meridian it meets, it immediately follows that the sum of the angles of the geodesic triangle $ABC$ is equal to two right angles. The infinitesimal geodesic geometry of the surface is, therefore, the same as that of the Euclidean plane.

The radius of the equidistant surface considered may be any whatever; and therefore it follows that a portion of the surface of one equidistant may be cut out and applied without stretching or tearing on an equidistant of different radius.

The existence of surfaces, in elliptic or hyperbolic space, whose infinitesimal geometry is the same as that of the Euclidean plane being thus demonstrated, their form, when surfaces of revolution, may be determined by the consideration that, in a given space, a portion of any one such surface must be applicable on any other.

Let the figure represent a finite portion cut out from an equidistant surface; $APB$, $CQD$, $ERF$ being a set of geodesics which meet in a point on the surface, while $ACE$, $PQR$, $BDF$ are a family of curves on the surface which meet the set of geodesics everywhere at right angles. Since the geometry of a finite portion of the surface is the same as that of the Euclidean plane, such a set of orthogonal curves must exist, and the relations




$$AP = CQ = ER,$$

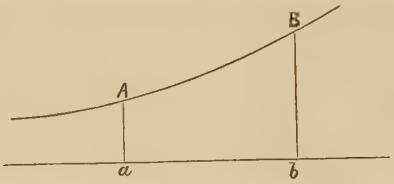
$$\text{arc } PQR - \text{arc } ACE = \alpha \,.\, AP,$$

must be satisfied for each curve $PQR$ of the family, $\alpha$ being a constant.

Now bring opposite edges to coincidence so that $A$ and $E$, $P$ and $R$, $B$ and $F$ coincide, while $ACE$, $PQR$, $BDF$ become circles with their centres on a straight line and their planes perpendicular to it. The surface will then form a portion of a surface of revolution whose infinitesimal geodesic geometry is that of the Euclidean plane.

Let $AB$ be a meridian section of the surface so formed, and $ab$ the

axis of the surface; $Aa$, $Bb$ being the perpendiculars from $A$ and $B$ on the axis.




Then, taking suitable units of length, the meridian section is given (i.) in elliptic space by

$$\sin Bb - \sin Aa = \frac{a}{2\pi} \text{ arc } AB;$$

and (ii.) in hyperbolic space by

$$\sinh Bb - \sinh Aa = \frac{a}{2\pi} \text{ arc } AB.$$

It will be found that for elliptic space the relation

$$\frac{a^2}{4\pi^2} < 1$$

is necessary in order that the surface may be real; and that, when $a \neq 0$, the surface necessarily cuts the axis. Reckoning the arc $s$ from a point where the curve cuts the axis and representing the ordinate by $y$, the equation is

$$\sin y = \frac{as}{2\pi}.$$

In hyperbolic space there is no limitation on $a$. If $a^2 < 4\pi^2$, the curve cuts the axis; and, with the previous notation, the equation is

$$\sinh y = \frac{as}{2\pi}.$$

If $a^2 > 4\pi^2$, the curve does not cut the axis. In this case, writing $\cosh \beta$ for $\frac{a}{2\pi}$, the equation can be put in the form

$$\sinh y - \sinh \beta = s \cosh \beta.$$

The dividing case $a = 2\pi$ gives the limit-surface, as it obviously should; for the limit-surface may be regarded as the equidistant of a straight line whose only real points are at infinity.

## *On Linear Homogeneous Continuous Groups whose Operations are Permutable.* *By* W. Burnside. Received March 7th, 1898. Read March 10th, 1898.

Two continuous groups with the same number of parameters, such that all the operations of each group are permutable with each other, are simply isomorphic. They are not, however, necessarily conjugate within a given more extensive group which contains them both. All such groups which are linear and homogeneous in $n$ variables are contained in the general linear homogeneous group. The investigation of this paper is directed towards a classification of the groups of permutable operations contained in the general linear homogeneous group into conjugate sets. Some of the properties of such groups, when intransitive, are contained in the latter part of the paper; but the main results obtained apply only to transitive groups. It is a result of known theorems, which, for convenience, are stated below, that for such groups, when transitive, the finite equations can be written in a form in which the parameters, as well as the variables, enter linearly and homogeneously. It is almost entirely from the point of view of the finite equations that the question is here treated.

In the *Vorlesungen über continuierliche Gruppen* of Lie-Scheffers (pp. 627–633) a series of theorems on linear homogeneous groups, due originally to Herr Study (*Leipziger Berichte*, 1889), are given, which may be stated as follows :—

(i) The totality of the linear homogeneous transformations of $n$ variables, which are permutable with every operation of a linear and homogeneous group $G$ of order $r$ in the $n$ variables, forms a linear and homogeneous group $G'$ of order $s$ in the finite equations of which the $s$ parameters enter linearly and homogeneously.

(ii) If $G$ is a simply transitive group, so that $r = n$, then $s = n$, and $G'$ is a simply transitive group and is also simply isomorphic with $G$. When $r = s = n$, the operations of $G$ form the totality of the operations, which are permutable with every operation of $G'$, and therefore the parameters of $G$ may be chosen so that its finite equations contain the parameters linearly and homogeneously. Hence :—

(iii) The finite equations of a simply transitive linear homogeneous group in $n$ variables can always be written so as to contain the $n$ parameters linearly and homogeneously.

1. Let $$x'_s = \Sigma\, a_{sik} x_i y_k, \quad (s = 1, 2, \ldots, n),$$

be the finite equations of a simply transitive linear homogeneous group* $G$ of order $n$; $y_1, y_2, \ldots, y_n$ being the $n$ parameters. The group must contain a substitution changing any one set of values (or point) $x_1, x_2, \ldots, x_n$ into any other, so long as the point does not belong to one of a finite number of multiplicities, given by equating homogeneous functions of the variables to zero, which are unchanged by every substitution of the group. It may be assumed that the variables are so chosen that the point $(1, 0, 0, \ldots, 0)$ does not belong to such an invariant multiplicity. Then, since there must be a substitution changing this point into any arbitrarily chosen point, the $n$ functions

$$\Sigma\, a_{s1k}\, y_k, \quad (s = 1, 2, \ldots, n),$$

must be independent. A new set of parameters $z_1, z_2, \ldots, z_n$ may therefore be introduced, such that

$$z_s = \Sigma\, a_{s1k} y_k.$$

Let $$x'_s = \Sigma\, \beta_{sik} x_i z_k$$

be the finite equations of the group expressed in terms of these parameters. Then

$$\beta_{s1k} = 0, \quad (s \neq k),$$

and

$$\beta_{s1s} = 1.$$

Such a form for the finite equations of the group will be called a *standard form.* It is not, of course, unique.

The finite equations must contain, for some value of the parameters, the identical substitution. From the forms of $\beta_{s1k}$, this clearly can only be

$$z_1 = 1, \quad z_2 = \ldots = z_n = 0.$$

Hence

$$\beta_{si1} = 0, \quad (s \neq i),$$

$$\beta_{ss1} = 1.$$

The operation $(z_1, 0, 0, \ldots, 0)$, *i.e.*, the operation for which the parameters have these values, is therefore

$$x'_1 = z_1 x_1, \quad x'_2 = z_1 x_2, \quad \ldots, \quad x'_n = z_1 x_n,$$

and the group necessarily contains the infinitesimal operation

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots + x_n \frac{\partial}{\partial x_n}.$$

* Throughout this paper the word "group" will be used in the sense of "linear homogeneous continuous group."

2. In a standard form the group is its own parameter group. For, if $(z_1'', z_2'', \ldots, z_n'')$ is the operation that results from first carrying out $(z_1, z_2, \ldots, z_n)$ and then $(z_1', z_2', \ldots, z_n')$, $z_s''$ is the coefficient of $x_1$ in

$$\sum_{i,k} \beta_{sik} z_k' \sum_{j,l} \beta_{ijl} x_j z_l.$$

Hence

$$\begin{aligned} z_s'' &= \sum_{i,k} \beta_{sik} z_k' \sum_{l} \beta_{i1l} z_l \\ &= \sum_{i,k} \beta_{sik} z_i z_k', \quad (s = 1, 2, \ldots, n), \end{aligned}$$

so that the equations connecting $z$, $z'$, and $z''$ are the same as those connecting $x$, $z$, and $x'$. From this a sufficient set of relations connecting the $\beta$'s to ensure that the equations shall have the group property may be at once written down (Lie-Scheffers, p. 637); namely, the values of $x_s''$, as formed by carrying out successively the operations $z$ and $z'$, and as formed by carrying out the single operation $z''$, must be the same.

This gives $$\sum_{i,j,k,l} \beta_{sik}\beta_{ijl} x_j z_l z_k' = \sum_{i,j,k,l} \beta_{sji}\beta_{ilk} x_j z_l z_k';$$

and therefore, if the relation

$$\sum_{i} (\beta_{sik}\beta_{ijl} - \beta_{sji}\beta_{ilk}) = 0$$

hold, for all sets of values of $s$, $j$, $k$, $l$ the group property is ensured. Interchanging $k$ and $j$, these relations become

$$\sum_{i} (\beta_{ski}\beta_{ilj} - \beta_{sij}\beta_{ikl}) = 0,$$

and therefore, the equations

$$x_s' = \Sigma \beta_{ski} x_i z_k, \quad (s = 1, 2, \ldots, n),$$

have the group property. They define a group $G'$ which is moreover expressed in standard form.

A linearly independent set of infinitesimal operations of $G$ is

$$X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots + x_n \frac{\partial}{\partial x_n},$$

and

$$X_k = \sum_{s,i} \beta_{sik} x_i \frac{\partial}{\partial x_s}, \quad (k = 2, 3, \ldots, n).$$

For $G'$, $X_1$ is the same, and

$$X_k' = \sum_{s,i} \beta_{ski} x_i \frac{\partial}{\partial x_s}, \quad (k = 2, 3, \ldots, n).$$

Hence

$$\begin{aligned} (X_k X_l') &= \sum_{s,i,t} (\beta_{sik}\beta_{tls} - \beta_{sli}\beta_{tsk}) x_i \frac{\partial}{\partial x_t} \\ &= 0, \end{aligned}$$

in consequence of the above relations between the $\beta$'s. Hence every operation of $G'$ is permutable with every operation of $G$; and $G'$ is the linear homogeneous group of order $n$ every one of whose operations is permutable with every operation of $G$.

3. Suppose, now, that the group $G$ of § 2, in standard form, is an Abelian* group. This will involve, in addition to the relations between the constants already obtained, further relations expressing that $(X_k X_l)$ shall vanish for all values of $k$ and $l$. Now

$$(X_k X_l) = \sum_{s,i,t} (\beta_{sik}\beta_{tsl} - \beta_{sil}\beta_{tsk}) x_i \frac{\partial}{\partial x_t};$$

and therefore

$$\sum_s (\beta_{sik}\beta_{tsl} - \beta_{sil}\beta_{tsk}) = 0$$

for all values of $i$, $t$, $k$, $l$.

If, in this equation, unity be written for $i$, and the previous relations

$$\beta_{s1k} = 0, \quad (s \neq \kappa),$$

$$\beta_{s1s} = 1,$$

be taken into account, the relation becomes

$$\beta_{tkl} = \beta_{tlk}$$

for all values of $t$, $k$, and $l$. Making use of this latter relation, the equation

$$\sum_s (\beta_{sik}\beta_{tsl} - \beta_{sil}\beta_{tsk}) = 0$$

may be replaced by

$$\sum_s (\beta_{ski}\beta_{tsl} - \beta_{skl}\beta_{tis}) = 0,$$

and the latter is identical, except in the letters used, with the form of the relation found in § 2 among the $\beta$'s for any simply transitive group.

Hence the necessary and sufficient relations among the $\beta$'s in order that

$$x'_s = \Sigma \beta_{sik} x_i y_k$$

should be the finite equations of a simply transitive Abelian group in standard form are

$$\beta_{s1s} = 0, \quad \beta_{s1t} = 0, \quad (s \neq t),$$

$$\sum_s (\beta_{sik}\beta_{tsl} - \beta_{sil}\beta_{tsk}) = 0,$$

for all values of $i$, $t$, $k$, and $l$. Involved in these equations are the relations

$$\beta_{tkl} = \beta_{tlk}$$

for all values of $t$, $k$, and $l$.

* The phrase "Abelian group" will be used, as in the theory of groups of finite order, to denote a group all of whose operations are permutable.

It follows at once from known theorems (*Theorie der Transformationsgruppen*, Lie-Engel, Vol. I, p. 376*) that a transitive Abelian group cannot contain a number of independent parameters greater than the number of variables; *i.e.*, that it cannot be more than simply transitive. Hence any transitive Abelian group can be brought to the form given by the above equations.

4. If

$$x'_s = \Sigma a_{si} x_i, \quad (s = 1, 2, \ldots, n),$$

are the finite equations of a group, the coefficients $a_{si}$ being functions of the parameters, the equation

$$\begin{vmatrix} a_{11}+\lambda, & a_{12}, & a_{13}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}+\lambda, & \ldots, & \ldots, & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n,1}, & a_{n,2}, & \ldots, & \ldots, & a_{nn}+\lambda \end{vmatrix} = 0$$

is called the characteristic equation† of the group in finite form. If $a_{si}$ are the coefficients of a particular substitution, the equation is called the characteristic equation of that substitution, and the determinant equated to zero is the characteristic determinant of the substitution. When zero is written for $\lambda$ in the characteristic determinant the result is called the determinant of the substitution. This must be different from zero, as otherwise the inverse substitution would not be definite.

Let the characteristic determinant itself be represented by $D(\lambda)$; and let $D_r(\lambda)$ be the greatest rational integral function of $\lambda$ which divides all the $r^{\text{th}}$ minors of $D(\lambda)$, the coefficient of the highest power of $\lambda$ in $D_r(\lambda)$ being taken as unity. Then $D_r(\lambda)$ is a factor of $D_{r-1}(\lambda)$, and, unless $D_r(\lambda)$ is unity, the degree of $D_{r-1}(\lambda)$ is greater than the degree of $D_r(\lambda)$. The functions

$$\frac{D(\lambda)}{D_1(\lambda)}, \quad \frac{D_1(\lambda)}{D_2(\lambda)}, \quad \frac{D_2(\lambda)}{D_3(\lambda)}, \quad \ldots\ldots$$

are therefore a set of rational integral functions of $\lambda$. They are called the *invariant-factors* of the determinant $D(\lambda)$. This conception was first introduced by Weierstrass in the memoir "Zur Theorie der

* I owe this reference, as well as much careful criticism which has helped me to avoid some errors, to the kindness of one of the referees.

† This is not to be confounded with the characteristic equation of an infinitesimal operation of the group.

bilinearen und quadratischen Formen" (*Berliner Monatsberichte*, 1868, and *Mathematische Werke*, Vol. II., pp. 19–44). The German term used is *Elementar-Theiler*. The conception may be extended to a determinant of any form. The earliest investigation with which I am acquainted into the properties of the invariant-factors of a determinant occurs in a paper of H. J. Stephen Smith, with the title "Arithmetical Notes" (*Proc. L. M. S.*, Vol. IV., 1873, pp. 236–249). The determinants there considered have integers for their elements; but the method is perfectly general. Herr Frobenius ("Über die Elementartheiler der Determinanten," *Berliner Sitzungsberichte*, 1894, pp. 31–44) has generalized the form of Stephen Smith's results and has somewhat added to them.*

It is shown in these memoirs that each of the invariant-factors is divisible by all that follow it; so that, if the factors are given, their sequence is necessarily determined by their degrees.

5. In the memoir of Weierstrass, quoted above, in which the conception of invariant-factors is first introduced, a theorem is proved which contains, as a special case, the conditions under which two linear homogeneous substitutions are conjugate within the general linear homogeneous group.

The special theorem may be stated as follows:—

If $$x'_s = \sum_i a_{si} x_i, \quad (s = 1, 2, \ldots, n),$$

or $A$, and $$x'_s = \sum_i b_{si} x_i, \quad (s = 1, 2, \ldots, n),$$

or $B$, are two linear homogeneous substitutions in $n$ variables $x_1$, $x_2$, ..., $x_n$, the necessary and sufficient conditions that it may be possible to find a substitution $C$, or

$$x'_s = \Sigma c_{si} x_i, \quad (s = 1, 2, \ldots, n),$$

in the $n$ variables, such that

$$C^{-1}BC = A,$$

are that the invariant-factors of the characteristic determinant of $A$ shall be identical with those of the characteristic determinant of $B$. This theorem is clearly of fundamental significance in dealing with the linear homogeneous group and its sub-groups. In particular, it gives a canonical form to which any linear substitution, the invariant-factors of whose characteristic determinant are known, may be reduced.

* Reference may also be made to a memoir by S. Pincherle "Sur le calcul fonctionnel distributif" (*Math. Ann.* Vol. XLIX, pp. 325–381), where the theory of invariant-factors is presented in a new and very simple form.

Thus let $a_1, a_2, a_3, \ldots$ be a set of integers, no one of which is greater than any that precede it, and let $b_1, b_2, b_3, \ldots$; $c_1, c_2, c_3, \ldots$; $\ldots$; $\ldots$; be similar sets. Suppose that $\Sigma a + \Sigma b + \Sigma c + \ldots = n$, and that $\alpha, \beta, \gamma, \ldots$ are quantities no two of which are equal. Then

$$
\begin{array}{llllll}
x'_1 = \alpha x_1, & x'_2 = x_1 + \alpha x_1, & \ldots & x'_{a_1} = x_{a_1-1} + \alpha x_{a_1}, & & \\
x_{a_1+1} = \alpha x_{a_1+1}, & x'_{a_1+2} = x_{a_1+1} + \alpha x_{a_1+2}, & \ldots & x'_{a_1+a_2} = x_{a_1+a_2-1} + \alpha x_{a_1+a_2}, & & \\
x'_{a_1+a_2+1} = \alpha x_{a_1+a_2+1}, & x_{a_1+a_2+2} = x_{a_1+a_2+1} + \alpha x_{a_1+a_2+2}, & \ldots & x'_{a_1+a_2+a_3} = x_{a_1+a_2+a_3-1} + \alpha x_{a_1+a_2+a_3}, & & \\
\ldots \quad \ldots & \ldots \quad \ldots & \ldots & \ldots \quad \ldots & \ldots & \ldots \\
y'_1 = \beta y_1, & y'_2 = y_1 + \beta y_2, & \ldots & y'_{b_1} = y_{b_1-1} + \beta y_{b_1}, & & \\
y'_{b_1+1} = \beta y_{b_1+1}, & y'_{b_1+2} = y_{b_1+1} + \beta y_{b_1+2}, & \ldots & y'_{b_1+b_2} = y_{b_1+b_2-1} + \beta y_{b_1+b_2}, & & \\
\ldots \quad \ldots & \ldots \quad \ldots & \ldots & \ldots \quad \ldots & \ldots & \ldots \\
z'_1 = \gamma z_1, & z'_2 = z_1 + \gamma z_2, & \ldots & z'_{c_1} = z_{c_1-1} + \gamma z_{c_1}, & & \\
\ldots \quad \ldots & \ldots \quad \ldots & \ldots & \ldots \quad \ldots & \ldots & \ldots
\end{array}
$$

is a linear substitution of $n$ variables the invariant factors of whose characteristic determinant are

$$(\alpha+\lambda)^{a_1}(\beta+\lambda)^{b_1}(\gamma+\lambda)^{c_1}\ldots, \quad (\alpha+\lambda)^{a_2}(\beta+\lambda)^{b_2}(\gamma+\lambda)^{c_2}\ldots,$$
$$(\alpha+\lambda)^{a_3}(\beta+\lambda)^{b_3}(\gamma+\lambda)^{c_3}, \quad \ldots, \quad \ldots.$$

In fact, in the simple form which the characteristic determinant takes for this substitution, the invariant factors are evident on inspection. Their formal calculation is carried out by Herr Netto in a memoir "Zur Theorie der linearen Substitutionen" (*Acta Mathematica*, Vol. XVII, pp. 267–271, 1893).

6. Let
$$x'_s = \sum_{i,k} \alpha_{sik} x_i y_k, \quad (s = 1, 2, \ldots, n),$$

be the finite equations of an Abelian group $G$; the $y$'s being the parameters. It is not at present assumed that the group is transitive, and the number, $m$, of parameters is not therefore necessarily equal to the number of variables. Let $S$ be any substitution of the group, and let $T$ be a linear homogeneous substitution which will transform $S$ into its canonical form. Then $T^{-1}GT$ is an Abelian group, conjugate to $G$ in the general linear homogeneous group and containing a substitution $T^{-1}ST$ in its canonical form. With the notation of the previous paragraph for the canonical form of $T^{-1}ST$, every substitution of the variables which is permutable with $T^{-1}ST$ must interchange the $x$'s, $y$'s, ... separately among themselves. For, if $U$, or

$$x'_1 = Ax_1 + \ldots + By_1 + \ldots,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

is permutable with $T^{-1}ST$, then $UT^{-1}ST$ and $T^{-1}STU$ are identical. But these substitutions replace $x_1$ by

$$\alpha(Ax_1 + \ldots + By_1 + \ldots)$$

and

$$A\alpha x_1 + \ldots + B\beta y_1 + \ldots,$$

respectively. Hence $B$ must be zero. So far as they affect the $x$'s, the substitutions of $T^{-1}GT$ will form an Abelian group in $a_1 + a_2 + a_3 + \ldots$ variables. Unless, for the general substitution of this group, the roots of the characteristic equation are all equal, the above process may be repeated, and the $x$'s divided up into a number of sets of variables, such that the variables of each set are transformed among themselves by all the operations of $T^{-1}GT$. The same process may be applied to the $y$'s, $z$'s, &c.

Hence, finally, by a suitable substitution, the original $n$ variables may be replaced by $n$ new variables divided into $r$ ($\leqslant n$) sets, such that every operation of the group transforms each of the sets among themselves; while for each set, the characteristic equation of the group, so far as it affects the set, has all its roots equal.

Now the characteristic determinant of the group in its original form is

$$\begin{vmatrix} \Sigma a_{11k} y_k + \lambda, & \Sigma a_{12k} y_k, & \ldots, & \Sigma a_{1nk} y_k \\ \Sigma a_{21k} y_k, & \Sigma a_{22k} y_k + \lambda, & \ldots, & \Sigma a_{2nk} y_k \\ \ldots \quad \ldots & \ldots \quad \ldots & \ldots & \ldots \\ \Sigma a_{n1k} y_k, & \Sigma a_{n2k} y_k, & \ldots, & \Sigma a_{nnk} y_k + \lambda \end{vmatrix}.$$

This is a homogeneous function of $y_1, y_2, \ldots, y_m, \lambda$ of the $n^{\text{th}}$ degree; and it is invariant for any transformation of the group.

In the final form of the group the characteristic determinant is the product of the $r$ characteristic determinants of the $r$ groups, obtained by neglecting in turn all but one of the $r$ sets into which the variables are divided. Each of these is at once the power of a linear function of $\lambda$, and a homogeneous function of $y_1, y_2, \ldots, y_m, \lambda$; *i.e.*, it is a power of a linear function of $y_1, y_2, \ldots, y_m, \lambda$.

Finally, therefore, the characteristic determinant of an Abelian group, in the finite equations of which the parameters enter linearly and homogeneously, is the product of $n$ linear homogeneous factors.*

* This result does not differ, except in form, from that given by Herr Frobenius "Über vertauschbare Matrizen" (*Berliner Sitzungsberichte*, 1896, p. 610). On account of its importance a formal proof of Herr Frobenius's result, from the point of view here taken, may be added.

Let $A$ or $\quad x'_s = \Sigma a_{si} x_i, \quad (s, i = 1, 2, \ldots, n),$

and $B$ or $\quad x'_s = \Sigma \beta_{si} x_i, \quad (s, i = 1, 2, \ldots, n),$

7. If $G$ is a transitive group, so that $m = n$, the groups by which each of the $r$ sets of variables are interchanged among themselves must also be transitive. Hence each must contain as many

---

be two permutable substitutions. Then the set of substitutions

$$x_s' = \Sigma\,(\alpha_{si}y + \beta_{si}z)\,x_i, \quad (s,\ i = 1,\ 2,\ \ldots,\ n),$$

where $y$ and $z$ are parameters, are all permutable with each other. They will not, however, in general, form a group, though they will generate a group of permutable operations. Let

$$x_s' = \Sigma\gamma_{si}x_i$$

be an operation of this group which is not contained in the above set. Then, $y$, $z$, $w$ being arbitrary parameters,

$$x_s' = \Sigma\,(\alpha_{si}y + \beta_{si}z + \gamma_{si}w)\,x_i$$

is a set of permutable substitutions. If they do not constitute a group, the same process may be repeated; and it must eventually lead to a linear homogeneous Abelian group which is linear and homogeneous in the parameters. The characteristic determinant of the general operation of this group is the product of $n$ factors, linear and homogeneous in $\lambda$ and the parameters $y$, $z$, .... Let the factors be

$$\begin{aligned} &\eta_1 y + \zeta_1 z + \ldots + \lambda, \\ &\eta_2 y + \zeta_2 z + \ldots + \lambda, \\ &\cdots \quad \cdots \quad \cdots \\ &\eta_n y + \zeta_n z + \ldots + \lambda. \end{aligned}$$

Then, putting $y=1$, $z = w = \ldots = 0$, $-\eta_1$, $-\eta_2$, ..., $-\eta_n$ are the roots of the characteristic equation of $A$; and, similarly, $-\zeta_1$, $-\zeta_2$, ..., $-\zeta_n$ are the roots of the characteristic equation of $B$. Also, putting $w = \ldots = 0$, the characteristic determinant of

$$x_s' = \Sigma\,(\alpha_{si}y + \beta_{si}z)\,x_i$$

is $$(\eta_1 y + \zeta_1 z + \lambda)(\eta_2 y + \zeta_2 y + \lambda)\ldots(\eta_n y + \zeta_n z + \lambda).$$

In other words, if

$$\begin{Bmatrix} \alpha_{11}, & \alpha_{12}, & \ldots, & \alpha_{1n} \\ \alpha_{21}, & \alpha_{22}, & \ldots, & \alpha_{2n} \\ \cdots & \cdots & & \cdots \\ \alpha_{n1}, & \alpha_{n2}, & \ldots, & \alpha_{nn} \end{Bmatrix} \text{ and } \begin{Bmatrix} \beta_{11}, & \beta_{12}, & \ldots, & \beta_{1n} \\ \beta_{21}, & \beta_{22}, & \ldots, & \beta_{2n} \\ \cdots & \cdots & & \cdots \\ \beta_{n1}, & \beta_{n2}, & \ldots, & \beta_{nn} \end{Bmatrix}$$

are two permutable matrices of non-vanishing determinants, and if $\eta_1, \eta_2, \ldots, \eta_n$ and $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the roots of

$$\begin{vmatrix} \alpha_{11}-\lambda, & \alpha_{12}, & \ldots, & \alpha_{1n} \\ \alpha_{21}, & \alpha_{22}-\lambda, & \ldots, & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1}, & \alpha_{n2}, & \ldots, & \alpha_{nn}-\lambda \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} \beta_{11}-\lambda, & \beta_{12}, & \ldots, & \beta_{1n} \\ \beta_{21}, & \beta_{22}-\lambda, & \ldots, & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1}, & \beta_{n2}, & \ldots, & \beta_{nn}-\lambda \end{vmatrix} = 0,$$

then a correspondence can be established between the $\eta$'s and the $\zeta$'s such that the roots of

$$\begin{vmatrix} \alpha_{11}+\beta_{11}-\lambda, & \alpha_{12}+\beta_{12}, & \ldots, & \alpha_{1n}+\beta_{1n} \\ \alpha_{21}+\beta_{21}, & \alpha_{22}+\beta_{22}-\lambda, & \ldots, & \alpha_{2n}+\beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1}+\beta_{n1}, & \alpha_{n2}+\beta_{n2}, & \ldots, & \alpha_{nn}+\beta_{nn}-\lambda \end{vmatrix} = 0$$

are $\eta_1+\zeta_1$, $\eta_2+\zeta_2$, .... $\eta_n+\zeta_n$. The extension of this to the case in which the determinants of one or both of the matrices vanish is so obvious that it need not be given at length.

parameters as variables. In this case, then, the original $n$ parameters may be replaced by $n$ new parameters, linear and homogeneous functions of the old ones, such that they separate into $r$ sets, corresponding to the $r$ sets of variables. The groups which interchange each of the $r$ sets of variables among themselves will then be independent in the sense that no one of them involves either a variable or a parameter occurring in any of the others. They are therefore sub-groups of the original group; and the original group is their direct product.

In a discussion of transitive Abelian groups it is therefore sufficient to deal with those groups for which the roots of the characteristic equation are all equal.

Suppose, then, that $G$ is a transitive Abelian group for which the roots of the characteristic equation are all equal. Suppose also that, for a general substitution of the group, *i.e.*, when the parameters are arbitrary, the invariant factors of the characteristic determinant are

$$(y_1+\lambda)^r,\ (y_1+\lambda)^s,\ (y_1+\lambda)^t,\ \ldots,$$

so that $r \geqslant s \geqslant t \geqslant \ldots$.

If the group be transformed by any substitution in the $n$ variables, these invariant factors of a general substitution necessarily remain unaltered; and therefore two transitive Abelian groups in $n$ variables, for which the sets of integers $r, s, t, \ldots$ are not the same, cannot be of the same type; that is, they are not conjugate within the general linear group. Supposing the set of integers $r, s, t, \ldots$ given, the question at once arises as to how many distinct types of Abelian groups correspond to them; and a complete answer to this question would be equivalent to a complete classification of transitive Abelian groups. With a view of determining the nature of the answer to this question I have considered the two simplest cases in which all but one, or all but two, of the integers $r, s, t, \ldots$ are zero. In the first case it appears, as is almost immediately obvious, that there is only one type of group. In the second case there are two types or one according as $s$ does or does not lie between 1 and $\frac{1}{2}(r+1)$.

The process used is as follows. First, a canonical form is obtained for the most general group $H$, every one of whose substitutions is permutable with a given substitution $S$. (To simplify the printed form it is assumed that the characteristic determinant of $S$ has only three invariant factors distinct from unity; but the result may be at once generalized). Every Abelian group in the $n$ variables which contains $S$ must itself be contained in $H$; and therefore it is sufficient

to consider the distinct types of transitive Abelian groups contained in $H$. These are discussed for the two cases in which $S$ has either one or two invariant factors distinct from unity.

8. Let $S'$ be a substitution the invariant factors of whose characteristic determinant are

$$(e+\lambda)^r,\ (e+\lambda)^s,\ (e+\lambda)^t,\ 1,\ 1,\ \ldots.$$

Within the general linear group this substitution is conjugate to, and can, therefore, be transformed into,

$$\begin{aligned} x_1' &= ex_1, \quad x_2' = x_1 + ex_2, \quad \ldots, \quad x_r' = x_{r-1} + ex_r, \\ y_1' &= ey_1, \quad y_1' = y_1 + ey_2, \quad \ldots, \quad y_s' = y_{s-1} + ey_s, \\ z_1' &= ez_1, \quad z_2' = z_1 + ez_2, \quad \ldots, \quad z_t' = z_{t-1} + ez_t. \end{aligned}$$

Let $T$, or

$$\begin{aligned} x_1' &= a_{11}x_1 + \ldots + a_{1r}x_r + b_{11}y_1 + \ldots + b_{1s}y_s + c_{11}z_1 + \ldots + c_{1t}z_t, \\ x_2' &= a_{21}x_1 + \ldots + a_{2r}x_r + b_{21}y_1 + \ldots + b_{2s}y_s + c_{21}z_1 + \ldots + c_{2t}z_t, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ y_1' &= d_{11}x_1 + \ldots + d_{1r}x_r + e_{11}y_1 + \ldots + e_{1s}y_s + f_{11}z_1 + \ldots + f_{1t}z_t, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ z_1' &= g_{11}x_1 + \ldots + g_{1r}x_r + h_{11}y_1 + \ldots + h_{1s}y_s + i_{11}z_1 + \ldots + i_{1t}z_t, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

be permutable with the previous substitution, which is represented by $S$. Then $ST$ and $TS$ must replace each of the symbols by the same linear function of them. Considering $x_1$,

$$\begin{aligned} e\,(a_{11}x_1 + \ldots + a_{1r}x_r + b_{11}y_1 + \ldots + b_{1s}y_s + c_{11}z_1 + \ldots + c_{1t}z_t) \\ \equiv\ & a_{11}ex_1 + a_{12}\,(x_1 + ex_2) + \ldots + a_{1r}\,(x_{r-1} + ex_r) \\ & + b_{11}ey_1 + b_{12}\,(y_1 + ey_2) + \ldots + b_{1s}\,(y_{s-1} + ey_s) \\ & + c_{11}ez_1 + c_{12}\,(z_1 + ez_2) + \ldots + c_{1t}\,(z_{t-1} + ez_t). \end{aligned}$$

Hence $a_{12} = \ldots = a_{1r} = b_{12} = \ldots = b_{1s} = c_{12} = \ldots = c_{1t} = 0$.

Considering $x_2$,

$$\begin{aligned} a_{11}x_1 + b_{11}y_1 + c_{11}z_1 + e\,(a_{21}x + \ldots + c_{2t}z_t) \\ \equiv\ & a_{21}ex_1 + a_{22}\,(x_1 + ex_2) + \ldots + a_{2r}\,(x_{r-1} + ex_r) \\ & + b_{21}ey_1 + b_{22}\,(y_1 + ey_2) + \ldots + b_{2s}\,(y_{s-1} + ey_s) \\ & + c_{21}ez_1 + c_{22}\,(z_1 + ez_2) + \ldots + c_{2t}\,(z_{t-1} + ez_t). \end{aligned}$$

Hence $$a_{22} = a_{11}, \quad b_{22} = b_{11}, \quad c_{22} = c_{11},$$

and $$a_{23} = \ldots = a_{2r} = b_{23} = \ldots = b_{2s} = c_{23} = \ldots = c_{2t} = 0.$$

Similarly, $x_3$ gives

$$a_{33} = a_{11}, \quad b_{33} = b_{11}, \quad c_{33} = c_{11},$$

$$a_{32} = a_{21}, \quad b_{32} = b_{21}, \quad c_{32} = c_{11},$$

and $$a_{34} = \ldots = a_{3r} = b_{34} = \ldots = b_{3s} = c_{34} = \ldots c_{3t} = 0.$$

In considering $x_{t+1}$, the terms in the $z$'s on either side of the identity are

$$c_{t1} z_1 + c_{t-1,1} z_2 + \ldots + c_{11} z_t + e\,(c_{t+1,1} z_1 + c_{t+1,2} z_2 + \ldots + c_{t+1,t} z_t)$$

$$\equiv c_{t+1,1}\, e z_1 + c_{t+1,2}\,(z_1 + e z_2) + \ldots + c_{t+1,t}\,(z_{t-1} + e z_t).$$

Hence, if $t < r$, $$c_{11} = 0.$$

Similarly, if $t < r-1$, $$c_{21} = 0,$$

and so on.

So, by considering $x_{s+1}$, it is seen that, if $s < r$, then

$$b_{11} = 0;$$

if $s < r-1$, then $$b_{21} = 0,$$

and so on.

By considering $y_1$, it is found that

$$d_{12} = \ldots = d_{1r} = e_{12} = \ldots = e_{1s} = f_{12} = \ldots = f_{1t} = 0,$$

and so on; the $d$'s, $e$'s, and $f$'s being independent of the $a$'s, $b$'s, and $c$'s. Moreover, if $t < s$, it is found that

$$f_{11} = 0,$$

and so on. Similar conclusions hold for the $g$'s, $h$'s, and $i$"s, which are also independent of the previous coefficients.

The notation may now be modified, so as to avoid double suffixes. Thus, writing

$$a_p \text{ for } a_{p,1}, \quad b_p \text{ for } b_{r-s+p,1}, \quad c_p \text{ for } c_{r-t+p,1},$$

$$d_p \text{ for } d_{p,1}, \quad e_p \text{ for } e_{p,1}, \quad f_p \text{ for } f_{s-t+p,1},$$

$$g_p \text{ for } g_{p,1}, \quad h_p \text{ for } h_{p,1}, \quad i_p \text{ for } i_{p,1},$$

the general form for a substitution $T$ which is permutable with $S$ becomes

$$
\begin{aligned}
x_1' &\equiv a_1x_1, \\
x_2' &\equiv a_2x_1 + a_1x_2, \\
&\dots\dots\dots\dots\dots \\
x_{r-s+1}' &\equiv a_{r-s+1}x_1 + a_{r-s}x_2 + \dots + a_1x_{r-s+1} \\
&\dots\dots\dots\dots\dots \\
x_{r-t+1}' &\equiv a_{r-t+1}x_1 + a_{r-t}x_2 + \dots\dots\dots\dots + a_1x_{r-t+1} \\
&\dots\dots\dots\dots\dots \\
x_r' &\equiv a_rx_1 + a_{r-1}x_2 + \dots\dots\dots\dots\dots + a_1x_r, \\
y_1' &\equiv d_1x_1 \\
y_2' &\equiv d_2x_1 + d_1x_2 \\
&\dots\dots\dots\dots\dots \\
y_{s-t+1}' &\equiv d_{s-t+1}x_1 + d_{s-t}x_2 + \dots + d_1x_{s-t+1} \\
&\dots\dots\dots\dots\dots \\
y_s' &\equiv d_sx_1 + d_{s-1}x_2 + \dots\dots\dots\dots\dots + d_1x_s \\
z_1' &\equiv g_1x_1 \\
z_2' &\equiv g_2x_1 + g_1x_2 \\
&\dots\dots\dots\dots\dots \\
z_t' &\equiv g_tx_1 + g_{t-1}x_2 + \dots\dots\dots + g_1x_t
\end{aligned}
$$

$$
\begin{aligned}
&+ b_1y_1 \\
&\dots\dots\dots \\
&+ b_{s-t+1}y_1 + b_{s-t}y_2 + \dots + b_1y_{s-t+1} \\
&\dots\dots\dots\dots\dots \\
&+ b_{s-1}y_2 + \dots\dots\dots\dots\dots\dots + b_1y_s + c_1x_1 + \dots\dots\dots \\
&+ e_1y_1, \\
&+ e_2y_1 + e_1y_2, \\
&\dots\dots\dots\dots\dots \\
&+ e_{s-t+1}y_1 + e_{s-t}y_2 + \dots + e_1y_{s-t+1} \\
&\dots\dots\dots\dots\dots \\
&+ e_sy_1 + e_{s-1}y_2 + \dots\dots\dots\dots\dots\dots + e_1y_s + f_1x_1 + \dots\dots\dots \\
&+ h_1y_1 \\
&+ h_2y_1 + h_1y_2 \\
&\dots\dots\dots\dots\dots \\
&+ h_ty_1 + h_{t-1}y_2 + \dots\dots\dots\dots + h_1y_t
\end{aligned}
$$

$$
\begin{aligned}
&+ c_1z_1, \\
&\dots\dots\dots \\
&+ c_1z_1, \\
&\dots\dots\dots \\
&+ f_1z_1 \\
&\dots\dots\dots \\
&+ f_1z_1, \\
&+ i_1z_1, \\
&+ i_2z_1 + i_1z_2, \\
&\dots\dots\dots \\
&+ i_1z_1 + i_{t-1}z_2 + \dots + i_1z_t.
\end{aligned}
$$

Now the totality of the substitutions which are permutable with $S$ form a group the finite equations of which contain the parameters linearly and homogeneously. Hence, regarding the $a$'s, $b$'s, ..., $i$'s as parameters, the totality of the substitutions of the above form constitute a group $H$, and this group is the most general linear homogeneous group which contains $S$ self-conjugately. Moreover, the form can be obviously extended to the case in which the invariant factors of $S$ are any whatever.*

9. Before discussing the transitive Abelian groups contained in $H$, it will be convenient to refer to certain sets of integers which are invariant for all transformations of an Abelian group. For such a group there must be, corresponding to each root of the characteristic equation, at least one linear function of the variables which is changed into a multiple of itself by every substitution of the group. If the roots of the characteristic equation are all equal, let $p_1$ be the number of independent linear functions of the variables, each of which is changed into a multiple of itself by every substitution of the group. The integer $p_1$ is then clearly invariant for all transformations of the group within the general linear group. If

$$x_{11},\ x_{12},\ \ldots,\ x_{1p_1}$$

are these $p_1$ functions of the variables, they may be taken for the first $p_1$ of a new set of variables in terms of which the group is expressed; and, if in the new equations of the group

$$x_{11},\ x_{12},\ \ldots,\ x_{1p_1},\quad x'_{11},\ \ldots,\ x'_{1p_1}$$

are all made zero, the equations will become the equations of an Abelian group in the remaining $n-p_1$ variables. Let $p_2$ be the number for this group that corresponds to $p_1$ for the original group, and let

$$x_{21},\ x_{22},\ \ldots,\ x_{2p_2}$$

---

* If the characteristic equation of $S$ has $m$ distinct roots, the most general group which contains $S$ self-conjugately will be the direct product of $m$ groups similar to that considered in the text. If, moreover, $\lambda_1, \lambda_2, \ldots$ be the distinct roots of the characteristic equation of $S$, and

$$(\lambda_1+\lambda)^{a_{1p}}(\lambda_2+\lambda)^{a_{2p}}\ldots$$

be the $p^{\text{th}}$ invariant factor of its characteristic determinant, then the order of the greatest group containing $S$ self-conjugately is

$$\sum_{r,s}(2r-1)\,a_{sr}.$$

be the corresponding $p_2$ linear functions. Then, by every substitution of the original group, each of these $p_2$ functions is changed into a multiple of itself added to a linear function of $x_{11}, x_{12}, \ldots, x_{1p_1}$; and $p_2$ is again invariant, however the original group be transformed. Continuing thus, the original variables may be replaced by $m$ sets of new variables

$$\begin{array}{llll} x_{11}, & x_{12}, & \ldots, & x_{1p_1}, \\ x_{21}, & x_{22}, & \ldots, & x_{2p_2}, \\ \ldots & \ldots & & \\ x_{m1}, & x_{m2}, & \ldots, & x_{mp_m}, \end{array}$$

such that each is changed into a multiple of itself added to a linear function of those in the lines above it by every substitution of the group; and the set of integers $p_1, p_2, \ldots, p_m$ is invariant for all transformations of the group. If the roots of the characteristic equation are not all equal, there will be such a set of invariant integers corresponding to each distinct root.

10. If $S$ has only one invariant factor distinct from unity, *i.e.*, if $r = n$, $s = t = \ldots = 0$, the group $H$ is given by

$$\begin{aligned} x_1' &= a_1 x_1, \\ x_2' &= a_2 x_1 + a_1 x_2, \\ x_3' &= a_3 x_1 + a_2 x_2 \quad + a_1 x_3, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' &= a_n x_1 + a_{n-1} x_2 + \ldots\ldots\ldots + a_1 x_n. \end{aligned}$$

It may be immediately and easily verified that this simply transitive group is Abelian. Moreover, the invariant factors of a general substitution of the group, or, as they will now be called, the invariant factors of the group, are from the form of the equations at once seen to be

$$(a_1 + \lambda)^n, \ 1, \ 1, \ \ldots.$$

It follows that all simply transitive groups whose invariant factors are

$$(a_1 + \lambda)^n, \ 1, \ 1, \ \ldots$$

are of the same type, or, in other words, are conjugate within the general linear homogeneous group of $n$ variables. For such a group each of the integers $p_1, p_2, \ldots$ is unity.

Suppose next that $S$ has two invariant factors distinct from unity,

so that $t = \ldots = 0$ and $s \leqslant r$. The group $H$ is then given by the equations:—

$$
\begin{aligned}
x_1' &= a_1 x_1, \\
x_2' &= a_2 x_1 + a_1 x_2, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
x_{r-s+1}' &= a_{r-s+1} x_1 + a_{r-s} x_2 + \ldots + a_1 x_{r-s+1} + b_1 y_1, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
x_r' &= a_r x_1 + a_{r-1} x_2 + \ldots\ldots\ldots\ldots + a_1 x_r + b_s y_1 + \ldots\ldots\ldots + b_1 y_s, \\
y_1' &= d_1 x_1 + e_1 y_1, \\
y_2' &= d_2 x_1 + d_1 x_2 + e_2 y_1 + e_1 y_2, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
y_s' &= d_s x_1 + d_{s-1} x_2 + \ldots + d_1 x_s + e_s y_1 + e_{s-1} y_2 + \ldots + e_1 y_s.
\end{aligned}
$$

If new variables are taken so that

$$\xi_\sigma = x_\sigma, \quad (\sigma = 1, 2, \ldots, r-s),$$

$$\xi_{r-s+\sigma} = x_{r-s+\sigma} + k y_\sigma, \quad \eta_\sigma = y_\sigma, \quad (\sigma = 1, 2, \ldots, s),$$

the form of $S$, the operation which is self-conjugate in $H$, is unaltered; and in the equations of $H$, for each value of $\sigma$ from 1 to $s$,

$$a_{r-s+\sigma} \text{ becomes } a_{r-s+\sigma} + k d_\sigma,$$

$$b_\sigma \text{ ,, } b_\sigma - k(a_\sigma - e_\sigma);$$

and, for each value of $\tau$ from 1 to $2s-r$,

$$e_{r-s+\tau} \text{ becomes } e_{r-s+\tau} - k d_\tau.$$

Suppose that $G$ is a transitive Abelian sub-group of $H$ whose characteristic determinant is the power of a single linear factor. Then the $a$'s and $d$'s are the independent parameters* of $G$, and the $b$'s and $e$'s are linear functions of them. If $T$ and $T'$ are any two operations

---

* If, for $G$, the $a$'s and the $d$'s were not linearly independent, the point

$$x_1 = 1, \; x_2 = \ldots = y_s = 0$$

would belong to an invariant multiplicity of $G$; and this is manifestly impossible.

of $G$, $TT'$ and $T'T$ replace every symbol by the same linear function. A consideration of the symbols $x_{r-s+1}$ and $y_1$ gives the identities

$$a_1 b_1' y_1 + b_1 (d_1' x_1 + e_1' y_1) \equiv a_1' b_1 y_1 + b_1' (d_1 x_1 + e_1 y_1)$$

and

$$d_1 a_1' x_1 + e_1 (d_1' x_1 + e_1' y_1) \equiv d_1' a_1 x_1 + e_1' (d_1 x_1 + e_1 y_1);$$

and therefore $\frac{b_1}{a_1 - e_1}$ must be an absolute constant for all operations of $G$. It follows that $H$ contains a sub-group $G'$, conjugate to $G$ in the general linear group, such that in the finite equations of $G'$ the terms in $b_1$ are wanting. By a repetition of this process all the $b$'s may be got rid of; and, since for the present purpose two groups which are conjugate in the general linear group are equivalent to each other, it is sufficient to consider transitive Abelian sub-groups of $H$ for which all the $b$'s are zero.

The conditions that $TT'$ and $T'T$ replace each $y$ by the same linear function give successively,

$$\frac{a_1 - e_1}{d_1} = \text{const.},$$

$$\frac{a_1 - e_1}{d_2} = \text{const.}, \quad \frac{a_2 - e_2}{d_1} = \text{const.},$$

$$\frac{a_1 - e_1}{d_3} = \text{const.}, \quad \frac{a_2 - e_2}{d_2} = \text{const.}, \quad \frac{a_3 - e_3}{d_1} = \text{const.},$$

and so on. Hence, since the $d$'s are arbitrary,

$$e_1 = a_1, \quad e_2 = a_2, \quad \ldots, \quad e_{s-1} = a_{s-1},$$

and

$$\ldots \qquad e_s = a_s + k d_1,$$

where $k$ is an absolute constant. The equations of $G$, with a slight change of notation, may now be written

$$\begin{aligned}
x_1' &= a_1 x_1, \\
x_2' &= a_2 x_1 + a_1 x_2, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \\
x_r' &= a_r x_1 + a_{r-1} x_2 + \ldots\ldots\ldots + a_1 x_r, \\
x_{r+1}' &= a_{r+1} x_1 + a_1 x_{r+1}, \\
x_{r+2}' &= a_{r+2} x_1 + a_{r+1} x_2 + a_2 x_{r+1} + a_1 x_{r+2}, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
x_{r+s}' &= a_{r+s} x_1 + a_{r+s-1} x_2 + \ldots + a_{r+1} x_s + (k a_{r+1} + a_s) x_1 + a_{s-1} a_2 + \ldots + a_1 x_{r+s}.
\end{aligned}$$

Further, if $k$ is not zero, it may clearly, by replacing each variable and parameter by a suitable multiple of itself, be made unity. There are therefore, at most, two distinct types of transitive Abelian subgroups contained in $H$, whose characteristic determinants are powers of a single linear factor. It remains to determine whether these groups are actually distinct or not, *i.e.*, whether a group for which $k$ is unity can or cannot be transformed into a group for which $k$ is zero.

Let $\Sigma$ be a substitution which transforms $G$ into a group whose finite equations differ from those of $G$ only in having a new value of $k$. Then $\Sigma$ must change $x_1$ into a multiple of itself, and without loss of generality may be supposed to leave $x_1$ unchanged. Also it must change $x_2$ and $x_{r+1}$ into linear functions of themselves and $x_1$, and so on. Suppose that of the symbols $x_1, x_2, \ldots, x_r$, the first affected by $\Sigma$ is $x_{\sigma+1}$, and let $\Sigma$ change $x_{\sigma+1}$ into $\xi_{\sigma+1}$, where

$$\xi_{\sigma+1} = p_{\sigma+1}x_{\sigma+1} + p_\sigma x_\sigma + \ldots + q_\sigma x_{r+\sigma} + q_{\sigma-1}x_{r+\sigma-1} + \ldots,$$

where the $p$'s and $q$'s are constants.

Since the form of $G$, except as regards the value of $k$, is unaltered by $\Sigma$, the coefficient of $x_2$ in $\xi'_{\sigma+1}$ must be $a_\sigma$. Hence

$$p_{\sigma+1} = 1, \quad p_\sigma = \ldots = p_2 = 0, \quad q_\sigma = \ldots = q_2 = 0,$$

and

$$\xi_{\sigma+1} = x_{\sigma+1} + p_1 x_1 + q_1 x_{r+1};$$

while the corresponding new parameter, namely, the coefficient of $x_1$ in $\xi'_{\sigma+1}$, is given by

$$a_{\sigma+1} = a_{\sigma+1} + q_1 a_{r+1}.$$

The functions by which $\Sigma$ replaces $x_{\sigma+2}, x_{\sigma+3}, \ldots$ may now be determined by a step-to-step process. Thus it will be found that

$$\xi_{\sigma+2} = x_{\sigma+2} + p_2 x_1 + p_1 x_2 + q_2 x_{r+1} + q_1 x_{r+2},$$

$$\xi_{\sigma+3} = x_{\sigma+3} + p_3 x_1 + p_2 x_2 + p_1 x_3 + q_3 x_{r+1} + q_2 x_{r+2} + q_1 r_{r+3},$$

and so on. In order that the form of the equation for $x_r$ may remain unchanged, $k$ being different from zero, it is clear that $\xi_r$ must not contain $x_{r+s}$. Hence

$$\sigma > r - s;$$

and, this condition being satisfied, the first $r$ of the equations of $\Sigma$ are

$$\xi_1 = x_1, \quad \xi_2 = x_2, \quad \ldots \quad \xi_\sigma = x_\sigma,$$

$$\xi_{\sigma+\tau} = x_{\sigma+\tau} + p_\tau x_1 + p_{\tau-1}x_2 + \ldots + p_1 x_\tau + q_\tau x_{r+1} + \ldots + q_1 x_{r+\tau},$$

$$(\tau = 1, 2, \ldots r - \sigma).$$

Let now the remaining equations of $\Sigma$ be taken to be

$$\xi_{r+\tau} = x_{r+\tau}, \qquad (\tau = 1, 2, \ldots, s).$$

A simple calculation then gives, for the transformed form of $G$,

$$\begin{aligned}
\xi'_1 &= a_1\xi_1, \\
\xi'_2 &= a_2\xi_1 + a_1\xi_2, \\
&\cdots \quad \cdots \quad \cdots \\
\xi'_r &= a_r\xi_1 + a_{r-1}\xi_2 + \ldots\ldots\ldots + a_1\xi_r, \\
x'_{r+1} &= a_{r+1}\xi_1 + \beta_1 x_{r+1}, \\
x'_{r+2} &= a_{r+2}\xi_1 + a_{r+1}\xi_2 + \beta_2 x_{r+1} + \beta_1 x_{r+2}, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
x'_{r+s} &= a_{r+s}\xi_1 + a_{r+s-1}\xi_2 + \ldots + a_{r+1}\xi_s + \beta_s x_{r+1} + \beta_{s-1} x_{r+2} + \ldots + \beta_1 x_{r+s},
\end{aligned}$$

where

$$\begin{aligned}
a_\tau &= a_\tau, && (\tau = 1, 2, \ldots, \sigma), \\
a_{\sigma+\tau} &= a_{\sigma+\tau} + q_\tau a_{r+1} + \ldots + q_1 a_{r+\tau}, && (\tau = 1, 2, \ldots, r-\sigma), \\
a_{r+\tau} &= a_{r+\tau}, && (\tau = 1, 2, \ldots, \sigma), \\
a_{r+\sigma+\tau} &= a_{r+\sigma+\tau} - p_\tau a_{r+1} - \ldots - p_1 a_{r+\tau}, && (\tau = 1, 2, \ldots, s-\sigma), \\
\beta_\tau &= a_\tau, && (\tau = 1, 2, \ldots, \sigma), \\
\beta_{\sigma+\tau} &= a_{\sigma+\tau} - p_\tau a_1 - \ldots - p_1 a_\tau - q_\tau a_{r+1} - \ldots - q_1 a_{r+\tau}, \\
& && (\tau = 1, 2, \ldots, s-\sigma);
\end{aligned}$$

and where, in the last equation, when $\tau = s-\sigma$, $a_s$ is replaced by $a_s + k a_{r+1}$.

Hence, with this form of $\Sigma$, the form of $G$ will be unchanged, except as regards the value of $k$, if and only if

$$\sigma = s-1.$$

Since

$$\sigma > r-s,$$

this condition can only be satisfied when

$$s > \tfrac{1}{2}(r+1).$$

Conversely, when the latter condition is satisfied, by taking $\sigma = s-1$ and all the $p$'s equal to zero, the form of $G$ is unaltered except that $k$ is changed into $k-2q_1$, and therefore, by suitably choosing $q_1$, $k$ may be made zero.

When

$$s \not> \tfrac{1}{2}(r+1),$$

the form assumed for $\Sigma$ will not leave the form of $G$ unchanged. Suppose, then, that of the symbols $x_{r+1}$, $x_{r+2}$, ..., $x_{r+s}$, the first changed by $\Sigma$ is $x_{r+\tau+1}$, and that this becomes $\xi_{r+\tau+1}$.

It may be shown, exactly as above, by considering the coefficient of $\xi_2$ in $\xi'_{r+\tau+1}$, that

$$\xi_{r+\tau+1} = x_{r+\tau+1} + p'_1 \xi_1 + q'_1 x_{r+1}.$$

This, however, leaves the coefficient of $x_{r+1}$ in $\xi'_{r+\tau+1}$ unchanged, and therefore, unless $\beta_{\tau+1} = \alpha_{\tau+1}$, the form of $G$ is changed. In other words, unless the original substitution $\Sigma$ for which $x_{r+1}, x_{r+2}, \ldots, x_{r+s}$ are unaltered leaves the form of $G$ unchanged, the new one cannot do so. Hence, when

$$s \not> \tfrac{1}{2}(r+1),$$

it is not possible to transform $G$ so that its form remains unchanged and at the same time to make $k$ zero.

The final result arrived at is that, if

$$s > \tfrac{1}{2}(r+1),$$

the group with $k$ equal to unity and that with $k$ equal to zero are conjugate; and the former is transformed into the latter by the substitution

$$\left.\begin{aligned} \xi_{s+\tau} &= x_{s+\tau} + \tfrac{1}{2} x_{r+1+\tau} \\ \alpha_{s+\tau} &= \alpha_{s+\tau} + \tfrac{1}{2} \alpha_{r+1+\tau} \end{aligned}\right\} \quad (\tau = 0, 1, \ldots, r-s);$$

but that, if

$$s \not> \tfrac{1}{2}(r+1),$$

the two groups are not conjugate.

It must, however, be noticed that, for the particular case when $s$ is unity, $k$ is necessarily zero, as otherwise the characteristic determinant of the group would have two distinct roots. There is then only one type.

11. For the group $G$ given by the tableau on p. 341, the invariant set of integers $p_1, p_2, \ldots$ consists of 1, then $s$ 2's, and then $(r-s-1)$ 1's, if $r > s$; and of 1, then $(s-1)$ 2's and then 1, if $r = s$.* Hence two such groups cannot be conjugate unless $r$ and $s$ are the same for both. From this it follows at once that the invariant factors of the characteristic determinant of the general operation of the group $G$ are

$$(a_1+\lambda)^r, \ (a_1+\lambda)^s, \ 1, \ 1, \ \ldots.$$

In fact, there cannot be three invariant factors distinct from unity,

* It is not to be inferred from this that the integers $p_1, p_2, \ldots$ are completely determined by $r, s; t, \ldots$. A consideration of the case in which the group has three invariant factors, distinct from unity, shows that this is not the case.

since for the particular substitution given by $a_3 = a_4 = \ldots = a_{r+s} = 0$, there are only two; and, since two groups for which $r$ and $s$ have different values are not conjugate, $G$ cannot contain a substitution for which

$$(a_1+\lambda)^r,\ (a_1+\lambda)^{s'},\ 1,\ 1,\ \ldots$$

are invariant factors, unless $r' = r,\ s' = s$.

Hence, finally, a transitive Abelian group, in $r+s$ variables ($s \leqslant r$), the roots of whose characteristic equation are all equal, and whose characteristic determinant has the invariant factors

$$(a_1+\lambda)^r,\ (a_1+\lambda)^s,\ 1,\ 1,\ \ldots,$$

is of one definite type if

$$s = 1 \quad \text{or} \quad s > \tfrac{1}{2}(r+1);$$

and belongs to one of two definite types if

$$1 < s \leqslant \tfrac{1}{2}(r+1).$$

The types are given by the tableau of p. 341, with either zero or unity for $k$.

By the remark made in § 7, the number of distinct types of transitive Abelian groups is thus determined when the characteristic determinant has any number of distinct roots, but not more than two invariant factors distinct from unity.

12. For particular cases the discussion of the number of types of group corresponding to given invariant factors may simplify materially. Thus, by an immediate extension of the methods of §§ 8, 10, a transitive Abelian group which contains a substitution whose invariant factors are

$$(a_1+\lambda)^r,\ a_1+\lambda,\ a_1+\lambda,\ \ldots \text{ to } n-r \text{ terms},\ 1,\ 1,\ \ldots$$

can be written in the form

$$\begin{aligned}
x_1' &= a_1 x_1, \\
x_2' &= a_2 x_1 + a_1 x_2, \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \\
x_r' &= a_r x_1 + a_{r-1} x_2 + \ldots + a_1 x_r, \\
x_{r+1}' &= a_{r+1} x_1 + a_1 x_{r+1}, \\
x_{r+2}' &= a_{r+2} x_1 + \alpha x_{r+1} + a_1 x_{r+2}, \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
x_n' &= a_n x_1 + \beta x_{r+1} + \gamma x_{r+2} + \ldots + a_1 x_n;
\end{aligned}$$

where $\alpha$, $\beta$, $\gamma$, ... are linear functions of $a_2, a_3, \ldots, a_n$. If the general

substitution of the group is to have the given invariant factors, every minor of $r$ rows and columns of the characteristic determinant must be divisible by $a_1+\lambda$; and this can clearly only be the case when $\alpha, \beta, \gamma, \ldots$ are all zero. There is therefore only a single type of transitive Abelian group with the invariant factors in question.

For values of $n$ up to and including $n=5$, the only case occurring that does not come under the previous heads is that in which the invariant factors are

$$(a_1+\lambda)^2,\ (a_1+\lambda)^2,\ a_1+\lambda,\ 1,\ 1;$$

and in this case the method of § 8 will show at once that there is only one type. The distinct types of transitive Abelian groups for 2, 3, 4, and 5 variables may therefore be written down at once from what precedes. The result is to show that, for $n=2$, there are two types; for $n=3$, four types; for $n=4$, nine types; and, for $n=5$, eighteen types. Of these last eighteen, seven cannot be expressed as direct products of Abelian groups in a smaller number of variables. They are :—

(i)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1 \qquad +a_1x_3, \\ x_4' &= a_4x_1 \qquad\quad +a_1x_4, \\ x_5' &= a_5x_1 \qquad\quad\ \ +a_1x_5; \end{aligned}$$

(ii)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1+a_2x_2+a_1x_3, \\ x_4' &= a_4x_1 \qquad\quad +a_1x_4, \\ x_5' &= a_5x_1 \qquad\quad\ \ +a_1x_5; \end{aligned}$$

(iii)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1+a_2x_2+a_1x_3, \\ x_4' &= a_4x_1+a_3x_2+a_2x_3+a_1a_4, \\ x_5' &= a_5x_1 \qquad\quad\ \ +a_1x_5; \end{aligned}$$

(iv)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1+a_2x_2+a_1x_3, \\ x_4' &= a_4x_1+a_3x_2+a_2x_3+a_1x_4, \\ x_5' &= a_5x_1+a_4x_2+a_3x_3+a_2x_4 \quad +a_1x_5; \end{aligned}$$

(v)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1 \qquad +a_1x_3, \\ x_4' &= a_4x_1+a_3x_2+a_2x_3+a_1x_4, \\ x_5' &= a_5x_1 \qquad\quad\ \ +a_1x_5; \end{aligned}$$

(vi) and (vii)

$$\begin{aligned} x_1' &= a_1x_1, \\ x_2' &= a_2x_1+a_1x_2, \\ x_3' &= a_3x_1+a_2x_2+a_1x_3, \\ x_4' &= a_4x_1 \qquad\quad +a_1x_4, \\ x_5' &= a_5x_1 \qquad\quad\ \ +(a_2+ka_4)x_4+a_1x_5; \end{aligned}$$

where for (vi) $k=0$ and for (vii) $k=1$.

13. The discussion of the preceding pages is concerned mainly with transitive Abelian groups; but the results of §§ 6, 9 apply directly to any Abelian group in the finite equations of which the parameters enter linearly and homogeneously. Suppose that for such a group the characteristic determinant is

$$(a_1+\lambda)^{m_1}(a_2+\lambda)^{m_2}\dots(a_r+\lambda)^{m_r},$$

and that the greatest common factor of all of its first minors is

$$(a_1+\lambda)^{m_1-1}(a_2+\lambda)^{m_2-1}\dots(a_r+\lambda)^{m_r-1},$$

so that the first invariant factor is

$$(a_1+\lambda)(a_2+\lambda)\dots(a_r+\lambda).$$

If $x_1, x_2, \dots, x_{m_1}$ are the set of symbols corresponding to the repeated factor $(a_1+\lambda)^{m_1}$ of the characteristic determinant, and if linear functions of these are chosen as in § 9, the finite equations of the group, so far as they affect these symbols, will be

$$\begin{aligned} x_1' &= a_1 x_1, \\ x_2' &= \beta x_1 + a_1 x_2, \\ \dots &\quad \dots \quad \dots \\ x_{m_1}' &= \gamma x_1 + \delta x_2 + \dots + a_1 x_{m_1}, \end{aligned}$$

where $\beta, \gamma, \delta, \dots$ are linear functions of the parameters other than $a_1$. These equations give an Abelian group whose characteristic determinant is $(a_1+\lambda)^{m_1}$. Moreover, since the first invariant factor is $a_1+\lambda$, every first minor of the characteristic determinant is divisible by $(a_1+\lambda)^{m_1-1}$, every second minor by $(a_1+\lambda)^{m_1-2}$, and so on. This can clearly only be the case if $\beta, \gamma, \delta, \dots$ are all zero.

Hence in this case the finite equations of the group may be written in the form

$$x_{is}' = a_s x_{is}, \quad \begin{pmatrix} i = 1, 2, \dots, m_s \\ s = 1, 2, \dots, r \end{pmatrix}.$$

This may clearly be regarded the most extreme case of an intransitive group corresponding to the given characteristic determinant.

In any case, if the characteristic determinant has more than one distinct root, the variables may be divided into sets, each of which are transformed among themselves; and therefore, as with transitive groups, it is natural to begin by a discussion of the case in which

the roots of the characteristic determinant are all equal. It must, however, be remembered that it is not now necessarily the case that the group itself is the direct product of the groups obtained by considering each set of variables alone. The finite equations of such a group may always be written in the form

$$\begin{aligned}
x_1' &= a_1 x_1, \\
x_2' &= \alpha_{21} x_1 + a_1 x_2, \\
x_3' &= \alpha_{31} x_1 + \alpha_{32} x_2 + a_1 x_3, \\
&\dots \quad \dots \quad \dots \quad \dots \\
x_n' &= \alpha_{n1} x_1 + \alpha_{n2} x_2 + \dots + \alpha_{n,n-1} x_{n-1} + a_1 x_n ;
\end{aligned}$$

where the $\alpha$'s are linear homogeneous functions of the parameters other than $a_1$.

14. From this form for the group it may be shown that the necessary and sufficient condition that the group may be intransitive is that there shall be at least two linear functions of the variables which are changed into multiples of themselves by every operation of the group. That the condition is sufficient is obvious, for, if $x_s'$ is changed into a multiple of itself by every operation of the group, then

$$\alpha_{s,1} = \alpha_{s,2} = \dots = \alpha_{s,s-1} = 0,$$

and

$$x_s' = a_1 x_s ;$$

so that $\frac{x_1}{x_s}$ is an absolute invariant for the group. To show that the condition is necessary, let $\alpha_{s+1,1}$ be the first of the leading coefficients in the equations which is not linearly independent of those that precede it. Then, since $a_1, \alpha_{2,1}, \alpha_{3,1}, \dots, \alpha_{s,1}$ are linearly independent, the first $s$ equations taken alone constitute a transitive group. For they permit of changing the point (1, 0, 0, 0, ...) into any arbitrary point, and therefore of changing any one arbitrary point into any other. Hence, if the group is intransitive,

$$s+1 \leqslant n.$$

The first $s$ equations constituting a transitive Abelian group, we may suppose it in standard form, so that $a_1, \alpha_{2,1}, \alpha_{3,1}, \dots, \alpha_{s,1}$ are the $s$ parameters $y_1, y_2, y_3, \dots, y_s$ of the group.

By supposition,

$$\alpha_{s+1,1} = c_2 \alpha_{2,1} + c_3 \alpha_{3,1} + \dots + c_s \alpha_{s,1} ;$$

where $c_2, c_3, \ldots, c_s$ are absolute constants. Hence, if a new variable $\xi_{s+1}$ be taken for $x_{s+1}$, such that

$$\xi_{s+1} = x_{s+1} - c_2 x_2 - c_3 x_3 \ldots - c_s x_s,$$

then the first $s+1$ equations which by themselves constitute a group have the form

$$\begin{aligned}
x_1' &= y_1 x_1 \\
x_2' &= y_2 x_1 + y_1 x_2 \\
x_3' &= y_3 x_1 + \alpha_{3,2} x_2 + y_1 x_3 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
x_s' &= y_s x_1 + \alpha_{s,2} x_2 + \ldots + y_1 x_s \\
\xi_{s+1}' &= \beta_2 x_2 + \ldots + \beta_s x_s + y_1 \xi_{s+1}.
\end{aligned}$$

If $\xi_{s+1}''$ is the value of $\xi_{s+1}$ which results by carrying out first the operation written and then the operation with accented parameters, the coefficient of $x_1$ in $\xi_{s+1}''$ is

$$\sum_2^s \beta_r' y_r.$$

This must be zero for all possible values of the $y$'s. Hence, since they are independent, all the $\beta$'s must be zero and

$$\xi_{s+1}' = y_1 \xi_{s+1}.$$

If the group is intransitive, this must hold for some value of the suffix $s+1$ equal to or less than $n$, and therefore there are at least two linear functions of the variables, namely, $x_1$ and $\xi_{s+1}$, each of which is changed into a multiple of itself by every operation of the group.

15. This result suggests at once that the process of § 9 may be used to obtain a preliminary classification of intransitive Abelian groups. If $(y_1+\lambda)^n$ is the determinant of the group, the first of the series of integers $p_1, p_2, p_3, \ldots$ is necessarily greater than 1, whereas, for a transitive group with the same characteristic determinant, it is necessarily equal to 1. Two intransitive groups with the same set of integers are not, of course, necessarily conjugate; but, as has already been pointed out, no two groups can be conjugate unless the set of integers is the same for each. For the smaller values of $n$, this remark makes the classification comparatively simple.

When $n$ is 2 there is only one intransitive group, namely,

$$x_1' = y_1 x_1, \quad x_2' = y_1 x_2.$$

When $n = 3$, the series of integers may be 2, 1, or 3. The corresponding groups are given by

$$x_1' = y_1 x_1, \quad x_2' = y_1 x_2, \quad x_3' = y_2 x_1 + y_3 x_2 + y_1 x_3.$$

There are three types of orders 1, 2, and 3, corresponding to

(i) $y_2 = y_3 = 0$;

(ii) $y_3 = 0$;

(iii) $y_2$ and $y_3$ both arbitrary.

When $n$ is 4, the series of integers may be 4; 3, 1; 2, 2; or 2, 1, 1. Corresponding to the first two cases, the groups are

$$x_1' = y_1 x_1, \quad x_2' = y_1 x_2, \quad x_3' = y_1 x_3, \quad x_4' = y_2 x_1 + y_3 x_2 + y_4 x_3 + y_1 x_4.$$

There are four types of orders 1, 2, 3, and 4, corresponding to

(i) $y_2 = y_3 = y_4 = 0$;

(ii) $y_3 = y_4 = 0$;

(iii) $y_4 = 0$;

(iv) $y_2$, $y_3$, and $y_4$ all arbitrary.

If the integers are 2, 2, the groups are

$$\begin{aligned} x_1' &= y_1 x_1, \\ x_2' &= \quad\quad y_1 x_2, \\ x_3' &= y_2 x_1 + y_3 x_2 + y_1 x_3, \\ x_4' &= y_4 x_1 + y_5 x_2 \quad\quad + y_1 x_4. \end{aligned}$$

There are five types of orders 2, 3, 3, 4, and 5, corresponding to

(i) $y_3 = y_4 = 0, \quad y_2 = y_5$,

(ii) $y_3 = y_4 = 0$,

(iii) $y_3 = y_5 = 0$,

(iv) $y_3 = 0$,

(v) $y_2$, $y_3$, $y_4$, $y_5$ all arbitrary.

If the integers are 2, 1, 1, the groups are

$$\begin{aligned} x_1' &= y_1 x_1, \\ x_2' &= \quad\quad y_1 x_2, \\ x_3' &= y_2 x_1 \quad\quad + y_1 x_3, \\ x_4' &= y_3 x_1 + y_4 x_2 + y_2 x_3 + y_1 x_4. \end{aligned}$$

There are five types of orders 2, 2, 3, 3, and 4, corresponding to

$$\text{(i)}\quad y_3 = y_4 = 0,$$

$$\text{(ii)}\quad y_3 = 0, \quad y_4 = y_2,$$

$$\text{(iii)}\quad y_3 = 0,$$

$$\text{(iv)}\quad y_4 = 0,$$

$$\text{(v)}\quad y_2,\ y_3,\ y_4 \text{ all arbitrary.}$$

For $n = 4$, there will be, in addition to the types already enumerated, one with determinant $(y_1+\lambda)^2\,(y_2+\lambda)\,(y_3+\lambda)$, three with determinant $(y_1+\lambda)^2\,(y_2+\lambda)^2$, and five with determinant $(y_1+\lambda)^3\,(y_2+\lambda)$. There are thus, in all, twenty-three distinct types of intransitive Abelian groups in four variables, which are linear and homogeneous both in the variables and in the parameters.

It may be specially noticed that the case $n = 4$ brings out that, for an intransitive group, the number of parameters may be greater than the number of variables; whereas, for a transitive group, the two numbers are necessarily the same. The same is clearly true for all values of $n$ which are greater than 3.

18. In his memoir "Über vertauschbare Matrizen" (*Berliner Sitzungsberichte*, 1896, pp. 601–614) Herr Frobenius asks and leaves unanswered the following question:—

If $A_1, A_2, A_3, \ldots, A_m$ are a set of matrices of $n^2$ elements every two of which are permutable, is it in general possible to find a matrix $R$ of $n^2$ elements, such that each matrix of the given set is expressible as a function of $R$?

The result just obtained shows that this question must be answered in the negative. In fact, since $R$ must satisfy an equation of the $n^{\text{th}}$ degree (or lower), every function of $R$ can be expressed in the form

$$y_1 E + y_2 R + y_3 R^2 + \ldots + y_\nu R^{\nu-1}, \quad (\nu \leqslant n),$$

where $y_1, y_2, \ldots, y_\nu$ are arbitrary constants, and $E$ is the unit matrix. To the matrix just written corresponds a linear substitution on $n$ variables with $\nu$ parameters entering in it linearly; and, since every two such matrices are permutable, while their product is another matrix of the same form, the set of substitutions corresponding to the matrices constitute a linear homogeneous Abelian group of order $\nu$. Hence, the given set of matrices cannot contain more than $n$ which are linearly independent. But it has been seen that an intransitive Abelian group in $n$ variables may contain more than $n$ parameters, and therefore that there may be more than $n$ linearly independent

matrices every two of which are permutable. For such a set no such matrix as $R$ can exist.

Further, let $A_1, A_2, \ldots, A_n$ be the matrices which correspond to the parameters $y_1, y_2, \ldots, y_n$ of a transitive Abelian group

$$x'_s = \Sigma a_{sik} x_i y_k,$$

so that the elements of $A_k$ are the $n^2$ constants $a_{sik}$, $(s, i = 1, 2, \ldots, n)$. These are a set of linearly independent permutable matrices. If a matrix $R$ can be found such that, for every value of $k$, $A_k$ is a function of $R$, then the simplest equation that $R$ satisfies must be of the $n^{\text{th}}$ degree, as otherwise $A_1, A_2, \ldots$ would not be linearly independent. But, if the simplest equation which $R$ satisfies is of the $n^{\text{th}}$ degree, the first minors of the characteristic determinant of $R$ have no common factor other than unity (Frobenius, *loc. cit.*, p. 606). Hence, if $D$ is the characteristic determinant of the group, $D$ has no invariant factors other than itself and unity. The $n$ linearly independent matrices given by the finite equations of a transitive Abelian group cannot therefore be expressed as functions of a single matrix unless the only invariant factor of the group is the characteristic determinant. On the other hand, when this condition is satisfied, it is easily seen that such a single matrix can be found. Thus, for the group

$$\begin{aligned} x'_1 &= y_1 x_1, \\ x'_2 &= y_2 x_1 + y_1 x_2, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots, \\ x'_n &= y_n x_1 + y_{n-1} x_2 + \ldots + y_1 x_n, \end{aligned}$$

the matrix

$$\begin{Bmatrix} 1, & 0, & 0, & 0, & \ldots, & 0 \\ 1, & 1, & 0, & 0, & \ldots, & 0 \\ 0, & 1, & 1, & 0, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0, & 0, & 0, & \ldots, & 1, & 1 \end{Bmatrix}$$

satisfies the condition.

*On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Divisions of a Scale.* *By* W. F. Sheppard, M.A., LL.M. Read March 10th, 1898. Received, in revised form, May 10th, 1898.

## I. General Formulæ.

1. Let

$$z = f(x) \quad \ldots \quad (1)$$

be the equation to a curve which lies wholly on the positive side of the axis of $x$, and extends continuously from the ordinate $z_0 = f(x_0)$ to the ordinate $z_p = f(x_p)$, and which is subject to the further condition that $f(x)$ is single-valued between $x = x_0$ and $x = x_p$. Let the range $x_p - x_0$ be divided into $p$ equal segments, each equal to $h$, by points whose abscissæ are $x_1, x_2, \ldots, x_{p-1}$, so that

$$x_r = x_0 + rh;$$

let the ordinates to the curve from the points of division be $z_1, z_2, \ldots, z_{p-1}$, so that

$$z_r = f(x_r);$$

and let $A_r$ denote the area between two consecutive ordinates $z_{r-1}$ and $z_r$, the curve, and the base $z = 0$, so that

$$A_r = \int_{x_{r-1}}^{x_r} f(x)\,dx.$$

For the mathematical treatment of statistics we require a convenient formula for calculating, approximately, the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, where—

(i.) $\phi(x)$ is a known function of $x$, which is single-valued, finite, and continuous from $x = x_0$ to $x = x_p$; and

(ii.) the form of $f(x)$ is unknown, but either

($a$) the values of the isolated ordinates $z_0 = f(x_0)$, $z_1 = f(x_1)$, ..., $z_p = f(x_p)$ are known, or

($b$) the values of the successive areas

$$A_1 = \int_{x_0}^{x_1} f(x)\,dx, \quad A_2 = \int_{x_1}^{x_2} f(x)\,dx, \quad \ldots, \quad A_p = \int_{x_{p-1}}^{x_p} f(x)\,dx$$

are known.

2. The Euler-Maclaurin formula

$$\int u_x\,dx = C+\Sigma u_x-\tfrac{1}{2}u_x-\frac{B_1}{2!}\frac{du_x}{dx}+\frac{B_2}{4!}\frac{d^3u_x}{dx^3}-\ldots \qquad (1)^*$$

may be made to meet both these cases. For the first case we have at once

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$=\tfrac{1}{2}f(x_0)\,\phi(x_0)+f(x_1)\,\phi(x_1)+\ldots+f(x_{p-1})\,\phi(x_{p-1})+\tfrac{1}{2}f(x_p)\,\phi(x_p)$$

$$-\left[\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}f(x)\,\phi(x)\right]_{x=x_0}^{x=x_p} \qquad (2)$$

$$=\tfrac{1}{2}z_0\varphi(x_0)+z_1\phi(x_1)+\ldots+z_{p-1}\phi(x_{p-1})+\tfrac{1}{2}z_p\varphi(x_p)$$

$$+\tfrac{1}{12}(\Delta-\tfrac{1}{2}\Delta^2+\tfrac{1}{60}\Delta^3-\ldots)f(x_0)\,\phi(x_0)$$

$$+\tfrac{1}{12}(\Delta'-\tfrac{1}{2}\Delta'^2+\tfrac{1}{60}\Delta'^3-\ldots)f(x_p)\,\phi(x_p), \qquad (2A)$$

the second set of differences $\Delta'$, $\Delta'^2$, $\Delta'^3$, ... being accented to show that they are taken backwards, *i.e.*, from $x=x_p$ towards $x=x_0$.

For the second case we write

$$F(x)=\int_x^{x_p} f(x)\,dx,$$

so that $F(x)$ denotes the whole area of the curve lying beyond the ordinate $z=f(x)$. Then

$$\left.\begin{aligned} F(x_0) &= A_1+A_2+A_3+\ldots+A_p\\ F(x_1) &= A_2+A_3+\ldots+A_p\\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ &\\ F(x_{p-1}) &= A_p\\ F(x_p) &= 0 \end{aligned}\right\}, \qquad (3)$$

so that $F(x)$ is a function for which the isolated values $F(x_0)$, $F(x_1)$, ..., $F(x_p)$ are known. By integration by parts,

$$\int_{x_0}^{x_p} f(x)\,\varphi(x)\,dx = F(x_0)\,\varphi(x_0)+\int_{x_0}^{x_p} F(x)\,\varphi'(x)\,dx, \qquad (4)\dagger$$

---

* Here $\Sigma u_x$ denotes the sum up to and including $u_x$. If $\Sigma$ is used as equivalent to $\Delta^{-1}$, $\Sigma u_x$ is the sum up to and including $u_{x-1}$, and we must replace $-\frac{1}{2}u_x$ by $+\frac{1}{2}u_x$.

† For instance, let $F(x)$ denote the proportion of individuals alive at age $x$ for every one alive at age $x_0$; then $f(x)dx$ will denote the proportion who die between ages $x-\frac{1}{2}dx$ and $x+\frac{1}{2}dx$. If

$$\phi(x)=P(1+i)^{-(x-x_0)},$$

and therefore

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \frac{1}{h}F(x_0)\,\phi(x_0) + \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$- \left[\left\{\frac{B_1}{2!}h\frac{d}{dx} - \frac{B_2}{4!}h^3\frac{d^3}{dx^3} + \ldots\right\} F(x)\,\phi'(x)\right]_{x=x_0}^{x=x_p} \qquad (5)$$

$$= \frac{1}{h}F(x_0)\,\phi(x_0) + \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$+ \tfrac{1}{12}\left(\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{60}\Delta^3 - \ldots\right) F(x_0)\,\phi'(x_0)$$

$$+ \tfrac{1}{12}\left(\Delta - \tfrac{1}{2}\Delta'^2 + \tfrac{1}{60}\Delta'^3 - \ldots\right) F(x_p)\,\phi'(x_p). \qquad (5\text{A})$$

3. The formulæ (2A) and (5A) are sufficient, but they involve the calculation of successive differences. In the cases which we have specially in view the curve $z = f(x)$ touches the base $z = 0$, to a very high order of contact, at the extreme points $x = x_0$ and $x = x_p$. In such a case $f(x_0)$ and $f(x_p)$ and their first few differential coefficients are zero. Hence the expression given by the square brackets in (2) is negligible, and we have, *approximately*,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx = \tfrac{1}{2}z_0\phi(x_0) + z_1\phi(x_1) + \ldots + z_{p-1}\phi(x_{p-1}) + \tfrac{1}{2}z_p\phi(x_p). \qquad (2\text{B})$$

Also the first few differential coefficients of $F(x_0)$ and $F(x_p)$ are zero, and

$$F(x_p) = 0;$$

but $F(x_0)$ is not zero. Hence the expression given by the square brackets in (5) is negligible, except as to the part involving $F(x_0)$. Taking this out, we have, approximately,

$$\frac{1}{h}\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \tfrac{1}{2}F(x_0)\,\phi'(x_0) + F(x_1)\,\phi'(x_1) + \ldots + F(x_{p-1})\,\phi'(x_{p-1})$$

$$+ \frac{1}{h}F(x_0)\left\{\phi(x_0) + \frac{B_1}{2!}h^2\phi''(x_0) - \frac{B_2}{4!}h^4\phi^{\text{iv}}(x_0) + \ldots\right\} \qquad (5\text{B})$$

---

$\int_{x_0}^{\infty} f(x)\,\phi(x)\,dx$ will be the present value of a reversion of value $P$, due on the death of a person aged $x_0$, interest being reckoned at $100i$ per cent. per annum, while $\int_{x_0}^{\infty} F(x)\,\phi(x)\,dx$ will be the present value of a continuous annuity at the rate of $P$ per annum, payable to a person of that age, and (4) shows that

(present value of reversion) = $P - \log_e(1+i)$ . (present value of annuity),

which is a well-known formula.

the expression omitted from the right-hand side being

$$\left[F(x)\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}\phi'(x)\right.$$
$$\left.-\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}F(x)\phi'(x)\right]_{x=x_0}^{x=x_p}. \qquad (5\text{C})$$

The formula (2B) is convenient for calculation, but (5B) is obviously inconvenient, on account of its unsymmetrical form. Our object is therefore to obtain a general formula for $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$, which shall be better suited for these particular cases.

4. Let $\xi_1, \xi_2, \ldots, \xi_p$ be the abscissæ of the middle points of the bases of the areas $A_1, A_2, \ldots, A_p$, so that

$$\xi_r = \tfrac{1}{2}(x_{r-1}+x_r).$$

We wish to obtain the value of $\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$ in the form

$$\Sigma A_r \phi_1(\xi_r) + h\left[\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

where $\phi_1(x)$ is a function of $x$ which can be derived from $\phi(x)$, and $\phi_2(x)$ is an expression which can be derived from $f(x)$ and $\phi(x)$, and which may be disregarded when $f(x)$ and its first few differential coefficients are zero. We must therefore express

$$\int_{x_{r-1}}^{x_r} f(x)\,\phi(x)\,dx \equiv \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,\phi(\xi_r+\theta)\,d\theta$$

in the form

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r+\theta)\,d\theta\,.\,\phi_1(\xi_r) + h\left[\phi_2(x)\right]_{x=x_{r-1}}^{x=x_r};$$

in other words, we must find such a value for $\phi_1(\xi)$ that

$$\Phi(\xi) \equiv \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta - \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\,\phi_1(\xi) \qquad (6)$$

may be capable of being expressed in the form

$$h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \qquad (7)$$

where $\phi_2(\xi)$ contains only $f(\xi)$ and $\phi(\xi)$ and their successive differential coefficients.

Now, in order that $\Phi(\xi)$ may be expressible in the form (7), it is

necessary and sufficient (subject to certain conditions as to convergence) that $\Phi(\xi)\,d\xi$ should be an exact differential. It is necessary, for if we expand (7) in ascending powers of $h$ we have only differential coefficients of $\phi_2(\xi)$. And it is sufficient, for if $\Phi(\xi)$ is equal to $h\Psi'(\xi)$ we can obviously choose the coefficients $C_1, C_2, \ldots$ so as to make it equal to

$$\left\{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\right\}+C_1\left\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^2 + C_2\left\{\Psi^{iv}(\xi+\tfrac{1}{2}h)-\Psi^{iv}(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^4+\ldots.$$

The values of these coefficients can be obtained most simply by the method of operators, which shows that

$$h\Psi'(\xi) = (e^{\frac{1}{2}h(d/d\xi)}-e^{-\frac{1}{2}h(d/d\xi)})\,\frac{\frac{1}{2}h\frac{d}{d\xi}}{\sinh\frac{1}{2}h\frac{d}{d\xi}}\,\Psi(\xi)$$

$$= \left\{\Psi(\xi+\tfrac{1}{2}h)-\Psi(\xi-\tfrac{1}{2}h)\right\} - \frac{P_1}{2!}\left\{\Psi''(\xi+\tfrac{1}{2}h)-\Psi''(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^2 + \frac{P_2}{4!}\left\{\Psi^{iv}(\xi+\tfrac{1}{2}h)-\Psi^{iv}(\xi-\tfrac{1}{2}h)\right\}(\tfrac{1}{2}h)^4 - \ldots, \tag{8}$$

where $P_1, P_2, \ldots$ are the coefficients in the expansion

$$\frac{\theta}{\sinh\theta} = 1-\frac{P_1}{2!}\theta^2+\frac{P_2}{4!}\theta^4-\ldots, \tag{9}$$

so that

$$P_r = (2^{2r}-2)\,B_r. \tag{10}$$

We consider that we are only dealing with cases in which the series (8) is initially convergent.

Now let us assume that

$$\phi_1(\xi) = \psi\left(\tfrac{1}{2}h\frac{d}{d\xi}\right)\phi(\xi). \tag{11}$$

Then if we denote $\frac{1}{2}h\frac{d}{d\xi}$ operating on $f(\xi)$ by $D_1$, and $\frac{1}{2}h\frac{d}{d\xi}$ operating on $\phi(\xi)$ by $D_2$, $\Phi(\xi)$ is equal to

$$h\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\psi(D_2)\right\}f(\xi)\,\phi(\xi). \tag{12}$$

In order that $\Phi(\xi)\,d\xi$ may be an exact differential, it is clearly necessary that (12) may be capable of being written in the form

$$h(D_1+D_2)\,\chi(D_1,\,D_2)\,f(\xi)\,\phi(\xi), \tag{13}$$

where $\chi(D_1,\,D_2)$ is a function containing only positive integral powers (including the power 0) of $D_1$ and $D_2$. It will be seen that

$$\psi(D_2)=\frac{D_2}{\sinh D_2}$$

satisfies these conditions. For

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\,\frac{D_2}{\sinh D_2}$$

can be expanded so as to include only positive integral powers of $D_1$ and $D_2$; and it vanishes when $D_1+D_2=0$, so that $D_1+D_2$ is a factor. Hence we have

$$\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,\phi(\xi+\theta)\,d\theta$$

$$=\int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi+\theta)\,d\theta\,.\left\{\phi(\xi)-\frac{P_1}{2!}\phi''(\xi)\,(\tfrac{1}{2}h)^2+\frac{P_2}{4!}\phi^{\mathrm{iv}}(\xi)\,(\tfrac{1}{2}h)^4-\dots\right\}$$

$$+h\left\{\phi_2(\xi+\tfrac{1}{2}h)-\phi_2(\xi-\tfrac{1}{2}h)\right\}, \tag{14}$$

where

$$\phi_2(\xi)=\tfrac{1}{2}\left\{\frac{1}{D_1+D_2}-\frac{D_2\sinh D_1}{D_1\sinh D_2\sinh(D_1+D_2)}\right\}f(\xi)\,\phi(\xi)$$

$$=\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\frac{D_1+D_2}{2}-\frac{1}{D_1+D_2}\right)-\left(\coth\frac{D_2}{2}-\frac{1}{D_2}\right)\right\}f(\xi)\,\phi(\xi).$$

Since

$$\tfrac{1}{2}\theta\coth\tfrac{1}{2}\theta=1+\frac{B_1}{2!}\theta^2-\frac{B_2}{4!}\theta^4+\dots, \tag{15}$$

this reduces to

$$\phi_2(\xi)=F(\xi)\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\dots\right\}\phi'(\xi)$$

$$-\left\{\frac{B_1}{2!}h\frac{d}{d\xi}-\frac{B_2}{4!}h^3\frac{d^3}{d\xi^3}+\dots\right\}F(\xi)\,\phi'(\xi). \tag{16}$$

Substituting in (14), writing $\xi=\xi_1,\,\xi_2,\,\dots,\,\xi_\rho$ successively, and adding

together the $p$ equations so obtained, we have, finally,

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h \left[ F(x) \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} \phi'(x) \right.$$

$$\left. - \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} F(x)\,\phi'(x) \right]_{x=x_0}^{x=x_p}. \quad (17)$$

This result can be verified by comparing it with (5B) and (5C). To give the expression in square brackets explicitly in terms of $f(x)$ and $\phi(x)$ and their derived functions, we write, for convenience,

$$Q_r = \frac{B_r}{2r}, \quad (18)$$

and we find that

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=1}^{r=p} A_r \left\{ \phi(\xi_r) - \frac{P_1}{2!}(\tfrac{1}{2}h)^2 \phi''(\xi_r) + \frac{P_2}{4!}(\tfrac{1}{2}h)^4 \phi^{\text{iv}}(\xi_r) - \dots \right\}$$

$$+ h^2 \left[ \left\{ \frac{Q_1}{0!}\phi'(x) - \frac{Q_2}{2!}h^2\phi'''(x) + \frac{Q_3}{4!}h^4\phi^{\text{v}}(x) - \dots \right\} f(x) \right.$$

$$- \frac{1}{2!}\left\{ \frac{Q_2}{1!}\phi''(x) - \frac{Q_3}{3!}h^2\phi^{\text{iv}}(x) + \frac{Q_4}{5!}h^4\phi^{\text{vi}}(x) - \dots \right\} h^2 f'(x)$$

$$- \frac{1}{3!}\left\{ \frac{Q_2}{0!}\phi'(x) - \frac{Q_3}{2!}h^2\phi'''(x) + \frac{Q_4}{4!}h^4\phi^{\text{v}}(x) - \dots \right\} h^2 f''(x)$$

$$+ \frac{1}{4!}\left\{ \frac{Q_3}{1!}\phi''(x) - \frac{Q_4}{3!}h^2\phi^{\text{iv}}(x) + \frac{Q_5}{5!}h^4\phi^{\text{vi}}(x) - \dots \right\} h^4 f'''(x)$$

$$\left. + \&\text{c.} \right]_{x=x_0}^{x=x_p}, \quad (19)$$

where

$$\left.\begin{aligned} P_1 &= \tfrac{1}{3}, \quad P_2 = \tfrac{7}{15}, \quad P_3 = \tfrac{31}{21}, \quad P_4 = \tfrac{127}{15}, \quad P_5 = \tfrac{2555}{33}, \quad \dots \\ Q_1 &= \tfrac{1}{12}, \quad Q_2 = \tfrac{1}{120}, \quad Q_3 = \tfrac{1}{252}, \quad Q_4 = \tfrac{1}{240}, \quad Q_5 = \tfrac{1}{132}, \quad \dots \end{aligned}\right\}. \quad (20)$$

5. The value of $\phi_1(\xi)$ might have been obtained without using the symbolic method, as follows.

Let us assume that

$$\phi_1(\xi) = \phi(\xi) + \frac{\kappa_1}{2!}(\tfrac{1}{2}h)^2\phi''(\xi) + \frac{\kappa_2}{4!}(\tfrac{1}{2}h)^4\phi^{\text{iv}}(\xi) + \dots .$$

(The odd differential coefficients are omitted, since it is clear that $\phi_1(\xi)$ is not altered by changing $\frac{1}{2}h$ into $-\frac{1}{2}h$). Expand $f(\xi+\theta)$ and $\phi(\xi+\theta)$ in ascending powers of $\theta$ by Taylor's theorem, substitute in (6), and perform the integrations; then substitute for $\phi_1(\xi)$ from the above expression, and arrange the result in a series proceeding by ascending powers of $h$. The successive coefficients will contain products as follows:—

| co. $h^3$ | co. $h^5$ | co. $h^7$ | |
|---|---|---|---|
| $f(\xi)\,\phi''(\xi)$, | $f(\xi)\,\phi^{\text{iv}}(\xi)$, | $f(\xi)\,\phi^{\text{vi}}(\xi)$, | ..., |
| $f'(\xi)\,\phi'(\xi)$, | $f'(\xi)\,\phi'''(\xi)$, | $f'(\xi)\,\phi^{\text{v}}(\xi)$, | ..., |
| | $f''(\xi)\,\phi''(\xi)$, | $f''(\xi)\,\phi^{\text{iv}}(\xi)$, | ..., |
| | $f'''(\xi)\,\phi'(\xi)$, | $f'''(\xi)\,\phi'''(\xi)$, | ..., |
| | | $f^{\text{iv}}(\xi)\,\phi''(\xi)$, | ..., |
| | | $f^{\text{v}}(\xi)\,\phi'(\xi)$, | ..., |
| | | | ... . |

The first coefficient will involve $\kappa_1$, the second $\kappa_1$ and $\kappa_2$, and so on. Hence the values of $\kappa_1, \kappa_2, \dots$ can be chosen successively so that each term in the expansion of $\Phi(\xi)\,d\xi$ in ascending powers of $h$ shall be an exact differential. The $n^{\text{th}}$ term, divided by $\dfrac{2}{2n+1}(\tfrac{1}{2}h)^{2n+1}$, is

$$\begin{aligned}\Big\{ &\frac{1}{0!}\,\frac{1}{(2n)!}\Big(1-\frac{2n+1}{1}\kappa_n\Big)f(\xi)\,\phi^{(2n)}(\xi) + \frac{1}{1!}\,\frac{1}{(2n-1)!}f'(\xi)\phi^{(2n-1)}(\xi) \\ &+ \frac{1}{2!}\,\frac{1}{(2n-2)!}\Big(1-\frac{2n+1}{3}\kappa_{n-1}\Big)f''(\xi)\phi^{(2n-2)}(\xi) + \dots \\ \dots &+ \frac{1}{(2n-2)!}\,\frac{1}{2!}\Big(1-\frac{2n+1}{2n-1}\kappa_1\Big)f^{(2n-2)}(\xi)\,\phi'(\xi) \\ &+ \frac{1}{(2n-1)!}\,\frac{1}{1!}\,f^{(2n-1)}(\xi)\,\phi'(\xi)\Big\}\,d\xi.\end{aligned}$$

Let us choose $\kappa_n$ so as to make this the differential of

$$C_1\frac{1}{1!}\,\frac{1}{(2n-1)!}f(\xi)\,\phi^{(2n-1)}(\xi) + C_2\frac{1}{2!}\,\frac{1}{(2n-2)!}f'(\xi)\,\phi^{(2n-2)}(\xi) + \dots$$

$$\dots + C_{2n-1}\frac{1}{(2n-1)!}\,\frac{1}{1!}f^{(2n-2)}(\xi)\,\phi'(\xi).$$

Differentiating this last expression, and equating coefficients,

$$\begin{aligned}
1-\frac{2n+1}{1}\kappa_n &= \frac{2n}{1}C_1,\\
1 &= C_1 + \frac{2n-1}{2}C_2,\\
1-\frac{2n+1}{3}\kappa_{n-1} &= C_2 + \frac{2n-2}{3}C_3,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots &\\
1-\frac{2n+1}{2n-1}\kappa_1 &= C_{2n-2}+\frac{2}{2n-1}C_{2n-1},\\
1 &= C_{2n-1}.
\end{aligned}$$

Eliminating, and remembering that

$$1-\frac{2n}{1}+\frac{2n\,(2n-1)}{1.2}-\ldots+\frac{2n\,(2n-1)\ldots 1}{1.2\ldots 2n}=(1-1)^{2n}=0,$$

we find that

$$\frac{1}{0!\,(2n+1)!}+\frac{\kappa_1}{2!\,(2n-1)!}+\frac{\kappa_2}{4!\,(2n-3)!}+\ldots+\frac{\kappa_n}{(2n)!\,1!}=0.$$

Now let $$\Omega(\theta)=1+\frac{\kappa_1}{2!}\theta^2+\frac{\kappa_2}{4!}\theta^4+\ldots.$$

Then the above relation shows that

$$\Omega(\theta)\,.\,\sinh\theta=\theta,$$

and therefore $$\Omega(\theta)=\frac{\theta}{\sinh\theta},$$

which agrees with (14).

6. In the general formula (17) or (19) the expressions to be summed are given in terms of $\xi_1, \xi_2, \ldots, \xi_p$, the abscissæ of the middle points of the bases of the compartments $A_1, A_2, \ldots, A_p$. If preferred, we can use $x_0, x_1, \ldots, x_p$. For

$$\begin{aligned}
\phi_1(\xi) &= \frac{\frac{1}{2}h\frac{d}{d\xi}}{\sinh\frac{1}{2}h\frac{d}{d\xi}}\phi(\xi)\\
&= \frac{h\frac{d}{d\xi}}{\sinh h\frac{d}{d\xi}}\tfrac{1}{2}\left\{\phi(\xi+\tfrac{1}{2}h)+\phi(\xi-\tfrac{1}{2}h)\right\}, \qquad (21)
\end{aligned}$$

and therefore

$$\int_{x_0}^{x_p} f(x)\,\phi(x)\,dx$$

$$= \sum_{r=0}^{r=p} \tfrac{1}{2}(A_r + A_{r+1}) \left\{ \phi(x_r) - \frac{P_1}{2!}h^2\phi''(x_r) + \frac{P_2}{4!}h^4\phi^{\text{iv}}(x_r) - \dots \right\}$$

$$+ h \left[ F(x) \left\{ \frac{B_1}{2!}h\frac{d}{dx} - \frac{B_2}{4!}h^3\frac{d^3}{dx^3} + \dots \right\} \phi'(x) \right.$$

$$\left. - \left\{ \frac{B_1}{2!}h\frac{d}{dx} - \frac{B_2}{4!}h^3\frac{d^3}{dx^3} + \dots \right\} F(x)\phi'(x) \right]_{x=x_0}^{x=x_p}, \qquad (22)$$

where $A_0$ and $A_{p+1}$ are each zero.

## II. Application to Calculation of $\Sigma x^m$.*

7. When an attribute common to a number of individuals is capable of exact measurement, and of representation by a linear magnitude $x$, whose variation is continuous, the result of a classification of the individuals according to this attribute is generally given in the form of a table showing the numbers which fall into consecutive classes corresponding to equal increments of $x$, *e.g.*,

(A) *Weights of* 5,552 *Englishmen.*†

| Weight. | Number. | Weight. | Number. |
|---|---|---|---|
| 90 lbs. to 100 lbs. | 2 | 180 lbs. to 190 lbs. | 304 |
| 100 lbs. to 110 lbs. | 26 | 190 lbs. to 200 lbs. | 174 |
| 110 lbs. to 120 lbs. | 133 | 200 lbs. to 210 lbs. | 75 |
| 120 lbs. to 130 lbs. | 338 | 210 lbs. to 220 lbs. | 62 |
| 130 lbs. to 140 lbs. | 694 | 220 lbs. to 230 lbs. | 33 |
| 140 lbs. to 150 lbs. | 1,240 | 230 lbs. to 240 lbs. | 10 |
| 150 lbs. to 160 lbs. | 1,075 | 240 lbs. to 250 lbs. | 9 |
| 160 lbs. to 170 lbs. | 881 | 250 lbs. to 260 lbs. | 3 |
| 170 lbs. to 180 lbs. | 492 | 260 lbs. to 270 lbs. | 1 |

* Some of the results of this section have been stated, without proof, in a paper printed in the *Journal of the Royal Statistical Society* (September, 1897, p. 698).

† *Report of the British Association*, 1883, p. 257.

(B) *Examination Marks of* 1,000 *Sandhurst Candidates.**

| Mark. | Number. | Mark. | Number. |
|---|---|---|---|
| 0 to 99 | 15 | 500 to 599 | 221 |
| 100 to 199 | 54 | 600 to 699 | 115 |
| 200 to 299 | 131 | 700 to 799 | 31 |
| 300 to 399 | 227 | 800 to 899 | 3 |
| 400 to 499 | 203 | 900 to 999 | 0 |

From the statistical point of view we are not concerned with particular individuals. We regard them—on grounds which need not be discussed here—as having been obtained by random selection from an indefinitely great community; and we investigate the distribution of the values of $x$ in this hypothetical community. Any such distribution can be expressed in terms of the mean values of $x, x^2, x^3, \ldots$ for the community, which we may denote by $[x], [x^2], [x^3], \ldots$; and therefore the problem is to determine, from the data, the most probable combination of values of $[x], [x^2], [x^3], \ldots$.

To do this, when the data take the form of a table such as (A) or (B), we adopt the same process as if we knew the exact value of $x$ for each of the observed individuals. We postulate an indefinite number of communities with different laws of distribution of $x$, and therefore with different values of $[x], [x^2], [x^3], \ldots$; we suppose that from each of these communities a random selection of $n$ individuals is made $N$ times, where $n$ is the number of individuals actually observed, and $N$ is an indefinitely great number; and from this double aggregate of sets of $n$ individuals we pick out those sets in which the numbers in the consecutive classes are the same as in the actual data, and consider how the different values of $[x], [x^2], [x^3], \ldots$ are distributed in these particular cases.

8. Let the range of values of $x$, in the original community, be from $x_0$ to $x_p$, this range being divided into $p$ segments, each equal to $h$; and let the actual data consist of a table giving numbers $n_1, n_2, \ldots n_p$ in the corresponding $p$ classes, where $n_1 + n_2 + \ldots + n_p = n$.

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* These are the marks in English Composition (a few candidates being omitted) at the examinations in November, 1894, and June, 1895. The table must not be taken as giving the law of distribution of the marks at either examination separately.

We assume that the data show that both the variation of $x$ and the frequency of the variation may be regarded as continuous; *i.e.*, that the true figure of frequency of $x$ may be regarded as bounded by a continuous curve. Let the equation to this unknown curve be

$$z = f(x),$$

and let the areas between the successive ordinates $f(x_0), f(x_1), \ldots, f(x_p)$, where $x_r = x_0 + xh$, be $A_1, A_2, \ldots, A_p$, the total area of the curve being $A$. Then our fundamental postulate is that there are an indefinite number of forms of $f(x)$, or, rather, of $f(x) \div A$; and we have to consider the result of a random selection of $n$ individuals when $f(x) \div A$ has any particular form, and thence to determine the distribution of the different values of

$$[x^m] \equiv \int_{x_0}^{x_p} f(x)\, x^m\, dx \div A,$$

when the random selection gives numbers $n_1, n_2, \ldots, n_p$ in the $p$ classes.

If in the expression contained in square brackets in (17) we replace the differential coefficients by differences, and if the resulting series is initially convergent when

$$\phi(x) = x^m,$$

we obtain $\int_{x_0}^{x_p} f(x)\, x^m\, dx$ in the form

$$C_1 A_1 + C_2 A_2 + \ldots + C_p A_p.$$

Let

$$n_r' = nA_r/A,$$

so that $n_1', n_2', \ldots, n_p'$ are the numbers (not necessarily integral) that would fall into the $p$ classes if we made a *representative* selection of $n$ individuals. Then

$$n\,[x^m] \equiv n \int_{x_0}^{x_p} f(x)\, x^m\, dx \div A$$

is of the form

$$C_1 n_1' + C_2 n_2' + \ldots + C_p n_p'.$$

By hypothesis, the numbers $n_1, n_2, \ldots, n_p$ are obtained by *random* selection, and they therefore differ from $n_1', n_2', \ldots, n_p'$ by amounts which, when $n$ is large, are distributed according to the normal law with mean values zero, and mean squares $n_1'(n - n_1')/n$, $n_2'(n - n_2')/n, \ldots, n_p'(n - n_p')/n$,

the $p$ distributions being normally correlated. It follows, by the ordinary methods of inverse probabability, that, if the *a priori* frequency of different values of $[x^m]$, in those cases in which $A_1, A_2, \ldots, A_p$ are nearly proportional to $n_1, n_2, \ldots n_p$, is continuous, the most probable value of $n[x^m]$ as deducible from the data is (approximately) $C_1n_1+C_2n_2+\ldots+C_pn_p$; and the possible values are distributed normally about this value with mean square $\{\Sigma C_r^2 n_r-(\Sigma C_r n_r)^2\}\div n$. Similarly, it may be shown that the different distributions obtained by giving different values to $m$ are normally correlated; and it follows that the most probable combination of values of $[x]$, $[x^2]$, $[x^3]$, ... is to be found by assuming the areas $A_1, A_2, \ldots, A_p$ of the figure of frequency to be proportional to $n_1, n_2, \ldots, n_p$. It is understood, throughout, that $n$ is a large number.

Denote the most probable value of $[x^m]$ by "$[x^m]$". Then, writing $\phi(x)=x^m$ in (17), multiplying both sides of the equation by $n/A$, and replacing $nA_r/A$ by $n_r$, we have

$$n\text{“}[x^m]\text{”}=\sum_{r=1}^{r=p} n_r\left(\xi_r^m-\tfrac{1}{4}P_1\frac{m!}{2!(m-2)!}h^2\xi_r^{m-2}+\tfrac{1}{16}P_2\frac{m!}{4!(m-4)!}h^4\xi_r^{m-4}-\ldots\right)$$

$$-n\left\{B_1\frac{m!}{2!\,(m-2)!}h^2x_0^{m-2}-B_2\frac{m!}{4!\,(m-4)!}h^4x_0^{m-4}+\ldots\right\}$$

$$-\frac{n}{A}\left[\left\{\frac{B_1}{2!}h\frac{d}{dx}-\frac{B_2}{4!}h^3\frac{d^3}{dx^3}+\ldots\right\}mhx^{m-1}\int_x^{x_p}f(x)\,dx\right]_{x=x_0}^{x=x_p}, \quad (23)$$

where $$\xi_r=\tfrac{1}{2}(x_{r-1}+x_r).$$

To express this in a convenient form, let $M_q$ denote the *rough* value of $\Sigma x^q$ for the $n$ individuals, obtained by massing the individuals in each class at the centre of the range of the class; *i.e.*, by taking every value of $x$ between $x_{r-1}$ and $x_r$ as equal to the arithmetic mean of $x_{r-1}$ and $x_r$. Thus

$$M_q=\sum_{r=1}^{r=p} n_r\xi_r^q.$$

Also let $$n_r+n_{r+1}+\ldots+n_p=N_{r-1},$$

so that $$N_0=n, \quad N_p=0;$$

and let $m(N_0x_0^{m-1}+N_px_p^{m-1})=S_0$, $m(N_1x_1^{m-1}+N_{p-1}x_{p-1}^{m-1})=S_1$,

$$m(N_2x_2^{m-}+N_{p-2}x_{p-2}^{m-1})=S_2, \ \ldots.$$

Then, if $S_1 - S_0 = \Delta S_0$, $S_2 - S_1 = \Delta S_1$, ..., $\Delta S_1 - \Delta S_0 = \Delta^2 S_0$, ...,

(23) becomes

$$n\text{“}[x^m]\text{”} = M_m - \tfrac{1}{4}P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{160} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \ldots$$
$$- n \left\{ \tfrac{1}{6} \frac{m!}{2!\,(m-2)!} h^2 x_0^{m-2} - \tfrac{1}{360} \frac{m!}{4!\,(m-4)!} h^4 x_0^{m-4} + \ldots \right\}$$
$$+ \tfrac{1}{12} h \left\{ \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{19}{60}\Delta^3 - \tfrac{9}{40}\Delta^4 + \tfrac{863}{50400}\Delta^5 - \ldots \right\} S_0. \qquad (24)$$

9. The data may be studied by another method, which avoids the difficulties incident to inverse probability, and is therefore incomplete. If we knew the exact value of $x$ for each of the $n$ individuals, we should calculate the values of $\Sigma x \div n$, $\Sigma x^2 \div n$, $\Sigma x^3 \div n$, ... ; and these calculated averages would constitute the data for determining the mean values $[x]$, $[x^2]$, $[x^3]$, ... . When we only know the numbers $n_1, n_2, \ldots, n_p$ in the $p$ classes, we cannot calculate the value of $\Sigma x^m$ exactly. We may, therefore, leave on one side the question of the amount of information that the value of $\Sigma x^m$ gives as to the frequency-constants of the original community, and consider only what is the most probable value of $\Sigma x^m$ itself, on the assumption that the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection from some particular (unknown) community, and how, on the same assumption, the different possible values are distributed about this most probable value.

Let
$$z = f(x)$$
be the equation to the curve of frequency in the original community; and, as before, let
$$n'_r = nA_r / A.$$

Since the numbers $n_1, n_2, \ldots, n_p$ are obtained by random selection, the difference $n_r \sim n'_r$ may be regarded as comparable with $\sqrt{n_r}\,(\overline{n - n_r})/\overline{n}$. To examine the distribution of possible values of $\Sigma x^m$, we suppose an indefinitely great number of sets of $n$ individuals to be obtained by random selection, and we pick out the cases in which the numbers in the $p$ classes are $n_1, n_2, \ldots, n_p$. Let $\Sigma_r x^m$ denote the portion of $\Sigma x^m$ which is due to the $n_r$ individuals for which $x$ lies between $x_{r-1}$ and $x_r$. Then it is clear, from the principles of random selection, that the distributions of $\Sigma_1 x^m$, $\Sigma_2 x^m$, ..., $\Sigma_p x^m$, and of their sum $\Sigma x^m$, in the special cases considered, are the same as if we made *independent* random selections of $n_1, n_2, \ldots, n_p$ individuals from the $p$ classes of the original community.

Let $(x^m)_r$ denote the mean value of $x^m$ for those individuals in the original community for which $x$ lies between $x_{r-1}$ and $x_r$, so that

$$(x^m)_r = \int_{x_{r-1}}^{x_r} f(x)\, x^m\, dx \div \int_{r-1}^{x_r} f(x)\, dx = \int_{x_{r-1}}^{x_r} f(x)\, x^m\, dx \div A_r.$$

Then the theory of error shows that, if $n_r$ individuals were taken at random from this particular class, the mean value of $\Sigma_r x^m$ would be $n_r (x^m)_r$, and the different values would be distributed about this mean with mean square $n_r \left[ (x^{2m})_r - \left\{ (x^m)_r \right\}^2 \right]$. It follows from the last paragraph that the different values of $\Sigma x^m$ are distributed about a mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$ with mean square $\sum_{r=1}^{r=p} n_r \left[ (x^{2m})_r - \left\{ (x^m)_r \right\}^2 \right]$. This is true whatever the value of $n$ may be. When $n$ is large, the distribution is approximately normal, and the most probable value of $\Sigma x^m$ is the same as the mean value $\sum_{r=1}^{r=p} n_r (x^m)_r$, while the actual value may be regarded as differing from this by an amount comparable with

$$\sqrt{\sum_{r=1}^{r=p} n_r \left[ (x^{2m})_r - \left\{ (x^m)_r \right\}^2 \right]},$$

*i.e.*, by an amount comparable with

$$\sqrt{n} \sqrt{\frac{\Sigma x^{2m}}{n} - \sum_{r=1}^{r=p} \frac{n_r'}{n} \left\{ (x^m)_r \right\}^2}.$$

Let "$\Sigma x^m$" denote the most probable value. Then, from (14), with the same notation as before,

$$\begin{aligned}
\text{"}\Sigma x^m\text{"} &= \sum_{r=1}^{r=p} n_r (x^m)_r \\
&= \sum_{r=1}^{r=p} \frac{n_r}{A_r} \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(\xi_r + \theta)(\xi_r + \theta)^m\, d\theta \\
&= \sum_{r=1}^{r=p} n_r \left\{ \xi_r^m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} h^2 \xi_r^{m-2} + \tfrac{1}{1\,6} P_2 \frac{m!}{4!\,(m-4)!} h^4 \xi_r^{m-4} - \dots \right\} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r}{A_r} \left[ h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r} \\
&= M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{1\,6} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \dots \\
&\quad + \frac{n}{A} \left[ h\phi_2(x) \right]_{x=x_0}^{x=x_p} \\
&\quad + \sum_{r=1}^{r=p} \frac{n_r - n_r'}{A_r} \left[ h\phi_2(x) \right]_{x=x_{r-1}}^{x=x_r}, \qquad \dots \qquad (25)
\end{aligned}$$

where

$$h\phi_2(x) = \left\{ B_1 \frac{m!}{2!\,(m-2)!} h^2 x^{m-2} - B_2 \frac{m!}{4!\,(m-4)!} h^4 x^{m-4} + \dots \right\} \int_x^{x_p} f(x)\,dx$$

$$- \left\{ \frac{B_1}{2!} h \frac{d}{dx} - \frac{B_2}{4!} h^3 \frac{d^3}{dx^3} + \dots \right\} mhx^{m-1} \int_x^{x_p} f(x)\,dx. \quad (26)$$

The value of "$\Sigma x^m$", as given by (25), and the value of $n$"$[x^m]$", as given by (23) or (24), agree in the terms

$$M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{1\,6} P_2 \frac{m!}{4!\,(m-4)!} M_{m-2} h^4 - \dots,$$

but the remaining terms are different. The difference, however, is a linear function of the differences $n_1 - n_1'$, $n_2 - n_2'$, ..., $n_p - n_p'$; and these are small in comparison with $n$. Hence $n$"$[x^m]$" and "$\Sigma x^m$" may be regarded as substantially the same, the value of either being given by (24).

10. The formula (24) applies to any distribution which satisfies the condition as to continuity. In the special cases we are considering, the extreme values $n_0$, $n_1$, ..., and $n_p$, $n_{p-1}$, ... are very small. We have then, approximately,

$$n\text{"}[x^m]\text{"} = \text{"}\Sigma x^m\text{"}$$
$$= M_m - \tfrac{1}{4} P_1 \frac{m!}{2!\,(m-2)!} M_{m-2} h^2 + \tfrac{1}{1\,6} P_2 \frac{m!}{4!\,(m-4)!} M_{m-4} h^4 - \dots, \quad (27)$$

which gives, as particular cases,

$$\left.\begin{aligned}
n\text{"}[x]\text{"} &= \text{"}\Sigma x\text{"} = M_1 \\
n\text{"}[x^2]\text{"} &= \text{"}\Sigma x^2\text{"} = M_2 - \tfrac{1}{1\,2} n h^2 \\
n\text{"}[x^3]\text{"} &= \text{"}\Sigma x^3\text{"} = M_3 - \tfrac{1}{4} M_1 h^2 \\
n\text{"}[x^4]\text{"} &= \text{"}\Sigma x^4\text{"} = M_4 - \tfrac{1}{2} M_2 h^2 + \tfrac{7}{2\,4\,0} n h^4 \\
n\text{"}[x^5]\text{"} &= \text{"}\Sigma x^5\text{"} = M_5 - \tfrac{5}{6} M_3 h^2 + \tfrac{7}{4\,8} M_1 h^4 \\
&\&c.
\end{aligned}\right\}, \quad (28)$$

since, of course, $M_0 = n$.

This is the most convenient form when we are able to take the zero at the middle point of one of the divisions of the range, so that $\xi_1$, $\xi_2$, ..., $\xi_p$ may be integral multiples of $h$. In some cases it may be desirable to take the zero at a point of division, so that $x_0$, $x_1$, $x_2$, ..., $x_p$

will be integral multiples of $h$. Let $M'_1$, $M'_2$, ..., $M'_m$ denote the rough values of $\Sigma x$, $\Sigma x^2$, ..., $\Sigma x^m$ obtained by massing $\frac{1}{2}n_1$ individuals at $x_0$, $\frac{1}{2}(n_1+n_2)$ at $x_1$, $\frac{1}{2}(n_2+n_3)$ at $x_2$, ..., $\frac{1}{2}n_p$ at $x_p$. Then, by (22),

$$\left.\begin{aligned}
n\text{“}[x]\text{”} &= \text{“}\Sigma x\text{”} = M'_1 \\
n\text{“}[x^2]\text{”} &= \text{“}\Sigma x^2\text{”} = M'_2 - \tfrac{1}{3}nh^2 \\
n\text{“}[x^3]\text{”} &= \text{“}\Sigma x^3\text{”} = M'_3 - M'_1 h^2 \\
n\text{“}[x^4]\text{”} &= \text{“}\Sigma x^4\text{”} = M'_4 - 2M'_2 h^2 + \tfrac{7}{15}nh^4 \\
n\text{“}[x^5]\text{”} &= \text{“}\Sigma x^5\text{”} = M'_5 - \tfrac{10}{3}M'_3 h^2 + \tfrac{7}{3}M'_1 h^4 \\
&\quad \&c.
\end{aligned}\right\}. \qquad (29)$$

If we use the rough values $M_1$, $M_2$, ... to determine $\mu_2$, $\mu_3$, ..., the mean square, mean cube, ... of the deviation from the mean, and if the common difference $h$ is the unit of measurement, we have, from (28), $\mu_1$ being zero,

$$\left.\begin{aligned}
\text{corrected value of } \mu_2 &= \mu_2 - \tfrac{1}{12} \\
&= \mu_2 - \cdot 083\dot{3} \\
\text{corrected value of } \mu_4 &= \mu_4 - \tfrac{1}{2}\mu_2 + \tfrac{7}{240} \\
&= \mu_4 - \tfrac{1}{2}\mu_2 + \cdot 02916\dot{6} \\
\text{corrected value of } \mu_5 &= \mu_5 - \tfrac{5}{6}\mu_3
\end{aligned}\right\}, \qquad (30)$$

the value of $\mu_3$ being unaltered. Similarly, if $\mu'_2$, $\mu'_3$, ... denote the values of $\mu_2$, $\mu_3$, ... as deduced from $M'_1$, $M'_2$, $M'_3$, ..., we have, from (29),

$$\left.\begin{aligned}
\text{corrected value of } \mu_2 &= \mu'_2 - \cdot\dot{3} \\
\text{corrected value of } \mu_4 &= \mu'_4 - 2\mu'_1 + \cdot 4\dot{6} \\
\text{corrected value of } \mu_5 &= \mu'_5 - \tfrac{10}{3}\mu'_3
\end{aligned}\right\}. \qquad (31)$$

11. The following are some numerical examples* showing the extent of the inaccuracy due to determining “$\Sigma x$”, “$\Sigma x^2$”, ... by the approximate formulæ (28), instead of by the accurate formula (25).

In the first example the selection is “representative,” the numbers being exactly proportional to the corresponding areas of the figure of frequency. The “most probable” values of $\Sigma x$, $\Sigma x^2$, ..., and of

* The first two of these illustrations are given in the paper referred to in note (*) on p. 362. A small error in one of the values has been corrected.

$\mu_2, \mu_3, \ldots$ are in this case obtained by a simple integration, and the differences between these values and the values given by (28) or (30) are due solely to the omitted term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

which is equal to the second and third terms in (24). In the second example, taken from Kramp's table of values of $\int_t^\infty e^{-t^2}dt$, the selection is approximately representative, the integer nearest to $n_r'$ being taken in each case. The most probable values are found by calculating each of the mean values $(x^m)_r$ separately, and taking the sum $\Sigma n_r (x^m)_r$; the differences between these and the values given by (28) or (30) are due principally to the term

$$\frac{n}{A}\left[h\phi_2(x)\right]_{x=x_0}^{x=x_p},$$

but partly also to the fractional differences $n_r - n_r'$ occurring in the last term of (25). The third example is formed from the second by adding to or subtracting from the number in each class a number comparable with the "probable error" of the total number in the class. It may, therefore, be taken as a typical illustration of the amount of error that would occur in an actual case of random selection. The values of $\mu_2, \mu_4$, and $\mu_8$ for the community selected being 2, 12, and 1680, the "probable errors" in $\mu_2$ and $\mu_4$ for a random selection of 88,622 individuals would be ·0064 and ·0888 respectively. The actual "errors" for the selected values of $n_1, n_2, \ldots, n_p$ are ·0156 and ·1815, so that the example is really an unfavourable one; but the formulæ in (30) give the values of $\mu_2$ and $\mu_4$ within ·0014 and ·0346 respectively, which is as close an approximation as we could expect to get.

In each case the "rough" values of $M_1, M_2, \ldots$ and the corresponding values of $\mu_2, \mu_3, \mu_4$ are given for comparison, so as to show the degree of inaccuracy in the even moments $\mu_2, \mu_4$ which would be caused by omitting the corrections afforded by (28) or (30).

(i.) *Curve of Frequency* $z \propto (x+4\frac{1}{2})^5 (5\frac{1}{2}-x)^6$.

$n = 10^{12}$.

(Representative Selection.)

| | |
|---|---|
| $x = -4\frac{1}{2}$ to $-3\frac{1}{2}$ ..... | 541,231,822 |
| $-3\frac{1}{2}$ to $-2\frac{1}{2}$ ..... | 18,864,047,410 |
| $-2\frac{1}{2}$ to $-1\frac{1}{2}$ ..... | 98,443,459,630 |
| $-1\frac{1}{2}$ to $-\frac{1}{2}$ ..... | 216,942,703,570 |
| $-\frac{1}{2}$ to $\frac{1}{2}$ ..... | 278,001,526,318 |
| $\frac{1}{2}$ to $1\frac{1}{2}$ ..... | 228,994,738,642 |
| $1\frac{1}{2}$ to $2\frac{1}{2}$ ..... | 119,611,449,550 |
| $2\frac{1}{2}$ to $3\frac{1}{2}$ ..... | 34,697,711,410 |
| $3\frac{1}{2}$ to $4\frac{1}{2}$ ..... | 3,852,951,310 |
| $4\frac{1}{2}$ to $5\frac{1}{2}$ ..... | 50,180,338 |
| Total ... ... | 1,000,000,000,000 |

| | $\Sigma x \div 10^6$ | $\Sigma x^2 \div 10^6$ | $\Sigma x^3 \div 10^6$ | $\Sigma x^4 \div 10^6$ | $\Sigma x^5 \div 10^6$ |
|---|---|---|---|---|---|
| Rough value ..... | 115387 | 1871774 | 827127 | 9429592 | 8085022 |
| Corrected value ..... | 115387 | 1788441 | 798281 | 8522872 | 7412577 |
| Most probable value ... | 115385 | 1788462 | 798077 | 8524038 | 7405402 |

| | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|---|---|---|---|
| Rough value ..... | 1·85847 | ·18227 | 9·19204 |
| Corrected value..... | 1·77513 | ·18227 | 8·29197 |
| Most probable value... | 1·77515 | ·18207 | 8·29803 |

(ii.) *Curve of Frequency* $z \propto e^{-\frac{1}{4}(x-2)^2}$.

$n = 88{,}622$.

(Approximately Representative Selection.)

| | | | |
|---|---|---|---|
| $x = -6\frac{1}{2}$ to $-5\frac{1}{2}$ ..... | 2 | $x = \frac{1}{2}$ to $1\frac{1}{2}$ ..... | 21,005 |
| $-5\frac{1}{2}$ to $-4\frac{1}{2}$ ..... | 37 | $1\frac{1}{2}$ to $2\frac{1}{2}$ ..... | 11,259 |
| $-4\frac{1}{2}$ to $-3\frac{1}{2}$ ..... | 355 | $2\frac{1}{2}$ to $3\frac{1}{2}$ ..... | 3,733 |
| $-3\frac{1}{2}$ to $-2\frac{1}{2}$ ..... | 2,098 | $3\frac{1}{2}$ to $4\frac{1}{2}$ ..... | 765 |
| $-2\frac{1}{2}$ to $-1\frac{1}{2}$ ..... | 7,670 | $4\frac{1}{2}$ to $5\frac{1}{2}$ ..... | 97 |
| $-1\frac{1}{2}$ to $-\frac{1}{2}$ ..... | 17,338 | $5\frac{1}{2}$ to $6\frac{1}{2}$ ..... | 8 |
| $-\frac{1}{2}$ to $\frac{1}{2}$ ..... | 24,255 | Total ... | 88,622 |

| | $\Sigma x$ | $\Sigma x^2$ | $\Sigma x^3$ | $\Sigma x^4$ | $\Sigma x^5$ |
|---|---|---|---|---|---|
| Rough value ..... | 17726 | 188168 | 111560 | 1196948 | 1169816 |
| Corrected value..... | 17726 | 180783 | $107128\frac{1}{2}$ | 1105449 | 1079434 |
| Most probable value..... | 17726 | 180783 | 107123 | 1105535 | 1079202 |

| | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|---|---|---|---|
| Rough value ..... | 2·08325 | ·00074 | 13·00394 |
| Corrected value..... | 1·99992 | ·00074 | 11·99148 |
| Most probable value... | 1·99992 | ·00068 | 11·99250 |

(iii.) *Curve of Frequency* $z \propto e^{-\frac{1}{2}(x-2)^2}$.

$n = 88{,}622.$

(Random Selection.)

| | | | |
|---|---|---|---|
| $x = -6\frac{1}{2}$ to $-5\frac{1}{2}$ ..... | 3 | $x = \frac{1}{2}$ to $1\frac{1}{2}$ ..... | 20,965 |
| $-5\frac{1}{2}$ to $-4\frac{1}{2}$ ..... | 31 | $1\frac{1}{2}$ to $2\frac{1}{2}$ ..... | 11,289 |
| $-4\frac{1}{2}$ to $-3\frac{1}{2}$ ..... | 374 | $2\frac{1}{2}$ to $3\frac{1}{2}$ ..... | 3,783 |
| $-3\frac{1}{2}$ to $-2\frac{1}{2}$ ..... | 2,153 | $3\frac{1}{2}$ to $4\frac{1}{2}$ ..... | 773 |
| $-2\frac{1}{2}$ to $-1\frac{1}{2}$ ..... | 7,657 | $4\frac{1}{2}$ to $5\frac{1}{2}$ ..... | 102 |
| $-1\frac{1}{2}$ to $-\frac{1}{2}$ ..... | 17,326 | $5\frac{1}{2}$ to $6\frac{1}{2}$ ..... | 12 |
| $-\frac{1}{2}$ to $\frac{1}{2}$ ..... | 24,154 | Total ... | 88,622 |

| | $\Sigma x$ | $\Sigma x^2$ | $\Sigma x^3$ | $\Sigma x^4$ | $\Sigma x^5$ |
|---|---|---|---|---|---|
| Rough value ..... | 17798 | 189716 | 113060 | 1218440 | 1216388 |
| Corrected value..... | 17798 | 182331 | $108610\frac{1}{2}$ | 1126167 | 1124767 |
| Most probable value..... | 17795 | 182202 | 108288 | 1122875 | 1116061 |

| | $\mu_2$ | $\mu_3$ | $\mu_4$ |
|---|---|---|---|
| Rough value ..... | 2·10040 | ·00198 | 13·23706 |
| Corrected value ..... | 2·01707 | ·00198 | 12·21603 |
| Most probable value ... | 2·01563 | −·00038 | 12·18146 |

## III. Extension to Two or More Variables.

12. The method of § 4 can be extended to any number of variables. The expression

$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}\cdots\frac{\sinh D_1}{D_1}\cdot\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\cdot\frac{D_2'}{\sinh D_2'}\cdots,$$

can be expanded so as to contain only positive integral powers (including the power 0) of $D_1, D_2, D_1', D_2', \ldots$; and it vanishes when $D_1+D_2, D_1'+D_2', \ldots$ are simultaneously equal to zero. Thus we see

that
$$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x, y, \ldots)\,\phi(x, y, \ldots)\,dx\,dy \ldots \tag{32}$$
is equal to
$$\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x, y, \ldots)\,dx\,dy \ldots \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\frac{P_2}{4!}(\tfrac{1}{2}h)^4\frac{d^4}{d\xi^4}-\ldots\right\}$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\frac{P_2}{4!}(\tfrac{1}{2}k)^4\frac{d^4}{d\eta^4}-\ldots\right\}\ldots\phi(\xi, \eta, \ldots)+R, \tag{33}$$
where $R$ consists of terms involving the boundary values of $f(x, y, \ldots)$, $\phi(x, y, \ldots)$, and their partial differential coefficients with regard to $x, y, \ldots$. Hence, if the values of $x, y, \ldots$ range from $x = x_0$ to $x = x_0+ph$, from $y = y_0$ to $y = y_0+qk$, ..., and if $f(x, y, \ldots)$ and its first few differential coefficients are negligible for the extreme values of *each* of the variables $x, y, \ldots$, we have, approximately,
$$\Sigma\Sigma\ldots\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k}\ldots f(x, y, \ldots)\,\phi(x, y, \ldots)\,dx\,dy\ldots$$
$$=\Sigma\Sigma\ldots\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k}\ldots f(x, y, \ldots)\,dx\,dy\ldots$$
$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\ldots\right\}\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\ldots\right\}\ldots\phi(\xi, \eta, \ldots), \tag{34}$$
the summations being made for the $p\times q\times\ldots$ terms corresponding to the different values of $\xi, \eta, \ldots$.

It will be sufficient to consider the case of two variables. Let $D_1$ and $D_2$ denote $\frac{1}{2}h\frac{d}{dx}$ operating on $f(x, y)$ and on $\phi(x, y)$ respectively, and let $D_1'$ and $D_2'$ have a similar meaning for $\frac{1}{2}k\frac{d}{dy}$. Then
$$\frac{\sinh(D_1+D_2)}{D_1+D_2}\cdot\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}$$
$$=\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\cdot\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}$$
$$+\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\right\}$$
$$+\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\right\}$$
$$+\left\{\frac{\sinh(D_1+D_2)}{D_1+D_2}-\frac{\sinh D_1}{D_1}\frac{D_2}{\sinh D_2}\right\}$$
$$\times\left\{\frac{\sinh(D_1'+D_2')}{D_1'+D_2'}-\frac{\sinh D_1'}{D_1'}\frac{D_2'}{\sinh D_2'}\right\}; \tag{35}$$

and therefore $$\frac{1}{hk}\int_{x_0}^{x_p}\int_{y_0}^{y_q} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$= \Sigma\Sigma \frac{1}{hk}\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy \left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}$$

$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta)$$

$$+\Sigma\frac{1}{h}\left[\chi(D_1',D_2')\cdot\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h} f(x,y)\,dx\cdot\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}\phi(\xi,y)\right]_{y=y_0}^{y=y_q}$$

$$+\Sigma\frac{1}{k}\left[\chi(D_1,D_2)\cdot\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dy\cdot\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(x,\eta)\right]_{x=x_0}^{x=x_p}$$

$$+\Phi(x_p,y_q)-\Phi(x_0,y_q)-\Phi(x_p,y_0)+\Phi(x_0,y_0), \tag{36}$$

where

$$\chi(D_1,D_2)\equiv\tfrac{1}{2}\frac{D_2}{D_1}\left\{\left(\coth\overline{D_1+D_2}-\frac{1}{D_1+D_2}\right)-\left(\coth D_2-\frac{1}{D_2}\right)\right\}, \tag{37}$$

$$\Phi(x,y)\equiv\chi(D_1,D_2)\,\chi(D_1',D_2')\,f(x,y)\,\phi(x,y). \tag{38}$$

The signs of summation in the first term denote, of course, that the expression is to be summed for each value of $\xi$ with each value of $\eta$; while in the second the summation is for all values of $\xi$, and in the third it is for all values of $\eta$. These terms, which denote values along the edges of the area of integration, with the four concluding terms, which denote values at the corners, can be expanded as in (19); or they can be expressed in terms of differences, as in (5A) and (24). They are negligible when $f(x,y)$ and its first few differential coefficients $\frac{df}{dx}$, $\frac{df}{dy}$, ... vanish for all values of $x$ when $y=y_0$ or $y_q$, and for all values of $y$ when $x=x_0$ or $x_p$; and in these cases we have, approximately,

$$\int_{x_0}^{x_p}\int_{y_0}^{y_p} f(x,y)\,\phi(x,y)\,dx\,dy$$

$$= \Sigma\Sigma\int_{\xi-\frac{1}{2}h}^{\xi+\frac{1}{2}h}\int_{\eta-\frac{1}{2}k}^{\eta+\frac{1}{2}k} f(x,y)\,dx\,dy\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}h)^2\frac{d^2}{d\xi^2}+\dots\right\}$$

$$\times\left\{1-\frac{P_1}{2!}(\tfrac{1}{2}k)^2\frac{d^2}{d\eta^2}+\dots\right\}\phi(\xi,\eta). \tag{39}$$

13. Now let the measurements of $x$ and of $y$ for $n$ individuals be given by a table of double entry, showing for values of $r$ from 1 to $p$, and for values of $s$ from 1 to $q$, the number of individuals for which $x$ lies between $x_0+(r-1)(x_p-x_0)/p$ and $x_0+r(x_p-x_0)/p$, and $y$ between $y_0+(s-1)(y_q-y_0)/q$ and $y_0+s(y_q-y_0)/q$. Let "$\Sigma x^m y^t$" denote the most probable value of $\Sigma x^m y^t$ for the $n$ individuals (or $n$ times the most probable mean value of $x^m y^t$ for the original community), and let $S_{mt}$ denote the rough value of $\Sigma x^m y^t$ obtained by taking the values of $x$ and of $y$ for all the individuals in each sub-class to be the arithmetic means of the values of $x$ and of $y$ respectively which determine the sub-class. Then, if $M_m (= S_{m0})$ and $N_t (= S_{0t})$ denote the "rough" values of $\Sigma x^m$ and of $\Sigma y^t$ obtained in this way, we have, approximately,

$$\left.\begin{aligned}
&\text{“}\Sigma xy\text{”} = S_{11}\text{*}\\
&\text{“}\Sigma x^2y\text{”} = S_{21}-\tfrac{1}{12}N_1h^2\\
&\text{“}\Sigma xy^2\text{”} = S_{12}-\tfrac{1}{12}M_1k^2\\
&\text{“}\Sigma x^3y\text{”} = S_{31}-\tfrac{1}{4}S_{11}h^2\\
&\text{“}\Sigma x^2y^2\text{”} = S_{22}-\tfrac{1}{12}M_2k^2-\tfrac{1}{12}N_2h^2+\tfrac{1}{144}nh^2k^2\\
&\text{“}\Sigma xy^3\text{”} = S_{13}-\tfrac{1}{4}S_{11}k^2\\
&\text{“}\Sigma x^4y\text{”} = S_{41}-\tfrac{1}{2}S_{21}h^2+\tfrac{7}{240}N_1h^4\\
&\text{“}\Sigma x^3y^2\text{”} = S_{32}-\tfrac{1}{4}S_{12}h^2-\tfrac{1}{12}M_3k^2+\tfrac{1}{48}M_1h^2k^2\\
&\text{“}\Sigma x^2y^3\text{”} = S_{23}-\tfrac{1}{4}S_{21}k^2-\tfrac{1}{12}N_3h^2+\tfrac{1}{48}N_1h^2k^2\\
&\text{“}\Sigma xy^4\text{”} = S_{14}-\tfrac{1}{2}S_{12}k^2+\tfrac{7}{240}M_1k^4\\
&\qquad\&c.
\end{aligned}\right\} \qquad (40)$$

## IV. The Spurious Curve of Frequency.

14. Let $\dfrac{1}{h}\displaystyle\int_{x-\frac{1}{2}h}^{x+\frac{1}{2}h} f(x)\,dx$ be denoted by $f_1(x)$. If we call $z=f(x)$ the principal curve, then $z=f_1(x)$ may be called the derived curve.

* There is an apparent contradiction in this and similar formulæ when the value of $y$ for each individual is equal or very nearly equal (or proportional) to that of $x$, *i.e.*, when the distributions of $x$ and of $y$ are very closely correlated; for the formula appears to give $M_2$ instead of $M_2-\frac{1}{12}h^2$. But, if $y$ were always equal to $x$, there would be a discontinuity all along the line $y=x$; and therefore the formula, which is based on the hypothesis of continuity, would not apply at all. When $y$ is only very nearly equal to $x$ it will be seen that the sets of planes corresponding to consecutive values of $x$ and $y$ cut the solid of frequency in such a way that the volumes, which in forming the values of $S_1$ are concentrated at points outside the line $y=x$, are appreciable.

The curves are related in such a way that the ordinate to the latter curve at any point $x = \xi$ is proportional to the area of the former curve comprised between the ordinates at $x = \xi - \frac{1}{2}h$ and $x = \xi + \frac{1}{2}h$. If these ordinates are $MP$ and $M'P'$, and if $h$ is relatively small, the extremity of the ordinate to the derived curve will lie inside the area formed by the arc $PP'$ and the chord $PP'$. Thus the derived curve lies inside the principal curve where the latter is convex, and outside it where it is concave; any point of intersection of the two curves being near a point of inflexion of the principal curve.

If $z = f(x)$ ranges from $x = x_0$ to $x = x_p$, then $z = f_1(x)$ only ranges from $x = x_0 + \frac{1}{2}h$ to $x = x_p - \frac{1}{2}h$. But, if $z = f(x)$ touches the base ($z = 0$) at its extremities, it may be considered to coincide with it beyond this range: and $z = f_1(x)$ may thus be regarded as extending from $x = x_0 - \frac{1}{2}h$ to $x = x_p + \frac{1}{2}h$; the curve, like the principal curve, touching the base at its extremities. We shall only consider this class of cases.

Let the areas of the principal curve between the successive ordinates corresponding to $x = x_0$, $x = x_0 + h$, $x = x_0 + 2h$, ..., $x = x_0 + ph$, be $A_1$, $A_2$, ..., $A_p$. Then the ordinates of the derived curve at the points $x = x_0 - \frac{1}{2}h$, $x = x_0 + \frac{1}{2}h$, ..., $x = x_p - \frac{1}{2}h$, $x = x_p + \frac{1}{2}h$ are $0$, $A_1/h$, $A_2/h$, ..., $A_p/h$, $0$. Hence, from (2B), we have, approximately,

$$\text{area of derived curve} = \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\, dx$$

$$= h\left\{\frac{A_1}{h} + \frac{A_2}{h} + \ldots + \frac{A_p}{h}\right\}$$

$$= \text{area of principal curve};$$

$$\text{first moment of derived curve} = \int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\, x\, dx$$

$$= h\left\{\frac{A_1}{h}\xi_1 + \frac{A_2}{h}\xi_2 + \ldots + \frac{A_p}{h}\xi_p\right\}$$

$$= \text{first moment of principal curve, by (17).}$$

Thus the two curves have the same area and the same mean value of $x$. From (2B) we have as the general formula relating to the derived curve, when $A_1$, $A_2$, ..., $A_p$ are given,

$$\int_{x_0 - \frac{1}{2}h}^{x_p + \frac{1}{2}h} f_1(x)\, \phi(x)\, dx = \sum_{r=1}^{r=p} A_r \phi(x_r). \qquad (41)$$

15. When the result of a random selection of $n$ individuals is to give numbers $n_1, n_2, \ldots, n_p$ in the successive classes corresponding to equidistant divisions of the scale of measurement, these numbers are, except for the "errors" due to random selection, proportional to the corresponding areas of the figure of frequency in the original community. In order, therefore, to represent the data graphically, we should draw a curve such that the areas between the successive ordinates at the $p+1$ points of the scale are proportional to $n_1, n_2, \ldots, n_p$. The calculation of the ordinates of this curve, which may be regarded as the true curve of frequency for the $n$ individuals, involves troublesome interpolations; and, to avoid this, statisticians often draw a curve whose ordinates at the middle points of the divisions of the scale are proportional to $n_1, n_2, \ldots, n_p$. This curve may be called the *spurious curve of frequency*. It bears the same relation to the true curve of frequency that the derived curve of § 14 bears to the principal curve. Thus it has the same area as the true curve, and its central ordinate starts from the same point on the base, but it is flattened down so as to be inside the true curve at or about the mean and outside it at the extremities; and therefore it has a greater mean square of deviation.

If we deal with the spurious curve instead of with the true curve, we must apply the formula (41) instead of the corresponding formula derived from (17). Hence, in order to find the values of $\mu_2$, $\mu_3$, &c., for the spurious curve, we must use the "rough" values $M_1$, $M_2$, $M_3$, ..., without the introduction of the corrections given by (28) or (30).

The divergence of the spurious curve from the true curve is greater or less according as $h$ is greater or less. The same facts may therefore be represented by any number of different spurious curves, according to the unit of measurement we adopt. Thus example (A) of § 7 would give as ordinates of the spurious curve, when the unit of measurement is 10 lbs.:—

| Weight. | Ordinate. |
|---|---|
| 115 lbs. | 133 |
| 145 lbs. | 1,240 |
| 175 lbs. | 492 |
| 205 lbs. | 75 |
| 235 lbs. | 10 |

but, if the unit had been 30 lbs., the ordinates (reduced in the ratio

of 1 : 3) would have been

| Weight. | Ordinate. |
|---|---|
| 115 lbs. | 166 |
| 145 lbs. | 1,003 |
| 175 lbs. | 559 |
| 205 lbs. | 104 |
| 235 lbs. | 17 |

which would have given a very different curve. Moreover, the results obtained by using the spurious curve are useless if we require an accurate test of any particular hypothesis. The arguments which lead us to expect to find a curve of a particular kind [*e.g.*, in Prof. Karl Pearson's recent researches, a curve whose equation is of the form $z \propto (x-x_0)^a (x_p-x)^\beta$] relate to the curve of frequency itself, not to the spurious curve; and, by attempting to fit a curve of this kind to the spurious curve, we shall obtain incorrect results.

## Appendix.

### *Moments of a Polygon.*

The question of the proper formula to be used for $\Sigma x^m$ has been incidentally discussed by Prof. Karl Pearson in a recent memoir.* His method is to draw ordinates proportional to the numbers given by the observations, and to join the tops of these ordinates by straight lines, so as to form a polygon. It will be seen that this polygon lies inside the spurious curve where the latter is convex, and outside it where it is concave; and that, when $z = f(x)$ has close contact with the base at its extremities, the area of the polygon is approximately equal to that of the curve, and the first moments of the two areas are also approximately equal. The "mean square of deviation" of the polygon is, therefore, greater than that of the spurious curve, which, again, is greater than that of the true curve.

It is interesting, though the result is of no practical value, to obtain the moments of such a polygon in terms of the areas, instead of in terms of the ordinates. Let ordinates $z_0, z_1, \ldots, z_p$ be drawn at points $x = x_0$, $x = x_1$, $\ldots$, $x = x_p$, the common difference of abscissa being $h$, and let the tops of the ordinates be joined so as to form a polygon. This polygon consists of a series of trapezia of areas $A_1, A_2, \ldots, A_p$, where

$$A_r = \tfrac{1}{2}h\,(z_{r-1}+z_r).$$

* *Phil. Trans.*, Vol. CLXXXVI. (1895), A, p. 349.

The $m^{\text{th}}$ moment of the polygon about $x=0$ is

$$\sum_{r=1}^{r=p}\int_{x_{r-1}}^{x_r} zx^m\,dx.$$

Let this be denoted by $P_m$. Then, as Prof. Pearson shows,

$$\int_{x_{r-1}}^{x_r} zx^m\,dx$$

$$=-\frac{1}{(m+1)(m+2)}\frac{1}{h}(z_r-z_{r-1})(x_r^{m+2}-x_{r-1}^{m+2})+\frac{1}{m+2}(z_r x_r^{m+1}-z_{r-1}x_{r-1}^{m+1}) \tag{42}$$

and this gives

$$\sum_{r=1}^{r=p}\int_{x_{r-1}}^{x_r} zx^m\,dx$$

$$=2\sum_{r=0}^{r=p}\left\{z_r\left(\frac{m!}{2!\,m!}x_r^m h+\frac{m!}{4!\,(m-2)!}x_r^{m-2}h^3+\ldots\right)\right\}$$

$$-\left[z\left(\frac{m!}{3!\,(m-1)!}x^{m-1}h^2+\frac{m!}{5!\,(m-3)!}x^{m-3}h^4+\ldots\right)\right]_{x=x_0}^{x=x_p}, \tag{43}$$

the expression under the sign of summation being only taken once for the extreme values, instead of twice. The term given by the square brackets vanishes when $z_0$ and $z_p$ are both zero.

To express $P_m$ in terms of $A_1, A_2, \ldots, A_p$, we have, from (42),

$$\int_{x_{r-1}}^{x_r} zx^m\,dx=-\frac{1}{(m+1)(m+2)}\frac{2}{h}(z_r x_r^{m+2}+z_{r-1}x_{r-1}^{m+2})$$

$$+\frac{1}{(m+1)(m+2)}\frac{1}{h}(z_r+z_{r-1})(x_r^{m+2}+x_{r-1}^{m+2})$$

$$+\frac{1}{m+2}(z_r x_r^{m+1}-z_{r-1}x_{r-1}^{m+1}). \tag{44}$$

To adapt the first term of the right-hand side, we use the formula*

$$(y+1)^{m+2}=y^{m+2}-E_1\frac{(m+2)!}{2!\,m!}y^m+E_2\frac{(m+2)!}{4!\,(m-2)!}y^{m-2}-\ldots$$

$$+T_1\frac{(m+2)!}{1!\,(m+1)!}(y+1)^{m+1}-T_2\frac{(m+2)!}{3!\,(m-1)!}(y+1)^{m-1}+\ldots, \tag{45}$$

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* "On the Relations between Bernoulli's and Euler's Numbers," *Quarterly Journal of Mathematics*, 1898, p. 18.

where $E_1, E_2, \ldots, T_1, T_2, \ldots$ are the coefficients in the expansions

$$\left.\begin{aligned} \operatorname{sech}\theta &= 1 - E_1\frac{\theta^2}{2!} + E_2\frac{\theta^4}{4!} - \ldots \\ \tanh\theta &= T_1\frac{\theta}{1!} - T_2\frac{\theta^3}{3!} + \ldots \end{aligned}\right\}. \tag{46}$$

If $$\xi_r = \tfrac{1}{2}(x_{r-1} + x_r),$$

this gives

$$x_r^{m+2} = \xi_r^{m+2} - E_1\frac{(m+2)!}{2!\,m!}(\tfrac{1}{2}h)^2\xi_r^m + E_2\frac{(m+2)!}{4!\,(m-2)!}(\tfrac{1}{2}h)^4\xi_r^{m-2} - \ldots$$
$$+ T_1\frac{(m+2)!}{1!\,(m+1)!}(\tfrac{1}{2}h)\,x_r^{m+1} - T_2\frac{(m+2)!}{3!\,(m-1)!}(\tfrac{1}{2}h)^3 x_r^{m-1} + \ldots,$$

with a similar equation obtained by changing $x_r$ into $x_{r-1}$ and $\frac{1}{2}h$ into $-\frac{1}{2}h$. Multiplying these equations by $z_r$ and $z_{r-1}$ respectively, and adding, substituting in (44), and adding the corresponding expressions for the other trapezia, we find, the summations being from $r = 1$ to $r = p$,

$$\Sigma\int_{x_{r-1}}^{x_r} zx^m\,dx$$
$$= \frac{E_1+1}{1.2}\Sigma A_r\xi_r^m - \frac{E_2-1}{3.4}\,\frac{m!}{2!\,(m-2)!}(\tfrac{1}{2}h)^2\Sigma A_r\xi_r^{m-2} + \ldots$$
$$+\left[\frac{T_2}{2.3}\,\frac{m!}{1!\,(m-1)!}(\tfrac{1}{2}h)^2 zx^{m-1} - \frac{T_3}{4.5}\,\frac{m!}{3!\,(m-3)!}(\tfrac{1}{2}h)^4 zx^{m-3} + \ldots\right]_{x=x_0}^{x=x_p}, \tag{47}$$

so that, when $z_0 = 0$ and $z_p = 0$,

$$P_m = \Sigma A_r\xi_r^m - \tfrac{1}{12}\frac{m!}{2!\,(m-2)!}\Sigma A_r\xi_r^{m-2}h^2 + \tfrac{31}{240}\frac{m!}{4!\,(m-4)!}\Sigma A_r\xi_r^{m-4}h^4 - \ldots, \tag{48}$$

which agrees with (27) in the second term, but not in subsequent terms.

# *On Aurifeuillians.*

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Chapter I., Aurifeuillians.

Chapter II., Bin-Aurifeuillians. Chapter III., Trin-Aurifeuillians.

*Notation.*—All symbols used in this paper are *integer* numbers: $p$ denotes a *prime*, $\omega$ an *odd* number, $\epsilon$ an *even* number; $\lambda$, $\mu$, $\nu$ denote *odd* numbers (except in Art. **17, 18, 31, 32,**); A.P.F. denotes *algebraic prime factor*.

## Chapter I., Aurifeuillians.

**1.** *Algebraic Prime Factors* (A.P.F.).

Let $n$ be an *odd* number; $n_1$, $n_2$, $n_3$, &c., *different odd primes*.

Let $\phi(n, U, V)$, or (for shortness), $\phi(n) = (U^n \pm V^n)$,* (1)

so that $\phi(1, U, V)$, or shortly, $\phi(1) = (U \pm V)$.* (1*a*)

Here $\phi(n)$ may be used, for shortness, whenever the *form* of the functions $U$, $V$ is not in question.

Then the *algebraic prime factor*, or, for shortness, A.P.F., of the function $\phi(n, U, V)$, or of $\varphi(n)$, is defined by

$$\mathfrak{F}\phi(n_1) = \frac{\phi(n_1)}{\phi(1)}, \quad \mathfrak{F}\phi(n_1 n_2) = \frac{\phi(n_1 n_2)\,\phi(1)}{\varphi(n_1)\,\varphi(n_2)}, \tag{2a}$$

$$\mathfrak{F}\phi(n_1 n_2 n_3) = \frac{\phi(n_1 n_2 n_3)\,\phi(n_1)\,\phi(n_2)\,\phi(n_3)}{\phi(1)\,\phi(n_2 n_3)\,\varphi(n_3 n_1)\,\phi(n_1 n_2)}, \text{ &c.,} \tag{2b}$$

and the law of formation is obvious.† When $n_1$, $n_2$, $n_3$, &c., are *not prime*, the function $\mathfrak{F}$ above formed is simply *an* algebraic factor of $\phi(n)$, but not the algebraic *prime* factor thereof.

As above defined, the function $\mathfrak{F}\phi(n, U, V)$ is *independent of the forms* of $U$, $V$. When, however, $U = u^\nu$, $V = v^\nu$, and $\nu$ is *odd*, a further reduction may be made, and the *Reduced algebraic prime factor* $\mathfrak{f}$ of the function $(u^{n\nu} \pm v^{n\nu})$, *i.e.*, the quantity $\mathfrak{f}\phi(n\nu, u, v)$, may be found

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* There is *no ambiguity* of sign; the sign used in $\phi(n)$ being used *in all its factors* including the A.P.F., *i.e.* in $\mathfrak{F}\phi(n)$.

† In fact $\phi(n, U, V)$ is the Norm of the binomial complex integer $(U - V\rho)$, where $\rho$ is a *proper* $n^{\text{th}}$ root of 1.

by a similar procedure. Of course, this quantity is of a *lower order* than the algebraic prime factor $\mathfrak{F}\phi(n, u^\nu, v^\nu)$ previously found. In this way a function $(u^{n\nu} \pm v^{n\nu})$ with *composite* index $(n\nu)$ may be said to have algebraic prime factors of *different orders*, viz.,

$$\mathfrak{F}\phi(n\nu, u, v), \quad \mathfrak{F}\phi(n, u^\nu, v^\nu), \quad \mathfrak{F}\phi(\nu, u^n, v^n),$$

according as it is considered as a function of $(u, v)$, or of $(u^\nu, v^\nu)$, or of $(u^n, v^n)$.

2. *Aurifeuillians.*—It is proposed to apply the name* *Aurifeuillian* of order $2n$ or $n$ respectively to the following functions (both algebraic prime factors) :—

$$N = \mathfrak{F}(X^{2n} + Y^{2n}), \qquad \text{where } 2nXY = z^2 \text{ (a square)}, \tag{3}$$

$$\left.\begin{aligned} N &= \mathfrak{F}(X^n + \iota^{n+1} Y^n), \text{ where } nXY = z^2 \text{ (a square)} \\ &\text{where } n \text{ is } \textit{odd} \text{ and } \iota = \sqrt{-1}, \text{ so that } \iota^{n+1} = \pm 1 \end{aligned}\right\}. \tag{4}$$

[Here $n$ is supposed to contain *no square† factors;* and $X$, $Y$ are supposed relatively † prime.]

These functions will be styled, for shortness, *Bin-*, *Trin-*, *Quint-*, &c., Aurifeuillians when the *order* $= 2, 3, 5,$ &c.

The condition $2nXY = z^2$, or $nXY = z^2$, taken together with the working condition ($X$ *prime* to $Y$) involves *one* of the following groups‡ of conditions :—

$X = x^2, \quad Y = 2ny^2$; $x$ *odd*, and *prime* to $n$, $y$, (3*a*)

$X = n_1x^2, \quad Y = 2n_2y^2$; $n = n_1n_2$§; $x$ *odd;* $n_1$ and $x$ *prime* to $n_2$ and $y$, (3*b*)

$X = x^2, \quad Y = ny^2$; $x$ prime to $n$, $y$, (4*a*)

$X = n_1x^2, \quad Y = n_2y^2$; $n = n_1n_2$§; $n_1$ and $x$ *prime* to $n_2$ and $y$. (4*b*)

Thus the most general forms are

$$N = \mathfrak{F}\left\{(x^2)^{2n} + (2ny^2)^{2n}\right\}, \tag{5a}$$

$$N = \mathfrak{F}\left\{(n_1x^2)^{2n} + (2n_2y^2)^{2n}\right\}, \quad \text{where } n_1n_2 = n, \tag{5b}$$

$$N = \mathfrak{F}\left\{(x^2)^n + \iota^{n+1}(ny^2)^n\right\}, \tag{6a}$$

$$N = \mathfrak{F}\left\{(n_1x^2)^n + \iota^{n+1}(n_2y^2)^n\right\}, \quad \text{where } n_1n_2 = n, \tag{6b}$$

---

* See Art. **3** for explanation of this term.

† These conditions greatly simplify the details, without any real loss of generality.

‡ Hereby $N$ will be an *odd* number, free from unnecessary factors.

§ The working condition as to $n$ above involves $n_1$, $n_2$ both free from square factors.

The forms (5*a*), (6*a*) are only simpler forms of the more general forms (5*b*), (6*b*); but, being the more important, are worth separate statement.

From (5*b*), (6*b*) it is seen that, when $n$ is a *composite* odd number, there arise *different species* of Aurifeuillians all of the *same order* ($2n$ or $n$), the number of species being the same as the number of ways in which $2n$ or $n$ can be resolved into a pair of (mutually prime) factors.

*Ex.* $\dfrac{x^{12}+6^6y^{12}}{x^4+6^2y^4}$ and $\dfrac{3^6x^{12}+2^6y^{12}}{3^2x^4+2^2y^4}$ are Sext-Aurifeuillians;

$\dfrac{x^{30}+15^{15}y^{30}}{x^{10}+15^5y^{10}}\ \dfrac{x^2+15y^2}{x^6+15^3y^6}$ and $\dfrac{3^{15}x^{30}+5^{15}y^{30}}{3^5x^{10}+5^5y^{10}}\ \dfrac{3x^2+5y^2}{3^3x^6+5^3y^6}$ are Quindec-Aurifeuillians.

In the abbreviated notation of Art. **1**, the Aurifeuillians (3), (4) would be written

$$N=\mathfrak{F}(n,\ X^2,\ Y^2),\ \text{or simply}=\mathfrak{F}(n),\ \text{when}\ 2nXY=\square, \quad (3,\ \text{i})$$

$$\left.\begin{array}{l} N=\mathfrak{F}(n,\ X,\ Y),\quad \text{or simply}=\mathfrak{F}(n),\ \text{when}\ \ nXY=\square \\ \text{the sign } (\pm) \text{ between } x^n,\ y^n \text{ being fixed, so that } \epsilon^{n+1}=\pm 1 \end{array}\right\}. \quad (4,\ \text{i})$$

**3.** *Aurifeuillian Factors* $(L, M)$.—The interest of these numbers lies chiefly in the fact of their being expressible* *algebraically* as the *difference of two squares* (say $P^2-Q^2$), so that they are *algebraically resolvable* into two† factors $(P\pm Q)$. These factors will be styled, for shortness, *Aurifeuillian Factors*, and will be denoted by $L$, $M$; so that $N=L.M$: their salient property is

$$\text{If}\ L=f(x,-y),\ \text{then}\ M=f(x,+y);\ \text{and}\ L<M. \qquad (7)$$

**4.** *Ant-Aurifeuillians.*—Changing $Y$ into $-Y$ in (4), and therewith

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* This property was discovered by M. Aurifeuille, of Toulouse. A table of the values of $P$ and $Q\div\sqrt{2nXY}$ or $Q\div\sqrt{nXY}$ is given in Ed. Lucas's paper, "Sur la Série récurrente de Fermat," Rome, 1879, p. 6, for the following cases:—

$n=3,\ 5,\ 7,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21,\ 23,\ 29,\ 31,\ 33$; $2n=2,\ 6,\ 10,\ 14,\ 22,\ 26,\ 30,\ 34$.

From these the factors $L$, $M$ can be at once found, as $(P\mp Q)$. The values of $L$, $M$ are also given in recent papers by Mr. C. E. Bickmore, "On the Numerical Factors of $(a^n-1)$," in the *Messenger of Mathematics*, Vol. xxv., 1896, pp. 1–44, and Vol. xxvi., 1897, pp. 1–37, for the following cases:—

$n=3,\ 5,\ 7,\ 11,\ 15$; $2n=2,\ 6,\ 10,\ 14,\ 30$.

† By this property, together with the use of "special divisors," it is possible to factorise many very large numbers *completely* (*i.e.*, into their *prime* factors). Many examples will be given in the sequel of numbers of over 20 figures; and some of 30, 40, 50, and (over) 60 figures completely factorised, see Tables, Arts. **16, 18, 31, 32.**

$y^2$ into $-y^2$ in $(4a, b)$, $(6a, b)$ a new number* $(N')$ results, which may be styled *Ant-Aurifeuillian*, from its being *algebraically* separable into a *sum of two squares* (say $P^2+Q^2$), in the same way as the Aurifeuillian itself is separable into a difference of squares, the expressions being in fact conjugate. Here

$$N' = \mathfrak{F}\left\{(x^2)^n \quad -\iota^{n+1}(ny^2)^n\right\}, \tag{8a}$$

$$N' = \mathfrak{F}\left\{(n_1x^2)^n - \iota^{n+1}(n_2y^2)^n\right\}. \tag{8b}$$

5. *Quadratic Forms.*—In common with the A.P.F. of numbers of form $(X^n \pm Y^n)$ in general, all Aurifeuillians can be expressed *algebraically* in various quadratic forms of the type† $(n_1\Xi^2+\iota^{n+1}n_2H^2)$, where $n_1n_2 = n$, and generally in more than one way in each type defined by a particular factorization $n_1n_2 = n$ ($n_1$ prime to $n_2$).

Moreover the Aurifeuillian Factors $L$, $M$ can themselves also be expressed *algebraically*‡ in quadratic forms *isomorphic* with each other, and *conformal* with those of $N$.

Thus, with Aurifeuillians of *odd* order $(n)$,

if $$L = n_1\left\{\phi_1(x,-y)\right\}^2+\iota^{n+1}n_2\left\{\phi_2(x,-y)\right\}^2, \tag{9a}$$

then, by (7), $$M = n_1\left\{\phi_1(x,y)\right\}^2 \quad +\iota^{n+1}n_2\left\{\phi_2(x,y)\right\}^2. \tag{9b}$$

Hence, by *conformal*§ *multiplication*,

$$N = \Phi_1^2+\iota^{n+1}n\Phi_2^2, \quad (\text{since } n = n_1n_2), \tag{10}$$

where $$\Phi_1 = n_1\phi_1(x,-y)\,\phi_1(x,y) \pm n_2\phi_2(x,-y)\,\phi_2(x,y), \tag{10a}$$

$$\Phi_2 = \quad \phi_1(x,-y)\,\phi_2(x,y) \mp \quad \phi_2(x,-y)\,\phi_1(x,y). \tag{10b}$$

(the signs being *unlike*, or *like* in $\Phi_1$, $\Phi_2$ according as $\iota^{n+1}$ is + or −).

Results $(9a, b)$, $(10a, b)$ apply, with slight modifications, (*e.g.*, omitting $\iota^{n+1}$, &c.,) to Aurifeuillians of *even* order $(2n)$.

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* This modification of the Aurifeuillian numbers (1) is due to Mr. Bickmore; see his papers quoted above, *passim*. Note that every Aurifeuillian of *even* order is obviously expressible in form $(a^2+b^2)$ (being itself the A.P.F. of a sum of two squares).

† When $n > 10$, the form $\frac{1}{4}(n_1\Xi^2+\iota^{n+1}n_2H^2)$ has sometimes to be used instead of that given.

‡ This has been shown in the two papers by Mr. Bickmore, quoted above, for most of the values of $n$ used by him; and, from the manner of composition of these numbers from their complex factors, it is seen to be generally true. This property greatly enhances the interest and utility of the Aurifeuillian numbers.

§ *I.e.*, multiplication of two *conformals* $(n_1\phi_1^2+n_2\phi_2^2)\cdot(n_1\psi_1^2+n_2\psi_2^2)$ to form the *conformal* product (10). See the author's paper "On the Connexion of Quadratic Forms," in *Proc. Lond. Math. Soc.*, Vol. XXVIII, 1897, Art. 6–13.

Hence every distinct partition of the factors $L$, $M$, defined by a particular set of values of $n_1$, $n_2$, $\phi_1$, $\phi_3$, yields *two* isomorphic partitions of $N$, in consequence of the double sign $(\pm)$ entering into $\phi_1$, $\phi_2$.

But, further, each distinct pair of values of $n_1$, $n_2$ yields in many cases either two or three *alternative* forms of the functions $\phi_1$, $\phi_2$, and therefore also of the pair $L$, $M$, corresponding to the sub-cases when *one* of $x$, $y$ is even (and the other odd), or *both* of $x$, $y$ odd; and each such alternative pair of $L$, $M$ yields—by *conformal multiplication* as above $(10, 10a, b)$—*two* forms of the product $(N)$ each conformal with $L$, $M$. This may be denoted as follows, where similar forms of $L$, $M$, $N$ are denoted by like accents,

$L'$, $M'$ yielding $N'$ and $N°$, when $y$ is *even*, (11*a*)
$L''$, $M''$ yielding $N''$ and $N°$, when $x$ is *even*, (11*b*)
$L'''$, $M'''$ yielding $N'''$ and $N°$, when $x, y$ are both *odd*; (11*c*)

or thus

$L'$, $M'$ yielding $N'$ and $N°$, when *one* of $x, y$ is *even*, (11*d*)
$L''$, $M''$ yielding $N''$ and $N°$, when $x, y$ are *both odd*. (11*e*)

[The forms differently accented (as $L'$, $L''$, $L'''$, &c.) are evidently *arithmetically distinct*: they differ, however, only by a unit-factor, the form of which depends on the value of $n$; *e.g.*, when $n = 3, 5$ the unit-factor is $\frac{1}{4}(1^2+3\,.\,1^2)$, $\frac{1}{4}(3^2-5\,.\,1^2)$ respectively; from this it comes that the products of the similarly accented $L'M'$, $L''M''$, $L'''M'''$ each yield *one* product form with the like accent $N'$, $N''$, $N'''$, and each yields also the *same* (general) form $N°$.]

*Ex.*—See Art. **21** for examples of (11 *a*, *b*, *c*), and Art. **30a** for examples of (11 *d*, *e*).

**6.** *Fermat's Divisors.*—This name may be applied to *prime* divisors $(p)$ depending directly on Fermat's theorem. Let

$$\mathfrak{A}_\epsilon = \left\{(n_1x^2)^{2n} + (2n_2y^2)^{2n}\right\}, \quad \text{where } n = n_1n_2, \tag{12}$$

$$\mathfrak{A}_\omega = \left\{(n_1x^2)^{n} + \iota^{n+1}(n_2y^2)^{n}\right\}, \quad \text{where } n = n_1n_2. \tag{13}$$

The prime divisors to be considered are of *two* forms $(2\rho+1)$ and $(4\rho+1)$, where $\rho$ denotes the Aurifeuillian *order*, *i.e.*, $\rho = 2n$ or $n$ in case of $\mathfrak{A}_\epsilon$, $\mathfrak{A}_\omega$ respectively.

**6a.** *Divisor* $p = (2\rho+1)$. By Fermat's theorem applied to $\mathfrak{A}_\epsilon$, for which $\rho = 2n$,

$$\mathfrak{A}_\epsilon \equiv n_1^{\frac{1}{2}(p-1)} + (2n_2)^{\frac{1}{2}(p-1)}, \pmod p, \quad [\text{here } x, y \text{ both prime to } p]$$

$$\equiv \left(\frac{n_1}{p}\right) + \left(\frac{2}{p}\right)\left(\frac{n_2}{p}\right), \text{ in Legendre's notation.}$$

Here $$n = 2n_1n_2, \quad p = 2\rho+1 = 4n+1.$$

Also, if $n_1$, $n_2$ be *primes*, then*

$$\left(\frac{n_1}{p}\right) = \left(\frac{n_2}{p}\right) = +1,$$

and, if $n_1$, $n_2$ be *composites*, viz.,

$$n_1 = \lambda_1.\lambda_2.\lambda_3 \ldots, \quad n_2 = \mu_1.\mu_2.\mu_3 \ldots .$$

Then, for every* factor $\lambda_z$, $\mu_z$,

$$\left(\frac{\lambda_z}{p}\right) = \left(\frac{\mu_z}{p}\right) = +1,$$

and $$\left(\frac{2}{p}\right) = -1;$$

therefore $$\mathfrak{Y}_\epsilon \equiv 0, \pmod p. \qquad (14a)$$

Next, by Fermat's theorem applied to $\mathfrak{Y}$, for which $\rho = n$,

$$\mathfrak{Y}_\omega = \frac{1}{n_1^n}\left\{(n_1x)^{2n} + \epsilon^{n+1}n^n y^{2n}\right\}$$

$$\equiv \frac{1}{n_1^n}\left\{1+\epsilon^{n+1}n^{\frac{1}{2}(p-1)}\right\}, \pmod p, \quad [\text{here } x, y \text{ both prime to } p]$$

$$\equiv \frac{1}{n_1^n}\left\{1+\epsilon^{n+1}\left(\frac{n}{p}\right)\right\}, \pmod p.$$

Now suppose $n = ST$, where $S$ contains only factors of form $(4s+1)$, and $T$ contains only factors of form $(4t-1)$, so that

$$S = s_1s_2s_3 \ldots s_\alpha, \quad \text{where } s_z = 4\sigma_z+1, \ (z = 1 \text{ to } z = \alpha),$$

$$T = t_1t_2t_3 \ldots t_\beta, \quad \text{where } t_z = 4\tau_z-1, \ (z = 1 \text{ to } z = \beta),$$

$$p = 2n+1 = 4\varpi-1, \ (\text{since } n \text{ is } \textit{odd}).$$

Therefore $\left(\frac{s_z}{p}\right) = +1$, for all† values of $z$; and $\left(\frac{S}{p}\right) = +1$,

$\left(\frac{t_z}{p}\right) = -1$, for all† values of $z$,

$\left(\frac{T}{p}\right) = +1$; and $n = 4\nu+1$, $\epsilon^{n+1} = -1$, when $\beta$ is *even*,

$\left(\frac{T}{p}\right) = -1$; and $n = 4\nu-1$, $\epsilon^{n+1} = +1$, when $\beta$ is *odd*.

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* This follows from the law of quadratic reciprocity, noting that $n_1$, $n_2$; $\lambda_z$, $\mu_z$ are factors of $(p-1)$.

† This follows from the law of quadratic reciprocity, noting that $s_z$, $t_z$ are factors of $(p-1)$.

Hence $$\mathfrak{A}_\omega \equiv 0, \pmod{p}, \text{ in all cases.} \qquad (14b)$$

Hence, by (14*a*, *b*),

$$\mathfrak{A}_\epsilon \text{ and } \mathfrak{A}_\omega \text{ are always divisible by } p = 2\rho+1, \qquad (14c)$$

(when $x$, $y$ prime to $p$),

and since, by definition, the Aurifeuillian $(N)$ is the A.P.F. of $\mathfrak{A}_\epsilon$ or $\mathfrak{A}_\omega$, it follows* that—

"Either the Aurifeuillian $(N)$, or some one of the other submultiples of $\mathfrak{N}_\epsilon$, $\mathfrak{N}_\omega$, is divisible by $p = 2\rho+1$ (when $x$, $y$ prime to $p$)." (14)

**6b.** *Divisor* $p = (4\rho+1)$. Applying Fermat's theorem to $\mathfrak{A}_\epsilon$, $\mathfrak{A}_\omega$,

$$\mathfrak{A}_\epsilon = \frac{1}{n_1^{2n}}\left\{(n_1x)^{4n}+(2n)^{2n}y^{4n}\right\}, \text{ where } \rho = 2n,\ p = 8n+1$$

$$\equiv \frac{1}{n_1^{2n}}\left\{\left(\frac{n_1}{p}\right)\left(\frac{x}{p}\right)+(2n)^{2n}\left(\frac{y}{p}\right)\right\}, \pmod{p},$$

[here $x$, $y$ both prime to $p$];

$$\mathfrak{A}_\omega \equiv \frac{1}{n_1^{n}}\left\{(n_1x)^{2n}+\iota^{n+1}.n^n.y^{2n}\right\}, \text{ where } \rho = n,\ p = 4n+1$$

$$\equiv \frac{1}{n_1^{n}}\left\{\left(\frac{n_1}{p}\right)\left(\frac{x}{p}\right)+\iota^{n+1}.n^n\left(\frac{y}{p}\right)\right\}, \pmod{p},$$

[here $x$, $y$ both prime to $p$].

Now, in both cases, since $p = 4\rho+1 = 8n+1$ in case of $\mathfrak{A}_\epsilon$, or $= 4n+1$ in case of $\mathfrak{A}_\omega$,

$$\left(\frac{n_1}{p}\right) = +1,\dagger \quad \text{and} \quad n^n \equiv +1,\ddagger \pmod{p},$$

and $\iota^{n+1} = +1$ or $-1$, according as $n = 4\nu-1$ or $4\nu+1$; therefore

$$\mathfrak{A}_\epsilon \equiv \frac{1}{(n_1)^{2n}}\left\{\left(\frac{x}{p}\right)+\left(\frac{y}{p}\right)\right\}, \pmod{p},$$

$$\mathfrak{A}_\omega \equiv \frac{1}{n_1^{n}}\left\{\left(\frac{x}{p}\right)+\iota^{n+1}\left(\frac{y}{p}\right)\right\}, \pmod{p}.$$

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* Part of this proof was communicated to the author by Mr. Bickmore.

† This follows from the law of quadratic reciprocity.

‡ See *Educational Times*, Vol. LXV., p. 78, Question 13058, by C. E. Bickmore, proving that $(\rho^\rho-1)$ is divisible by $p = 4\rho+1$.

Hence, using the abbreviations Q.R. = quadratic residue of $p$, Q.N.R. = quadratic non-residue of $p$,

$\mathfrak{N}_\epsilon \equiv 0$, (mod $p$), when one of $x$, $y$ is Q.R. and the other Q.N.R. of $p$, (15$a$)

$\mathfrak{N}_\omega \equiv 0$, (mod $p$), if $n = 4\nu - 1$, when one of $x$, $y$ is Q.R., and the other Q.N.R. of $p$

also, if $n = 4\nu + 1$, when $x$, $y$ are both Q.R., or both Q.N.R. of $p$ (15$b$)

Hence it follows that—

"When $\mathfrak{N}_\epsilon$ or $\mathfrak{N}_\omega$ is divisible by $p = (4\rho + 1)$, either the Aurifeuillian ($N$), or some of the other submultiples of $\mathfrak{N}_\epsilon$, $\mathfrak{N}_\omega$, is divisible by $p$." (15)

Also, writing $N'$ for the Ant-Aurifeuillian (Art. 4) conjugate to $N$, and $\mathfrak{N}'$ for the multiple thereof conjugate to $N_\omega$; then, by similar reasoning to above,

$$\mathfrak{N}'_\omega = \left\{(n_1 x^2)^n - \iota^{n+1}(n_2 y^2)^n\right\}, \text{ where } n = n_1 n_2$$

$$\equiv \frac{1}{n^n}\left\{\left(\frac{x}{p}\right) - \iota^{n+1}\left(\frac{y}{p}\right)\right\}, \text{ (mod } p\text{), where } p = 4n+1$$

[here both $x$, $y$ prime to $p$]

$\equiv 0$, (mod $p$), if $n = 4\nu - 1$, when $x$, $y$ are both Q.R., or both Q.N.R. of $p$, (15$a'$)

also, if $n = 4\nu + 1$, when one of $x$, $y$ is Q.R., and the other Q.N.R. of $p$; (15$b'$)

and therefore

"When $\mathfrak{N}'_\omega$ is divisible by $p = 4\rho + 1$, either the Ant-Aurifeuillian $N'$, or some of the other submultiples of $\mathfrak{N}'_\omega$, is divisible by $p$." (15')

7. *Quotient-Aurifeuillians.*—If an Aurifeuillian $\mathfrak{A}$ be divisible by another Aurifeuillian ($N$) of the *same order*, then the quotient ($Q$) will be, in general, decomposable into two factors, say $S$, $T$. For, if $\mathfrak{L}$, $\mathfrak{M}$ and $L$, $M$ be the Aurifeuillian factors of $\mathfrak{A}$, $N$ respectively, then

$$Q = \mathfrak{A} \div N = \mathfrak{L} \,.\, \mathfrak{M} \div L \,.\, M = S \,.\, T,$$

so that the factors $\mathfrak{L}$, $\mathfrak{M}$ contain between them *all* the factors of $L$, $M$.

Also the quadratic forms in which $\mathfrak{A}$, $\mathfrak{L}$, $\mathfrak{M}$; $N$, $L$, $M$ are expressible are *conformal* (because $\mathfrak{A}$, $N$ are Aurifeuillians of same

order). Hence also, the quadratic forms in which $Q$, $S$, $T$ are expressible are *conformal* with them; and $Q$, $S$, $T$ can be actually expressed in those quadratic forms by the process* of *conformal division*, (because they are quotients of conformals). The quotient $Q$ thus has properties quite similar to those of Aurifeuillians.

**7a.** *Quotient-Aurifeuillians* (continued).—A specially important case of Quotient-Aurifeuillians is obtained by writing

$$X^\omega,\ Y^\omega,\ x^\omega,\ y^\omega \text{ for } X,\ Y,\ x,\ y \text{ respectively,}$$

in the Aurifeuillian functions of Art. 2. Hereby, in the abbreviated notation of (3, i), (4, i),

$$N = \mathfrak{F}(n,\ X^2,\ Y^2) \quad \text{gives} \quad \mathfrak{N} = \dagger\mathfrak{F}(n,\ X^{2\omega},\ Y^{2\omega}), \tag{16a}$$

$$N = \mathfrak{F}(n,\ X,\ Y) \quad \text{gives} \quad \mathfrak{N} = \dagger\mathfrak{F}(n,\ X^{\omega},\ Y^{\omega}). \tag{16b}$$

Thus the new $\mathfrak{N}$ is seen to be an Aurifeuillian of the *same order* ($2n$ or $n$) as the old $N$, and the new $\mathfrak{L}$, $\mathfrak{M}$ are found by changing $X$, $Y$, &c., into $X^\omega$, $Y^\omega$, &c. Also $\mathfrak{N}$ is plainly divisible by $N$ (since $\omega$ is *odd*). And, in this case, it happens in general that

$$\mathfrak{L} \text{ contains } L \text{ or } M; \quad \text{and } \mathfrak{M} \text{ contains } M \text{ or } L,$$

so that the quotient $Q$ is resolvable into factors $(S, T)$, such that

$$Q = \mathfrak{N} \div N = S.T;\ S = \mathfrak{L} \div L, \text{ or } \mathfrak{L} \div M;\ T = \mathfrak{M} \div M, \text{ or } \mathfrak{M} \div L. \tag{17}$$

Hence also $Q$, $S$, $T$ are expressible (by the process of conformal division) in quadratic forms conformal with those of $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$. And, lastly, the factors $S$, $T$ are interrelated by the same relation (7), which connects $\mathfrak{L}$ and $\mathfrak{M}$, or $L$ and $M$. Thus *this* Quotient-Aurifeuillian has all the properties of an Aurifeuillian.

**7b.** *Reduced Quotient-Aurifeuillians.*—The Quotient-Aurifeuillians $(Q)$, just treated of, are not the *Reduced* A.P.F. of their subject-functions $(X^2, Y^2)$ or $(X, Y)$ with regard to the complete index $n\omega$, except when $\omega$ is *prime;* the extension to the case when $\omega$ is *composite* is made as follows.

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* *I.e.,* division with *preservation of form;* see Art. 15–23 of the author's paper "On the Connexion of Quadratic Forms," above quoted.

† Here the new $\mathfrak{N}$ is not the A.P.F. of the subject-function with regard to the complete index $n\omega$, but only with regard to the index $n$, (compare Art. 1).

Let $N(\omega)$* be what the Aurifeuillian functions of Art. **2** become when $X^\omega$, $Y^\omega$, $x^\omega$, $y^\omega$ are substituted for $X$, $Y$, $x$, $y$ respectively therein,

$$\left.\begin{aligned} N(\omega) &= \mathfrak{F}(n, X^{2\omega}, Y^{2\omega}) \\ \text{so that}\quad N(1) &= \mathfrak{F}(n, X^2, Y^2) \end{aligned}\right\}, \quad \text{when } 2nXY = \square;$$

$$\left.\begin{aligned} N(\omega) &= \mathfrak{F}(n, X^\omega, Y^\omega) \\ \text{so that}\quad N(1) &= \mathfrak{F}(n, X, Y) \end{aligned}\right\}, \quad \text{when } nXY = \square.$$

Then, changing $\omega$ into $\lambda$, $\lambda\mu$, $\lambda\mu\nu$, &c. in succession, ($\lambda$, $\mu$, $\nu$, &c., being *unequal odd* primes), the rules of Art. **1** give

$$\mathfrak{F}N(\lambda) = N(\lambda) \div N(1), \tag{18a}$$

$$\mathfrak{F}N(\lambda\mu) = N(\lambda\mu)\,N(1) \div N(\lambda)(N\mu), \tag{18b}$$

$$\mathfrak{F}N(\lambda\mu\nu) = \frac{N(\lambda\mu\nu)\,N(\lambda)\,N(\mu)\,N(\nu)}{N(1)\,N(\mu\nu)\,N(\nu\lambda)\,N(\lambda\mu)}, \text{ &c.,} \tag{18c}$$

and the law of formation is obvious. Now each of these functions $\mathfrak{F}$ is (by Art. **1**) the *Reduced* A.P.F. of its subject-function with regard to the *complete* index $n\omega$: and is composed of the product of the Quotient-Aurifeuillians described in Art. **7a**, divided by a product of similar functions, *e.g.*,

$$\mathfrak{F}N(\lambda\mu\nu) = \left(\frac{N(\lambda\mu\nu)}{N(\mu\nu)}\,\frac{N(\lambda)}{N(1)}\right) \div \left(\frac{N(\nu\lambda)}{N(\nu)}\,\frac{N(\lambda\mu)}{N(\mu)}\right). \tag{18c'}$$

Hence the result, which may be termed a *Reduced Quotient-Aurifeuillian* ($Q$), is (for reasons similar to those used in Art. **7**, **7a**) resolvable into two factors ($S$, $T$), and the three functions $Q$, $S$, $T$ are expressible as before in quadratic forms *conformal* with those of $N(\omega)$ and of its factors $\mathfrak{X}$, $\mathfrak{M}$. And $S$, $T$ are connected by the relation (7). Thus these Reduced Quotient-Aurifeuillians have the *same* properties as Aurifeuillians.

*Examples.*—The above properties are of great use in factorising high numbers of form $(u^t \pm v^t)$ where $t = 2\omega$ or $\omega$, and $\omega$ is composite. Thus, if $t = 105 = 3 \cdot 5 \cdot 7$, then the *Reduced* A.P.F. of $N = (u^{105} \pm v^{105})$ is only of order

$$(105 + 3 + 5 + 7) - (15 + 21 + 35 + 1) = 48$$

in the subject-functions $u, v$. And writing $n\lambda\mu = 105$, $N = (u^n + i^{n+1} v^n)$ will be an Aurifeuillian of order $n$ when $nuv = \square$. And the *Reduced* A.P.F. of $N$, *i.e.*, $\mathfrak{F}N(\lambda\mu)$, will be a Quotient-Aurifeuillian, and therefore resolvable into factors $S$, $T$ as above. The table shows the resolvable number $N$.

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* Here $N(\omega)$ is not the A.P.F. of the subject-function with regard to the *complete* index $n\omega$, but only with regard to the index $n$, (compare Art. **1**).

$$\begin{array}{lccccccc} \text{Order } n = & 3, & 5, & 7, & 15, & 21, & 35, & 105 \\ \lambda\mu = & 35, & 21, & 15, & 7, & 5, & 3, & 1 \end{array}$$

$$N = \begin{cases} 3^{105}+1, & 5^{105}-1, & 7^{105}+1, & 15^{105}+1, & 21^{105}-1, & 35^{105}+1, & 105^{105}-1 \\ & & & 5^{105}+3^{105}, & 7^{105}-3^{105}, & 7^{105}+5^{105}, & 35^{105}-3^{105} \\ & & & & & & 21^{105}-5^{105} \\ & & & & & & 15^{105}-7^{105} \end{cases}$$

**7c.** *Quotient Ant-Aurifeuillians.* — The quotient of one Ant-Aurifeuillian $\mathfrak{A}$ by another ($N$) of the same order may be treated of in a manner quite similar to that of Art. **7**—**7b**, and the results will be, *mutatis mutandis*, quite similar: the difference being that the quotient ($Q$) will be expressible as a *sum of two squares* (instead of as a difference of squares = product of two factors). And $Q$ will possess all the properties of an Ant-Aurifeuillian.

## Chapter II., Bin-Aurifeuillians.

**8.** *Bin-Aurifeuillians.*—When, in Results (3; 3*a*, *b*; 5*a*, *b*),

$$n = n_1 = n_2 = 1, \quad X = x^2, \quad Y = 2y^2,$$

the function $N$ becomes the *Bin-Aurifeuillian*

$$N = x^4 + 2^2.y^4 \quad [x \text{ taken}^* \textit{ odd}, \text{ and prime to } y; \text{ see } (3a, b)], \qquad (19)$$

and its quadratic partitions are

$$N^\circ = (x^2)^2 + (2y^2)^2; \quad N' = (x^2 - 2y^2)^2 + (2xy)^2. \qquad (19a)$$

Its Aurifeuillian factors ($L$, $M$), and their quadratic partitions, are

$$L = x^2 - 2xy + 2y^2, \quad M = x^2 + 2xy + 2y^2, \qquad (20)$$

$$L' = (x-y)^2 + y^2, \quad M' = (x+y)^2 + y^2. \qquad (20a)$$

Also, under Art. 6, $N$ is divisible by 5 (except when one of $x$, $y$ contains 5); and thus *one* of $L$, $M$ contains 5, viz.,

$$\left.\begin{array}{l} L \text{ is divisible by } 5, \text{ when } x+y = 5k, \text{ or } x+2y = 5k \\ M \text{ is divisible by } 5, \text{ when } x-y = 5k, \text{ or } x-2y = 5k \end{array}\right\}. \qquad (21)$$

[It is worth noting that 5 is itself a Bin-Aurifeuillian (since $5 = 1^4 + 4 \cdot 1^4$); and its $L = 1$, $M = 5$].

Again, since every 4th power ends in 1, 5, or 6, and $x$ is taken *odd*,

Every Bin-Aurifeuillian ($x^4 + 4y^4$) ends in 1, 5, or 9. (21*a*)

**9.** *Bin-Aurifeuillian Dimorphs.*—The question now arises whether the Bin-Aurifeuillian partition of a number ($N$) is *unique*, or whether

* Hereby $N$ will be of form $(4f+1)$.

it may be *dimorphic.* Let, if possible,

$$N = x^4+4y^4 = \xi^4+4\eta^4, \quad (\text{where } \xi \neq x,\ \eta \neq y); \tag{22}$$

therefore $$N = (x^2-2y^2)^2+(2xy)^2 = (\xi^2-2\eta^2)^2+(2\xi\eta)^2. \tag{22a}$$

A *general* solution of these seems difficult: but a somewhat limited solution may be obtained by equating separately, the *parts* (a, b) of two partitions $(a^2+b^2)$ of *different* kinds, say

$$\xi^2-2\eta^2 = x^2, \quad \xi\eta = y^2.$$

These involve ($\xi$, $\eta$ being supposed *relative primes*)

$$\xi = \tau^2, \quad \eta = v^2, \quad x = t, \quad y = \tau v,$$

whence $$\tau^4-2v^4 = t^2. \tag{23}$$

Any solution of this last (23) gives a solution of (22); now the *general* solution of (23) is known* to be

$$\tau = \tau_0^4+2v_0^4, \quad v = 2\tau_0 v_0 . t_0, \quad \text{where } t_0 = \sqrt{\tau_0^4 \sim 2v_0^4}. \tag{23a}$$

Hence every solution of $\tau_0^4 \sim 2v_0^4 = t_0^2$ gives a solution of (23); and, since $t$ is of the same form as one of the forms of $t_0$, each set of values of $\tau$, $v$, $t$ so found from the base-set $\tau_0$, $v_0$, $t_0$ may be used as a new base-set (*i.e.*, in place of $\tau_0$, $v_0$, $t_0$) from which to generate a *new* set of $\tau$, $v$, $t$; and this process may be continued *ad infinitum.* Hence, every base-solution ($\tau_0$, $v_0$) yields an infinite series of *dimorphic* Bin-Aurifeuillians.

*Ex.*—The elements of the *lowest* solutions are given in the table below: the two solutions $\tau_0 = 1$, $v_0 = 1$; $\tau_0 = 1$, $v_0 = 13$ are †fundamental, each giving rise to an infinite number of solutions; but the numbers increase in magnitude so very rapidly that it is not worth continuing them.

| Data. | Auxiliaries. | Aurifeuillian Partitions. | | Bin-Aurifeuillian. |
|---|---|---|---|---|
| $\tau_0$ ; $v_0$ ; $t_0$ | $\tau$ ; $v$ ; $t$ | $x$ ; $y$ | $\xi$ ; $\eta$ | $N$ |
| 1 ; 1 ; 1 | 3 ; 2 ; 7 | 7 ; 6 | 9 ; 4 | 7585 |
| 3 ; 2 ; 7 | 113 ; 84 ; 7967 | 7967 ; 9492 | 12769 ; 7056 | $x^4+4y^4$ |
| 1 ; 13 ; 239 | 57123 ; 6214 ; $t$ | $t$ ; $\tau v$ | $\tau^2$ ; $v^2$ | $\tau^8+4v^8$ |

* Euler's *Elements of Algebra*, (Hewlett's transl., 1822), Part II, Art. 211.

† One other ($\tau_0 = 1525$, $v_0 = 1343$) is given in Mr. Lebesgue's paper, "Résolution des Équations Biquadratiques," in the *Journal des Math. Pures et Appliquées*, t. XVIII, p. 83.

9a. *Factorisation of above.*—The resolution of these numbers into their Aurifeuillian Factors in *two distinct ways*, $N = L_1M_1 = L_2M_2$, greatly helps their ultimate resolution into *prime* factors; thus, taking the second number above

$$N = 7967^4 + 4\cdot 9492^4 = (1525^2 + 9492^2)\,(17459^2 + 9492^2) = L_1 . M_1$$

$$= 12769^4 + 4\cdot 7056^4 = (5713^2 + 7056^2)\,(19825^2 + 7056^2) = L_2 . M_2.$$

Here $L_1 = 92{,}423{,}689$, $M_1 = 394{,}914{,}745$; $L_2 = 82{,}425{,}505$; $M_2 = 442{,}817{,}761$. And it will be found that the G.C.M. of $L_1, M_2$ is 1553, and the G.C.M. of $L_2, M_1$ is $5 . 277$. Hence it easily results

$$N = (1553 . 59513)\,(5 . 277 . 389 . 733) = (5 . 277 . 59513)\,(389 . 733 . 1553).$$

[This large number, > 36400 billion, would be otherwise difficult to factorise.]

10. *Higher Bin-Aurifeuillians.*—The results of imposing *one** of the conditions

$$\text{i. } x^2 \mp 2y^2 = z^2; \qquad \text{ii. } 2y^2 - x^2 = z^2, \tag{24}$$

on the Bin-Aurifeuillian $N = x^4 + 4y^4 = LM$, are interesting. The results are

$$\left.\begin{aligned} &\text{i. } N = 2x^4 - 2x^2z^2 + z^4 = \mathfrak{L} \text{ of } \mathfrak{N} \\ &\text{ii. } N = 2x^4 + 2x^2z^2 + z^4 = \mathfrak{M} \text{ of } \mathfrak{N} \end{aligned}\right\}, \tag{25}$$

where $\mathfrak{L}$, $\mathfrak{M}$ are the Aurifeuillian Factors of the Higher Bin-Aurifeuillian

$$\mathfrak{N} = z^8 + 4x^8 = \mathfrak{L}\mathfrak{M}.$$

This gives *three* (easily recognizable) cases in which $\mathfrak{N}$ is resolvable into *three* factors, viz.,

$$\text{i. } \mathfrak{N} = LM . \mathfrak{M}; \qquad \text{ii. } \mathfrak{N} = \mathfrak{L} . LM.$$

The general solutions of the Diophantines i, ii of (24), in terms of two arbitrary integers $(m, n)$, take the following four forms†

$$\left.\begin{aligned} &\text{i}a.\ x = m^2 - 2n^2, && y = 2mn, && z = m^2 + 2n^2, \ (m \text{ odd}) \\ &\text{i}b.\ x = m^2 + 2n^2, && y = 2mn, && z = m^2 - 2n^2, \ (m \text{ odd}) \\ &\text{ii}a.\ x = m^2 + 2mn - n^2, && y = m^2 + n^2, && z = n^2 + 2mn - m^2 \\ &\text{ii}b.\ x = n^2 + 2mn - m^2, && y = m^2 + n^2, && z = m^2 + 2mn - n^2 \end{aligned}\right\}. \tag{26}$$

[$m$ *odd* and $n$ *even* in ii$a$, ii$b$]

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* The results obtained by changing $z^2$ into $2z^2$ will be found reducible to those given by i and ii.

† Noting that Cases $a$, $b$ are interconvertible by interchange of $x$, $z$.

*Examples.*—The following are the *lowest* examples of each case:—

| Case. | Data. | Parts. | $\mathfrak{N} = (x^8 + 4x^8)$. | $\mathfrak{N} = \mathfrak{L}\mathfrak{M}$. | $\mathfrak{L}$ or $\mathfrak{M} = N = LM$. |
|---|---|---|---|---|---|
| | $m$ ; $n$ | $x$ ; $y$ ; $z$ | | $\mathfrak{L}$ ; $\mathfrak{M}$ | $N$ $L$ $M$ |
| i*a* | 1 ; 1 | −1 ; 2 ; 3 | 6,565 | 65 ; 101 | $\mathfrak{L} = 5$ ; 13 |
| i*b* | 1 ; 1 | 3 ; 2 ; −1 | 26,245 | 145 ; 181 | $\mathfrak{L} = 5$ ; 29 |
| i*a* | 1 ; 2 | −7 ; 4 ; 9 | 66,105,925 | 3,425 ; 19,301 | $\mathfrak{L} = 25$ ; 137 |
| i*b* | 1 ; 2 | 9 ; 4 ; −7 | 177,955,685 | 7,585 ; 23,461 | $\mathfrak{L} = 41$ ; 185 |
| ii*a* | 1 ; 2 | 1 ; 5 ; 7 | 5,764,805 | 2,305 ; 2,501 | $\mathfrak{M} = 41$ ; 61 |
| ii*b* | 1 ; 2 | 7 ; 5 ; 1 | 23,059,205 | 4,705 ; 4,901 | $\mathfrak{M} = 29$ ; 169 |
| ii*a* | 3 ; 2 | 17 ; 13 ; 7 | 27,908,794,565 | 141,121 ; 197,765 | $\mathfrak{M} = 185$ ; 1069 |
| ii*b* | 3 ; 2 | 7 ; 13 ; 17 | 7,008,816,645 | 60,001 ; 116,645 | $\mathfrak{M} = 205$ ; 569 |

**11.** *Bin-Aurifeuillians; Quartic Forms.*—A Bin-Aurifeuillian being a quartic form, it becomes interesting to inquire whether it can be expressed in *other* quartic forms. In the first place, it is known that a Bin-Aurifeuillian $(x^4+4y^4)$ cannot be* a perfect square, and *a fortiori* not a perfect fourth power. The forms considered below are

Form $(X^4-Y^4)$, Art. **11a**, Form $(X^4+Y^4)$, Art. **11b**,

Form $(U^4+W^4)(V^4+W^4)$, Art. **11c**.

**11a.** *Form* $(X^4-Y^4)$.—Let, if possible,

$$N = x^4+4y^4 = X^4-Y^4. \tag{27}$$

A *general* solution of this seems difficult; but a somewhat limited solution may be obtained by expressing $(x^4+4y^4)$ as a difference of squares, and assuming this expression to be *identical* with the other form $(X^4-Y^4)$: thus

$$N = (x^2+2y^2)^2-(2xy)^2 = (X^2)^2-(Y^2)^2;$$

whence $$x^2+2y^2 = X^2, \quad 2xy = Y^2.$$

These involve (since $x$, $y$ are supposed *mutually prime*)

$$x = f^2, \quad y = 2g^2, \quad X = h, \quad Y = 2fg,$$

where $$f^4+8g^4 = h^2. \tag{28}$$

Any solution of this last (28) gives a solution of (27): now the *general* solution of (28) may be shown† to be

$$f = f_0^4-8g_0^4, \quad g = 2f_0g_0h_0, \quad \text{where } h_0 = \sqrt{f_0^4+8g_0^4}. \tag{28a}$$

* See Euler's *Elements of Algebra*, Art. 209–1.

† By a process similar to that used in Euler's *Elements of Algebra* (Howlett's transl.), Part II, Art. 211, for solution of $r^4-2v^4 = t^2$: compare Art. 9 above.

Hence every solution of $f_0^4+8g_0^4=h_0^2$ gives a solution of (28); and, since $h$ is of the same form as $h_0$, each set of values of $f, g, h$ so formed from the base-set $f_0, g_0, h_0$ may be used as a new base-set (*i.e.*, in place of $f_0, g_0, h_0$) from which to generate a a *new* set of $f, g, h$; and this process may be continued *ad infinitum*. Hence every solution $f_0, g_0, h_0$ yields an infinite series of Bin-Aurifeuillians expressible in the required form $X^4-Y^4$.

*Examples.*—The elements of the *lowest* solutions are given in the table below: the solutions $f_0=1$, $g_0=1$; $f_0=239$, $g_0=13$ are fundamental, each giving rise to an infinite number of solutions; but the numbers increase so very rapidly that it is not worth while continuing them.

| Data. | Auxiliaries. | Aurifn. Parts. | Quartic Parts. | Bin-Aurifn. |
|---|---|---|---|---|
| $f_0$ ; $g_0$ ; $h_0$ | $f$ ; $g$ ; $h$ | $x$ ; $y$ | $X$ ; $Y$ | $N$ |
| . ; . ; . | 1 ; 1 ; 3 | 1 ; 2 | 3 ; 2 | 65 |
| 1 ; 1 ; 3 | 7 ; 6 ; 113 | 49 ; 72 | 113 ; 84 | 113,260,225 |
| 7 ; 6 ; 113 | 7967 ; 9492 ; $(f^4+8g^4)^{\frac{1}{2}}$ | $f^2$ ; $2g^2$ | $h$ ; $2fg$ | $f^8+64g^8$ |

**11a′.** *Connexion with Dimorph Aurifeuillians.*—The two Diophantine equations

$$(23),\ t^2=\tau^4\sim 2v^4\ ;\quad (28),\ h^2=f^4+8g^4$$

are known* to be so connected that every solution of either gives one solution of the other. Now the former has been shown to lead to the *dimorph* Bin-Aurifeuillian

$$N_1=x_1^4+4y_1^4=(\tau^2)^4+4(v^2)^4,$$

and the latter has been shown to lead to the Bin-Aurifeuillian

$$N_2=x_2^4+4y_2^4=X^4-Y^4:$$

so that these two kinds of Bin-Aurifeuillians must be connected. Comparing Art. **9, 11**, it is seen that

$$\left.\begin{array}{lll}\text{either} & x_1=f,\ y_1=g\ ; & x_2=f^2=x_1^2\ ;\quad y_2=2g^2=2y_1^2\\ \text{or} & \tau=h=X, & v=Y\end{array}\right\}. \qquad (29)$$

Thus "Every dimorph Bin-Aurifeuillian ($N_1$) leads to a Bin-Aurifeuillian ($N_2$) expressible as ($X^4-Y^4$), and *vice versa*." (29*a*)

**11b.** *Form* $(X^4+Y^4)$.—Let, if possible,

$$N=x^4+4y^4=(x^2-2y^2)^2+(2xy)^2=X^4+Y^4.$$

As $Y^2\neq 2y^2$, the algebraic solution involves

$$x^2\sim 2y^2=X^2,\quad 2xy=Y^2.$$

These involve (since $x$ is supposed *odd*, and prime to $y$)

$$x=t^2,\quad y=2u^2,\quad Y=2tu,\quad t^4\sim 8u^4=X^2.$$

But the latter results, $(t^4\sim 8u^4)$ = square, are known* to be impossible forms: thus a Bin-Aurifeuillian *cannot* take the form $(X^4+Y^4)$.

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* See Mr. Lebesgue's paper quoted above, (footnote † of Art. **9**).

**11c.** *Form* $(U^4+W^4)(V^4+W^4)$.—Let, if possible,

$$N = x^4+4y^4 = (U^4+W^4)(V^4+W^4). \qquad (30)$$

A somewhat limited solution may be obtained* by equating the Aurifeuillian Factors $(L, M)$ of $N$ separately to the factors $(U^4+W^4)$, $(V^4+W^4)$; thus

$$L = (x-y)^2+y^2 = U^4+W^4, \quad M = (x+y)^2+y^2 = V^4+W^4,$$

giving $$x-y = U^2, \quad x+y = V^2, \quad y = W^2,$$

whence $$x = \tfrac{1}{2}(V^2+U^2), \quad y = \tfrac{1}{2}(V^2-U^2) = W^2. \qquad (30a)$$

The *general* solution of the latter Diophantine in terms of two arbitraries $(s, t)$ is

$$U = s^2-2t^2, \quad V = s^2+2t^2, \quad W = 2st, \qquad (30b)$$

whence $$x = s^4+4t^4, \quad y = (2st)^2. \qquad (30c)$$

And, since $x$ is supposed *odd* and prime to $y$, therefore $s$ must be *odd* and prime to $t$. Every such pair of values of $s$, $t$ gives a value of $N$ of the required form; thus the computation of such is *direct*, and the number of such forms is infinite.

Note that $x$ itself is a Bin-Aurifeuillian, and is therefore a multiple of 5 (Art. 8), except when one of $s$, $t$ contains 5 (in which case $y$ is a multiple of 5): hence $N$ cannot contain 5 in any case.

*Examples.*—The *lowest* solutions are given in the table below: the numbers will be seen to rise very rapidly; most of the results $(N)$ are so high that it has not been thought worth while to multiply out the product $L \,.\, M = N$.

| Data. | Quartic Parts. | Aurifn. Parts. | Factors of $N$. |
|---|---|---|---|
| $s$ ; $t$ | $U$ ; $V$ ; $W$ | $x$ ; $y$ | $L$ ; $M$ |
| 1 ; 1 | 1 ; 3 ; 2 | 5 ; 4 | 17 ; 97 |
| 1 ; 2 | 7 ; 9 ; 4 | 65 ; 16 | 2657 ; 6817 |
| 1 ; 3 | 17 ; 19 ; 6 | 325 ; 36 | 84,817 ; 131,617 |
| 3 ; 1 | 7 ; 11 ; 6 | 85 ; 36 | 3697 ; 15937 |
| 3 ; 2 | 1 ; 17 ; 12 | 145 ; 144 | 20,737 ; 104,257 |
| 5 ; 1 | 23 ; 27 ; 10 | 629 ; 100 | 289,841 ; 541,441 |
| 5 ; 2 | 17 ; 33 ; 20 | 689 ; 400 | 243,521 ; 1345,921 |

12. *Bin-Aurifeuillian Factors.*—Let $N_1$, $N_2$ be two numbers whose product $(N)$ is a Bin-Aurifeuillian. The conditions that $N_1$, $N_2$ should be themselves the Aurifeuillian Factors $(L, M)$ of $N$ are

$$N_1 = L = x^2-2xy+2y^2, \quad N_2 = M = x^2+2xy+2y^2. \qquad (31)$$

Hence $$x^2+2y^2 = \tfrac{1}{2}(N_2+N_1), \quad 2xy = \tfrac{1}{2}(N_2-N_1),$$

whence $$x^2-2y^2 = \tfrac{1}{2}\left\{4N_1N_2-(N_2-N_1)^2\right\}^{\frac{1}{2}}.$$

* This solution was given by the author in his Paper "On 2 as a 16-ic Residue," Art. 19, in *Proc. Lond. Math. Soc.*, Vol. xxvii.

Hence *three* conditions must obtain that any two numbers $N_1$, $N_2$ may be the Aurifeuillian Factors of a Bin-Aurifeuillian ($N$), viz.,

$$\{4N_1N_2-(N_2-N_1)^2\} = \{2(x^2-2y^2)\}^2 = (2Z)^2, \text{ an } \textit{even square}, \quad (32a)$$

$$(N_1+N_2) \pm 2Z = (2x)^2, \text{ an } \textit{even square}, \quad (32b)$$

$$\tfrac{1}{2}\{(N_1+N_2) \mp 2Z\} = (2y)^2, \text{ an } \textit{even square}. \quad (32c)$$

When $N_1$, $N_2$ are given numbers (satisfying the above conditions), the values of $x$, $y$ giving the Aurifeuillian partition of $N$, and therewith, also, the quadratic partitions of $N$, $L$, $M$ (Art. **5**), are given directly.

**13.** *Conformal Bin-Aurifeuillian Products.*—The question now arises whether the *product* of two or more Bin-Aurifeuillians can be a Bin-Aurifeuillian; or, conversely, whether a Bin-Aurifeuillian can be resolved into the product of two or more Bin-Aurifeuillians.

Let $$N = x^4+4y^4, \quad N_1 = x_1^4+4y_1^4, \quad N_2 = x_2^4+4y_2^4, \quad (33)$$

and, if possible, let $$N = N_1N_2, \text{ (where } N_1 < N_2). \quad (34)$$

The *general* solution of (34) seems difficult; but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors ($L$, $M$) of $N$ separately to $N_1$, $N_2$, thus obtaining *two* equations

$$L = N_1, \quad M = N_2. \quad (35)$$

Let $L$, $M$, $N_1$, $N_2$ be now expressed in the quadratic form $(a^2+b^2)$; herein the forms of $L$, $M$ are algebraically *unique* ($L'$, $M'$), whilst those of $N_1$, $N_2$ are algebraically *dimorph* ($N_1^\circ$, $N_1'$; $N_2^\circ$, $N_2'$); see Art. **5**, **8**. Hence the pair of equations (35) lead to the four systems

$$\textit{Systems} \begin{cases} \text{I. } L' = N_1^\circ, \quad M' = N_2^\circ; & \text{II. } L' = N_1', \quad M' = N_2^\circ \\ \text{III. } L' = N_1^\circ, \quad M' = N_2'; & \text{IV. } L' = N_1', \quad M' = N_2' \end{cases}. \quad (35a)$$

By now equating the *parts* (a, b) of $L'$ separately to the *parts* (a, b) of the corresponding $N_1$ (*i.e.*, of $N_1^\circ$ or $N_1'$); and also the *parts* (a, b) of $M'$ separately to the *parts* (a, b) of the corresponding $N_2$ (*i.e.*, of $N_2^\circ$ or $N_2'$), each system (I, II, III, IV) yields a group of *four* equations between the six quantities $x$, $y$, $x_1$, $y_1$, $x_2$, $y_2$. Also, in each system, four such groups (of four equations each) may be formed; since *either* part (a, b) of $L'$, $M'$ may be equated to *either* part (a, b) of the corresponding $N_1^\circ$, $N_1'$, $N_2^\circ$, $N_2'$. Also, since the parts (a, b) are defined only by their squares ($a^2$, $b^2$), each of the parts (a, b) may have either a $\pm$ sign; so that every group (as above formed) may be

formed in *four* ways, by the mere change of sign of each a, b. Thus finally 64 sets of *four* equations each can be formed, 16 in each system.

The four equations in each of the 64 sets are linear in $x$, $y$, and quadratic in $x_1$, $y_1$, $x_2$, $y_2$. Hence, $x$, $y$ can always be *linearly* eliminated, thus leaving *two* equations quadratic in $x_1$, $y_1$, $x_2$, $y_2$. From these two the value of either pair $(x_1, y_1)$, $(x_2, y_2)$ can be solved (by solution of quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus, *three* conditions must, in general, be satisfied (viz., that the expressions under the three radicals should be perfect squares), in order that this form of solution may be *possible*.

Thus, whenever either of $N_1$, $N_2$ is *given*, and also the Aurifeuillian partition $(x_1, y_1)$ or $(x_2, y_2)$ thereof, these criteria can be *directly* applied; and, if satisfied, the other pair $(x_2, y_2)$, or $(x_1, y_1)$, and thence also $N_2$ or $N_1$, and, lastly, $x$, $y$, and $N$ can be *directly* calculated. So that this problem is *direct*, and the solution is *unique*.

The solution of the problem *ab initio* is, however, much more difficult. In fact, in the 48 cases of Systems II, III, IV, the radicals are such complex functions that a solution seems difficult; but the 16 cases of System I are so simple in form that it is easy to recognize that 12* of the 16 cases involve some impossibility. The remaining *four* cases (here denoted i, ii, iii, iv) of System I turn out to be *completely* solvable, and to be interrelated in an interesting way.

**13a.** *System I.*—This system is defined (see 35*a*) by

$$L' = N_1^\circ, \quad M' = N_2^\circ,$$

or $$(x-y)^2+y^2 = (x_1^2)^2+(2y_1^2)^2, \quad (x+y)^2+y^2 = (x_2^2)^2+(2y_2^2)^2.$$

The equation of the separate parts ($\pm$a, $\pm$b) gives rise to the four possible Cases i, ii, iii, iv.

i. $x-y = x_1^2, \quad x+y = x_2^2, \quad y = 2y_1^2 = 2y_2^2, \quad x = \frac{1}{2}(x_2^2+x_1^2)$
with *two* conditions $\quad x_2^2-x_1^2 = (2y_1)^2 = (2y_2)^2$ } , (36)

ii. $y-x = x_1^2, \quad y+x = x_2^2, \quad y = 2y_1^2 = 2y_2^2, \quad x = \frac{1}{2}(x_2^2-x_1^2)$
with *two* conditions $\quad x_2^2+x_1^2 = (2y_1)^2 = (2y_2)^2$ } , (36)

iii. $x-y = 2y_1^2, \quad x+y = 2y_2^2, \quad y = x_1^2 = x_2^2, \quad x = y_2^2+y_1^2$
with *two* conditions $\quad y_2^2-y_1 = x_1^2 = x_2^2$ } , (36)

iv. $y-x = 2y_1^2, \quad y+x = 2y_2^2, \quad y = x_1^2 = x_2^2, \quad x = y_2^2-y_1^2$
with *two* conditions $\quad y_2^2+y_1^2 = x_1^2 = x_2^2$ } . (36)

* It seems unnecessary to give the detail.

Thus there are *two* conditions of similar form in each of the four cases,

One of type $y_1 = y_2$ (in i, ii), or $x_1 = x_2$ (in iii, iv);

One of type $x_2^2 \mp x_1^2 = (2y_1)^2$ (in i, ii), or $y_2^2 \mp y_1^2 = x_1^2$ (in iii, iv).

This latter condition is equivalent to $c^2 = a^2 + b^2$.

Now the *general* solution of this Pythagorean equation is expressible in terms of *two* arbitrary integers $m$, $n$, viz.,

$$c = m^2 + n^2, \quad a = m^2 \sim n^2 \text{ or } 2mn, \quad b = 2mn \text{ or } m^2 \sim n^2.$$

Here $m$, $n$ should be so taken that $x_1, x_2, x$ may be all *odd*, and $x_1, y_1$; $x_2, y_2$; $x, y$ should be relatively prime; this involves

$m$, $n$ mutual primes; and either one *odd*, one *even*, or both *odd*. (37*a*)

The general solutions of i, ii, iii, iv are

i. $$\left.\begin{aligned} &x_2 = m^2 + n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = mn \\ &x = m^4 + n^4, \quad y = 2m^2n^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}, \tag{37}$$

ii. $$\left.\begin{aligned} &2y_1 = 2y_2 = m^2 + n^2, \quad x_1 \text{ or } x_2 = m^2 \sim n^2, \quad x_2 \text{ or } x_1 = 2mn \\ &\text{[This solution rejected, as giving } x_1, x_2, x \text{ all } even\text{]} \end{aligned}\right\}, \tag{37}$$

iii. $$\left.\begin{aligned} &y_2 = m^2 + n^2, \quad x_1 = x_2 = m^2 \sim n^2, \quad y_1 = 2mn \\ &x = m^4 + 6m^2n^2 + n^4, \quad y = (m^2 \sim n^2)^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}, \tag{37}$$

iv. $$\left.\begin{aligned} &x_1 = x_2 = m^2 + n^2, \quad y_1 \text{ or } y_2 = m^2 \sim n^2 \quad y_2 \text{ or } y_1 = 2mn \\ &x = m^4 - 6m^2n^2 + n^4, \quad y = (m^2 + n^2)^2; \quad x_1, x_2, x \text{ all } odd \end{aligned}\right\}. \tag{37}$$

The solutions when $m$, $n$ are *both odd* lead eventually only to the *same* results as above, so need not be here detailed; in fact, writing for distinctness $m'$, $n'$ in this case, the two solutions of i, ii, iii, iv are interconvertible by the relations

$$m \text{ or } n = \tfrac{1}{2}(m' + n'), \quad n \text{ or } m = \tfrac{1}{2}(m' \sim n'); \quad m' = m + n, \quad n' = m \sim n.$$

Thus there are only *three* distinct general solutions (Nos. i, iii, iv) practically available. These solutions are *general* and *complete*, as they give the means of *direct* computation of Bin-Aurifeuillians $N_1$, $N_2$, $N$, such that $N_1 . N_2 = N$ for any arbitrary values of $m$, $n$ (one *odd*, one *even*, and relatively prime).

*Examples.*—The following table shows the elements and the results of a few of the *lowest* of such numbers $N_1 . N_2 = N$ in each of the solvable cases (i, iii, iv). As $N$ is hereby resolved algebraically into *four* factors, it becomes possible to factorise such numbers ($N$) *completely*, when $< (9 \text{ million})^4 = 6561 \times 10^{24}$, as the present factor-tables extend to 9 million.

| Case. | Data. | Aurifeuillian Parts. Of $N_1$. | Aurifeuillian Parts. Of $N_2$. | Aurifeuillians. | Aurif. Parts of $N$. $[N = N_1 N_2]$ |
|---|---|---|---|---|---|
| | $m$ ; $n$ | $x_1$ ; $y_1$ | $x_2$ ; $y_2$ | $N_1$ ; $N_2$ | $x$ ; $y$ |
| i | 1 ; 2 | 3 ; 2 | 5 ; 2 | 145 ; 689 | 17 ; 8 |
| | 1 ; 4 | 15 ; 4 | 17 ; 4 | 51,649 ; 84,545 | 257 ; 32 |
| | 1 ; 6 | 35 ; 6 | 37 ; 6 | 1,505,809 ; 1,879,345 | 1297 ; 72 |
| | 3 ; 2 | 5 ; 6 | 13 ; 6 | 5,809 ; 33,745 | 97 ; 72 |
| | 3 ; 4 | 7 ; 12 | 25 ; 12 | 85,345 ; 473,569 | 337 ; 288 |
| | 5 ; 2 | 21 ; 10 | 29 ; 10 | 234,481 ; 747,281 | 641 ; 200 |
| iii | 1 ; 2 | 3 ; 4 | 3 ; 5 | 1105 ; 2581 | 41 ; 9 |
| | 1 ; 4 | 15 ; 8 | 15 ; 17 | 67,009 ; 384,709 | 353 ; 225 |
| | 3 ; 2 | 5 ; 12 | 5 ; 13 | 83,569 ; 114,869 | 313 ; 25 |
| | 3 ; 4 | 7 ; 24 | 7 ; 25 | 1,329,505 ; 1,564,901 | 1201 ; 49 |
| | 5 ; 2 | 21 ; 20 | 21 ; 29 | 834,481 ; 3,023,605 | 1241 ; 441 |
| iv | 1 ; 2 | 5 ; 4 | 5 ; 3 | 1649 ; 949 | 7 ; 25 |
| | 1 ; 4 | 17 ; 8 | 17 ; 15 | 99,905 ; 286,021 | 161 ; 289 |
| | 3 ; 2 | 13 ; 12 | 13 ; 5 | 111,505 ; 31,061 | 119 ; 169 |
| | 3 ; 4 | 25 ; 24 | 25 ; 7 | 1,717,729 ; 400,229 | 527 ; 625 |
| | 5 ; 2 | 29 ; 20 | 29 ; 21 | 1,347,281 ; 1,485,205 | 41 ; 841 |

**14.** *Conformal Product of three Bin-Aurifeuillians.*—Let $N_1$, $N_2$, $N_3$; $N_{12}$, $N_{123}$ be Bin-Aurifeuillians, defined by

$$\left.\begin{aligned} N_1 = x_1^4 + 4y_1^4, \quad N_2 = x_2^4 + 4y_2^4, \quad N_3 = x_3^4 + 4y_3^4 \\ N_{12} = x_{12}^4 + 4y_{12}^4, \quad N_{123} = x_{123}^4 + 4y_{123}^4 \end{aligned}\right\}. \tag{38}$$

Also let $N_1$, $N_2$ be such that

$$N_1 . N_2 = N_{12}. \tag{38a}$$

Also, if possible, let $N_{12}$, $N_3$ be such that

$$N_{12} . N_3 = N_{123}. \tag{39}$$

Then $N_1$, $N_2$, $N_{12}$ are here related precisely as the $N_1$, $N_2$, $N$ of Art. **13, 13a**; the $N$, $x$, $y$ of that Article being here replaced by $N_{12}$, $x_{12}$, $y_{12}$, the suffixes $_{(12)}$ being introduced to show distinctly that $N_{12}$, $x_{12}$, $y_{12}$ arise from $N_1$, $N_2$.

Then, multiplying together $(x-y)$, $(x+y)$ of the results (36) of Art. (**13a**), and changing $x$, $y$ thereof into the $x_{12}$, $y_{12}$ of the new notation,

$$\left.\begin{aligned} &\text{Case i.} \quad x_{12}^2 - y_{12}^2 = (x_1 x_2)^2; \qquad \text{Case ii.} \quad y_{12}^2 - x_{12}^2 = (x_1 x_2)^2 \\ &\text{Case iii.} \quad x_{12}^2 - y_{12}^2 = (2y_1 y_2)^2; \qquad \text{Case iv.} \quad y_{12}^2 - x_{12}^2 = (2y_1 y_2)^2 \end{aligned}\right\}. \tag{40}$$

These four results may be included under $x_{12}^2 \sim y_{12}^2 = z^2$ (a square).

Next, taking $\qquad y_3 = x, \quad x_3 = x_{12},$

it will be found that *both* the conditions that $N_{123} = N_{12} \cdot N_3$, *i.e.*, that $N_{123}$, $N_{12}$, $N_3$ should be related similarly to the $N$, $N_1$, $N_2$ of Art. **13a**, are satisfied at once in Cases iii, iv of Art. **13a** (as may be seen by writing $x_{12}$, $y_{12}$, $x_3$, $y_3$ for $x_1$, $y_1$, $x_2$, $y_2$ in the conditions there formed); and the $x$, $y$ of those results (36, iii, iv) become the $x_{123}$, $y_{123}$ here required. Thus

i. After forming $N_1$, $N_2$, $N_{12}$ by Case i, Art. **13a**, then

$$x_{12}^2 - y_{12}^2 = (x_1 x_2)^2; \quad \text{take} \quad y_3 = x_1 x_2, \quad x_3 = x_{12}. \tag{41}$$

Then $x_{12}$, $y_{12}$, $x_3$, $y_3$ (substituted for $x_1$, $y_1$, $x_2$, $y_2$ in Case iv, Art. **13a**) satisfy the two conditions of Case iv, Art. **13a**, so that the $x$, $y$ of the latter become the required

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{41a}$$

ii. This caseis rejected as leading to $x_1$, $x_2$, $x_{12}$, all *even*; see (37, ii).

iii. After forming $N_1$, $N_2$, $N_{12}$ by Case iii, Art. **13a**, then

$$x_{12}^2 - y_{12}^2 = (2y_1 y_2)^2; \quad \text{take} \quad y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \tag{42}$$

Then, $x_{12}$, $y_{12}$, $x_3$, $y_3$ satisfy the two conditions of Case iv, as before; and similarly

$$x_{123} = y_3^2 - y_{12}^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{42a}$$

iv. After forming $N_1$, $N_2$, $N_{12}$ by Case iv, Art. **13a**, then

$$y_{12}^2 - x_{12}^2 = (2y_1 y_2)^2; \quad \text{take} \quad y_3 = 2y_1 y_2, \quad x_3 = x_{12}. \tag{43}$$

Then $x_{12}$, $y_{12}$, $x_3$, $y_3$ (substituted for $x_1$, $y_1$, $x_2$, $y_2$ in Case iii, Art. **13a**) satisfy the two conditions of Case iii, Art. **13a**; and the $x$, $y$ of the latter become the required

$$x_{123} = y_{12}^2 + y_3^2, \quad y_{123} = x_{12}^2 = x_3^2. \tag{43a}$$

The general values of $x_{12}$, $y_{12}$; $x_3$, $y_3$; $x_{123}$, $y_{123}$ in terms of the arbitraries $m$, $n$ of Art. **13a** are now easily written down; thus the present $x_{12}$, $y_{12}$ are simply the $x$, $y$ of Results (37, i, iii, iv), and the rest $x_3$, $y_3$, $x_{123}$, $y_{123}$ can be obtained from above.

*Examples.*—See end of next Article.

**15.** *Conformal Continued Product of Bin-Aurifeuillians.*—The law of formation of a Bin-Aurifeuillian ($N_{123\ldots r}$) which shall be the product of any number ($r$) of Bin-Aurifeuillians ($N_1 N_2 N_3 \ldots N_r$) is now easily seen. Starting with any values of $m$, $n$ (mutual primes, one *odd*, one

*even*), and using any of the three known Cases (37, i, iii, iv) to form the first product $N_{12} = N_1 . N_2$; then (by Art. **14**) Cases i, iii lead to Case iv, whilst Case iv leads to Case iii; hence $N_3$, and $N_{123}$ may be computed as in Art. **14**. The formation of the successive new factors $N_4$, $N_5$, &c., and new products $N_{1234}$, $N_{12345}$, &c., falls alternately under Cases iii, iv (Art. **14**). It will be seen that the process may be continued *ad infinitum*, and that the result at each step is *unique*, *i.e.*, any pair of values of $m$, $n$ yields a *definite* continued product.

*Examples.*—Three examples are given below, starting from the *lowest* values of $m$, $n$ ($m = 1$, $n = 2$) in each of Cases i, ii, iii, and carried to the fourth order ($N_{1234}$), showing also the intermediate steps, (thus giving *eight* algebraic factors):

$$N_{12} = N_1 . N_2 \; ; \; N_{123} = N_{12} . N_3 = N_1 N_2 N_3 \; ; \quad N_{1234} = N_{123} . N_4 = N_1 N_2 N_3 N_4 . \quad (44)$$

The numbers rise so rapidly that it is not worth while pushing to higher orders. The larger numbers ($N_{1234}$) here factorised would be otherwise very difficult to resolve completely; (Ex. 3 runs to 26 figures).

*Ex.* 1.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case i),

$$N_1 = 3^4 + 4 . 2^4 = 145 = 5 . 29 \; ; \quad N_2 = 5^4 + 4 . 2^4 = 689 = 13 . 53 \; ;$$

therefore $$N_{12} = N_1 N_2 = 17^4 + 4 . 8^4 = 99{,}905 = (5 . 29)(13 . 53).$$

Hence, (Art. **14**, Case iv),

$$N_3 = 17^4 + 4 . 15^4 = 286{,}021 = 229 . 1249 \; ;$$

therefore $$N_{123} = N_{12} . N_3 = 161^4 + 4 . 289^4.$$

Hence, (Art. **14**, Case iii),

$$N_4 = 161^4 + 4 . 240^4 = 63{,}841 . 218{,}401$$

$$N_{1234} = N_{123} . N_4 = 141{,}121^4 + 4 . 25{,}921^4$$

$$= (5 . 29)(13 . 53)(229 . 1249)(63{,}841 . 218{,}401).$$

*Ex.* 2.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case iii),

$$N_1 = 3^4 + 4 . 4^4 = 1105 = (17)(5 . 13) \; ; \quad N_2 = 3^4 + 4 . 5^4 = 2581 = 29 . 89 \; ;$$

therefore $$N_{12} = N_1 . N_2 = 41^4 + 4 . 9^4 = 2{,}852{,}005 = \{17 . (5 . 13)\} . (29 . 89).$$

Hence, (Art. **14**, Case iv),

$$N_3 = 41^4 + 4 . 40^4 = 13{,}065{,}761 = 1601 . 8161 \; ;$$

therefore $$N_{123} = N_{12} . N_3 = 1519^4 + 4 . 1681^4 = \{17 . (5 . 13)\}(29 . 89)(1601 . 8161).$$

Hence, (Art. **14**, Case iii), $N_4 = 1519^4 + 4 . 134{,}480^4$;*

therefore $$N_{1234} = N_{123} . N_4 = 2{,}827{,}361^4 + 4 . 2{,}307{,}361^4$$

$$= \{17 . (5 . 13)\}(29 . 89)(1601 . 8161) . N_4.$$

*Ex.* 3.—Take $m = 1$, $n = 2$.

Then, (Art. **13a**, Case iv),

$$N_1 = 5^4 + 4 . 3^4 = 949 = 13 . 73 \; ; \quad N_2 = 5^4 + 4 . 4^4 = 1649 = 17 . 97 \; ;$$

therefore $$N_{12} = N_1 . N_2 = 7^4 + 4 . 25^4 = 1{,}564{,}901 = (13 . 73)(17 . 97).$$

* Here $N_4 = L_4 . M_4 = (132{,}961^2 + 134{,}480^2)(135{,}999^2 + 134{,}480^2)$; but the further resolution of $L_4$, $M_4$ is not obvious.

Hence, (Art. **14**, Case iii),

$$N_3 = 7^4 + 4.24^4 = 1{,}329{,}505 = (5.173).(29.53);$$

therefore $$N_{123} = N_{12}.N_3 = 1201^4 + 4.49^4.$$

Hence, (Art. **14**, Case iv),

$$N_4 = 1201^4 + 4.1200^4 = (337.4273).(7{,}204{,}801);$$

therefore $$N_{1234} = N_{123}.N_4 = 1{,}437{,}599^4 + 4.1{,}442{,}401^4$$
$$= (13.73)(17.97)\{(5.173)(29.53)\}.\{(337.4273).(7{,}204{,}801)\}.$$

**16.** *Quotient Bin-Aurifeuillians.*—Writing $2^\omega$, $x^\omega$, $y^\omega$ for $2, x, y$ in the expression (19); (compare Art. **7**),

$$N = x^4 + 2^2y^4 \text{ becomes } \mathfrak{N} = x^{4\omega} + 2^{2\omega}y^{4\omega} = (x^\omega)^4 + 2^2.(2^{\frac{1}{2}(\omega-1)}y^\omega)^4. \quad (45)$$

Hence $\mathfrak{N}$ is a Bin-Aurifeuillian, and it obviously contains $N$, so that $Q = \mathfrak{N} \div N$ is an algebraic integer, being in fact the A.P.F. of $\mathfrak{N}$ when $\omega$ is a *prime*.

One of its $(a^2+b^2)$ quadratic forms is given by *conformal division* (of $\mathfrak{N}$ by $N$) in the form

$$Q = \frac{\mathfrak{N}}{N} = \left(\frac{x^{2\omega+2} \pm 2^{\omega+1}y^{2\omega+2}}{x^4+2^2y^4}\right)^2 + (2x^2y^2)^2.\left(\frac{x^{2\omega-2} \mp 2^{\omega-1}y^{2\omega-2}}{x^4+2^2y^4}\right)^2, \quad (46)$$

using the upper signs when $\omega = 4\mu+1$, and lower signs when $\omega = 4\mu+3$.

The other $(a^2+b^2)$ quadratic form is also similarly expressible, but the criterion of sign is not so easily expressible in a general form, *i.e.*, except for particular values of $\omega$.

The Aurifeuillian Factors $\mathfrak{L}$, $\mathfrak{M}$ of $\mathfrak{N}$ are

$$\mathfrak{L} = x^{2\omega} - 2^{\frac{1}{2}(\omega+1)}x^\omega y^\omega + 2^\omega y^{2\omega}, \qquad \mathfrak{M} = x^{2\omega} + 2^{\frac{1}{2}(\omega+1)}x^\omega y^\omega + 2^\omega y^{2\omega} \quad (47)$$

$$= (x^\omega - 2^{\frac{1}{2}(\omega-1)}y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)}y^\omega)^2, \quad = (x^\omega + 2^{\frac{1}{2}(\omega-1)}y^\omega)^2 + (2^{\frac{1}{2}(\omega-1)}y^\omega)^2. \quad (48)$$

The factors $S$, $T$ of $Q$ are the *integer* quotients of $\mathfrak{L}$ by $L$ or $M$, and of $\mathfrak{M}$ by $M$ or $L$. To discover which quotients are integral, it suffices to write $x = 1$, $y = 1$ in $\mathfrak{L}$, $\mathfrak{M}$, $L$, $M$; whereon $L$, $M$ reduce to 1, 5; showing* that

$$S = \mathfrak{L} \div L, \quad T = \mathfrak{M} \div M, \quad \text{when } \omega = 8i \pm 1 = \textit{prime},$$
$$\textit{i.e.}, \text{ when } \left(\frac{2}{\omega}\right) = +1, \quad (49a)$$

$$S = \mathfrak{L} \div M, \quad T = \mathfrak{M} \div L, \quad \text{when } \omega = 8i \pm 3 = \textit{prime},$$
$$\textit{i.e.}, \text{ when } \left(\frac{2}{\omega}\right) = -1. \quad (49b)$$

* Adapted from a similar rule for Aurifeuillian Factors, given in Art. 30 of Mr. Bickmore's first paper, above quoted.

Expressing $\mathfrak{L}$, $\mathfrak{M}$, $L$, $M$ in the $(a^2+b^2)$ quadratic form, $S$ and $T$ can now also be expressed in that form by *conformal division*, in a form similar to (48); but the criterion of sign cannot be so easily expressed in general, *i.e.*, except for particular values of $\omega$.

All the quantities $\mathfrak{N}$, $N$; $\mathfrak{L}$, $\mathfrak{M}$, $L$, $M$; $Q$, $S$, $T$ are also expressible in at least two other quadratic forms depending on the nature of $\omega$, when $\omega$ is a *given small* number (Art. **5**).

When $\omega$ is not a prime, the Reduced A.P.F. of $\mathfrak{N}$ may be formed by the formulæ of Art. **7b**, and is a *Reduced Quotient Bin-Aurifeuillian* having the *same* properties as a Bin-Aurifeuillian.

As the quadratic partitions of $Q$, $S$, $T$ are completely reducible only when $\omega$ is a *given* small number, it is better to pass on at once to the simple cases when $\omega = 3, 5, 7,$ &c., as these alone are of *practical* interest. These may be styled *Cubic*, *Quintic*, &c., *Bin-Aurifeuillians.*

**16a.** *Cubic Bin-Aurifeuillians.*—Writing $\omega = 3$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{12}+2^6y^{12}}{x^4+2^2y^4} = x^8-2^2x^4y^4+2^4y^8. \tag{50}$$

Here $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also $Q$, $S$, $T$, are capable of the three quadratic partitions $(a^2+b^2)$, $(A^2+3B^2)$, $(C^2-3D^2)$; those of $S$, $T$ are *monomorph*,* whilst those of $Q$ are* *dimorph*, (since $Q = ST$). The results are

$$\begin{aligned} Q &= (x^4-4y^4)^2+(2x^2y^2)^2 = (x^4-2x^2y^2+4y^4)^2+(2xy)^2\,(x^2-2y^2)^2 && (50a)\\ &= (x^4-2y^4)^2+3\,(2y^4)^2 = (x^4-6x^2y^2+4y^4)^2+3\,(2xy)^2\,(x^2-2y^2)^2 && (50b)\\ &= (x^4+4y^4)^2-3\,(2xy)^2 = (x^4+6x^2y^2+4y^4)^2-3\,(2xy)^2\,(x^2+2y^2)^2. && (50c) \end{aligned}$$

$$S = \frac{\mathfrak{L}}{M} = \frac{x^6-4x^3y^3+8y^6}{x^2+2xy+2y^2}; \qquad T = \frac{\mathfrak{M}}{L} = \frac{x^6+4x^3y^3+8y^6}{x^2-2xy+2y^2} \tag{51}$$

$$\begin{aligned} &= (x^4+2x^2y^2+4y^4)-2xy\,(x^2+2y^2); && = (x^4+2x^2y^2+4y^4)+2xy\,(x^2+2y^2) && (51')\\ &= x^2\,(x-y)^2+y^2\,(x-2y)^2; && = x^2\,(x+y)^2+y^2\,(x+2y)^2 && (51a)\\ &= (x^2-xy-y^2)^2+3y^2\,(x-y)^2; && = (x^2+xy-y^2)^2+3y^2\,(x+y)^2 && (51b)\\ &= (x^2-xy+2y^2)^2-3\,(xy)^2; && = (x^2+xy+2y^2)^2-3\,(xy)^2. && (51c) \end{aligned}$$

**16b.** *Quintic Bin-Aurifeuillians.*—Writing $\omega = 5$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{20}+2^{10}y^{20}}{x^4+2^2y^4} = x^{16}-2^2x^{12}y^4+2^4x^8y^8-2^6x^4y^{12}+2^8y^{16}. \tag{52}$$

Here $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also $Q$, $S$, $T$, are capable of the *three* quadratic partitions $(a^2+b^2)$, $(u^2-5v^2)$, $(U^2+5V^2)$; those of $S$, $T$ are *monomorph*,* whilst those of $Q$ are *dimorph*,* (since $Q = ST$). The results are

---

* Any $(C^2-3D^2)$, $(u^2-5v^2)$ or $(W^2-7Z^2)$ partition can of course be transformed into the *automorphs* $(C_k^2-3D_k^2)$, $(u_k^2-5v_k^2)$, or $(W_k^2-7Z_k^2)$, by conformal multiplication by any integer power $(k)$ of the unit-form $(2^2-3.1^2)$, $(9^2-5.4^2)$, or $(8^2-7.3^2)$ respectively (*see* the author's paper quoted in footnote § to Art. **5**).

$$Q = (x^8-2^2x^4y^4+2^4y^8)^2+(2xy)^2\,(x^4-2^2y^4)^2 \tag{52a}$$
$$= (x^8-2x^6y^2-2^2x^4y^4-2^3x^2y^6+2^4y^8)^2+(2xy)^2\,(x^6-2^3y^6)^2 \tag{52b}$$
$$= (x^8-2x^4y^4+2^4y^8)^2-5\,(2x^4y^4)^2 \tag{52c}$$
$$= (x^8+10x^6y^2+28x^4y^4+40x^2y^4+16y^8)^2-5\,(2xy)^2\,(x^6+4x^4y^2+8x^2y^4+8y^6)^2 \tag{52d}$$
$$= (x^8-12x^4y^4+16y^8)^2+5\,(2x^2y^2)^2\,(x^4-4y^4)^2 \tag{52e}$$
$$= (x^8-10x^6y^2+28x^4y^4-40x^2y^6+16y^8)^2+5\,(2xy)^2\,(x^6-4x^4y^2+8x^2y^4-8y^6)^2, \tag{52f}$$
$$S = \mathfrak{L} \div M = (x^{10}-2^3x^5y^5+2^5y^{10}) \div (x^2+2xy+2y^2) \tag{53}$$
$$= x^8-2x^7y+2x^6y^2-4x^4y^4+8x^2y^6-16xy^7+16y^8 \tag{53a}$$
$$= (x^4-x^3y+2xy^3-4y^4)^2+(xy)^2\,(x^2-2xy+2y^2)^2 \tag{53b}$$
$$= (x^4-x^3y+3x^2y^2-2xy^3+4y^4)^2-5\,(xy)^2\,(x^2-xy+2y^2)^2 \tag{53c}$$
$$= (x^4-x^3y-2x^2y^2-2xy^3+4y^4)^2+5\,(xy)^2\,(x^2-2y^2)^2, \tag{53d}$$
$$T = \mathfrak{M} \div L = (x^{10}+2^3x^5y^5+2^5y^{10}) \div (x^2-2xy+2y^2) \tag{53'}$$
$$= x^8+2x^7y+2x^6y^2-4x^4y^4+8x^2y^6+16xy^7+16y^8 \tag{53a'}$$
$$= (x^4+x^3y-2xy^3-4y^4)^2+(xy)^2\,(x^2+2xy+2y^2)^2 \tag{53b'}$$
$$= (x^4+x^3y+3x^2y^2+2xy^3+4y^4)^2-5\,(xy)^2\,(x^2+xy+2y^2)^2 \tag{53c'}$$
$$= (x^4+x^3y-2x^2y^2+2xy^3+4y^4)^2+5\,(xy)^2\,(x^2-2y^2)^2. \tag{53d'}$$

**16c.** *Septimic Bin-Aurifeuillians.*—Writing $\omega = 7$ in the formulæ of Art. **16**,

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{28}+2^{14}y^{28}}{x^4+2^2y^4} = x^{24}-2^2x^{20}y^4+2^4x^{16}y^8-2^6x^{12}y^{12}+2^8x^8y^{16}-2^{10}x^4y^{20}+2^{12}y^{24}. \tag{54}$$

Here $\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{M}$, and therefore also $Q$, $S$, $T$, are capable of the *three* quadratic partitions $(a^2+b^2)$, $(w^2+7z^2)$, $(W^2-7Z^2)$; those of $Q$ being *dimorph*,* whilst those of $S$, $T$ are *monomorph** (since $Q = ST$). The results are

$$Q = \left\{(x^4-2^2y^4)(x^8+2^4y^8)\right\}^2+(2x^2y^2)^2\,(x^8-2^2x^4y^4+2^4y^8)^2 \tag{54a}$$
$$= \left\{x^{12}+2^6y^{12}-2x^4y^4\,(x^4+2^2y^4)\right\}^2+7\,(2x^4y^4)^2\,(x^4-2^2y^4)^2 \tag{54b}$$
$$= (x^4+2^2y^4)^6-7\,(2x^2y^2)^2\,(x^8+2^2x^4y^4+2^4y^8)^2, \tag{54c}$$
$$S = \mathfrak{L} \div L = (x^{14}-2^4x^7y^7+2^7y^{14}) \div (x^2-2xy+2y^2), \tag{55}$$
$$T = \mathfrak{M} \div M = (x^{14}+2^4x^7y^7+2^7y^{14}) \div (x^2+2xy+2y^2) \tag{55'}$$
$$= x^{12}-2x^{11}y+2x^{10}y^2-4x^8y^4+8x^7y^5-8x^6y^6+16x^5y^7-16x^4y^8+32x^2y^{10}-64xy^{11}+64y^{12} \tag{55a'}$$
$$= x^2\,(x^5-x^4y+2x^2y^3-4xy^4+4y^5)^2+y^2\,(x^5-2x^4y+2x^3y^2-4xy^4+8y^5)^2 \tag{55b'}$$
$$\dagger = (x^6-x^5y-3x^4y^2+4x^3y^3-6x^2y^4+4xy^5+8y^6)^2+7x^2y^2\,(x^4-x^3y+2xy^3-4y^4)^2 \tag{55c'}$$
$$\dagger = (x^6-x^5y+4x^4y^2-10x^3y^3+8x^2y^4-4xy^5+8y^6)^2-7x^2y^2(x^4-2x^3y+2x^2y^2-4xy^3+4y^4)^2 \tag{55d'}$$

The value of $S$ and the quadratic partitions of $S$ may be obtained by writing $-y$ for $y$ in those of $T$; the other (dimorph) partitions of $Q$ may be obtained by conformal multiplication of the isomorphs of $S$, $T$.

**17.** *Fermat's Numbers.* (Base 2).—These are numbers of form

$$N_n = 2^n+1.$$

Writing $x = 1$, $y = 2^\nu$, (where $\nu$ may be *any* integer), the Bin-Aurifeuillian $N = x^4+4y^4$ becomes

$$N = 1+4\,.\,2^{4\nu} = 1+2^{4\nu+2} = N_{4\nu+2}. \tag{56}$$

* *See* the footnote * to Art. 16a, 16b.

† These two partitions were communicated by Mr. Bickmore.

Hence *all* Fermat's numbers of this form are resolvable into two factors $L$, $M$, of which *one* is always a multiple of 5, (Art. 8),

$$L = (2^{2\nu+1} - 2^{\nu+1} + 1), \quad M = (2^{2\nu+1} + 2^{\nu+1} + 1), \qquad (56a)$$

by the formulæ of Art. 8; and can be completely resolved into their *prime* factors up to $n = 46$ (since at this limit the larger factor $M < 9^*$ million). When $2\nu+1 = \omega = 3, 5, 7,$ &c., they can be completely resolved into their *prime* factors by the formulæ of Arts. **16-16c** up to $n = 66$ (since at this limit the larger factor $S$ or $T$ is still $< 9^*$ million. With the aid of "special divisors"† the resolution into *prime* factors can be carried far beyond this limit.

*Examples.*—The following Table‡ shows the *complete* factorisation of the *Reduced A.P.F.* of Fermat's Numbers $N_n = 2^n + 1$, where $n = 4\nu + 2$, complete as far as $n = 102$, and—at intervals—as far as $n = 210$. The factorisation of the Aurifeuillian factors $L$, $M$ or $S$, $T$ is shown separately, and prime factors § only are shown; the semicolon (;) indicates that the factorisation is *complete*.

[*Note*, that the highest number $(2^{210}+1)$, here *completely* factorised, contains 64 figures.]

Factors of the Reduced A.P.F. of $(2^n+1)$, [when $n = 4\nu+2$].

| $n$ | $L$ or $S$ | $M$ or $T$ | $n$ | $L$ or $S$ | $M$ or $T$ | $n$ | $L$ or $S$ | $M$ or $T$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1; | 5; | 26 | 1613; | 53.157; | 50 | 101.8101; | 5.268,501; |
| 6 | 1; | 13; | 30 | 61; | 1321; | 54 | 246,241; | 279,073; |
| 10 | 5; | 41; | 34 | 137.953; | 26,317; | 58 | 107,367,629; | 536,903,681; |
| 14 | 113; | 29; | 38 | 229.457; | 525,313; | 62 | 5581.384,773; | 8681.49,477; |
| 18 | 37; | 109; | 42 | 14,449; | 1429; | 66 | 312,709; | 4,327,489; |
| 22 | 397; | 2113; | 46 | 277.30,269; | 1013.1657; | 70 | 47,392,381; | 7,416,361; |

* The present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, Stuttgart, 1856, contains (pp. 42-53) a table of "special prime divisors" of $(2^n-1)$, up to $p < 5000$. The author has a MS. extension (incomplete) to a much higher limit.

‡ Adapted from the Table at pp. 9, 10 of Lucas's *Memoir* (of 1879) above quoted. The Table is there (p. 7) ascribed to M. Le Lasseur (the results as far as $n = 62$ had been given by M. Landry in his *Décomposition des nombres* $(2^n \pm 1)$ *en leurs facteurs premiers*, Paris, 1869). It has been necessary to *recompute* it entirely, as Lucas does not show the factors of $L$, $M$ or $S$, $T$ separately, and shows only the *special* divisors for each value of $n$, (*i.e.*, divisors not belonging to any *lower* values of $n$); thus Lucas's Table omits the factor 5 when $n = 10$ and 50, and the factor 13 when $n = 78$. Only one mistake was discovered in the part of Lucas's Table hereby verified, viz., the omission of the factor 12,112,549 when $n = 82$. The large factor of $M$ when $n = 146$ was found in Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11; it is not in the 1879 Memoir.

§ The determination of most of the large primes (of twelve to sixteen figures) is ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 11. The author's work has verified the large primes up to 36 millions.

FACTORS OF THE REDUCED A.P.F. OF $(2^n+1)$, [when $n = 4\nu+2$]—*continued.*

| $n$ | $L$ or $S$ | $M$ or $T$ |
|---|---|---|
| 74 | 149 . 184,481,113 ; | 593 . 231,769,777 ; |
| 78 | 13 . 313 . 1249 ; | 3121 . 21,841 ; |
| 82 | 181,549 . 12,112,549 ; | 10169 . 43,249,589 ; |
| 86 | 1,759,217,765,581 ; | 173 . 101,653 . 500,177 ; |
| 90 | 181 . 54,001 ; | 29,249,661 ; |
| 94 | 140,737,471,578,113 ; | 3761 . 7,484,047,069 ; |
| 98 | 4,981,857,697,937 ; | 197 . 19,707,683,773 ; |
| 102 | 409 . 3061 . 13669 ; | 1,326,700,741 ; |
| 110 | 415,878,438,361 ; | 3,630,105,520,141 ; |
| 114 | 131,101 . 160,969 ; | 275,415,303,169 ; |
| 118 | 1181 . 3541 . 157,649 . 174,877 ; | |
| 122 | 733 . 1709 . 368,140,581,013 ; | |
| 126 | 118,750,098,349 ; | 40,388,473,189 ; |
| 130 | 108,140,989,558,681 ; | 521 . 51,481 . 34,110,701 ; |
| 138 | 70,334,392,823,809 ; | 5,415,624,023,749 ; |
| 146 | | 293 . 9929 . 649,301,712,182,209 ; |
| 150 | 63,901 . 13,334,701 ; | 1201 . 1,182,468,601 ; |
| 174 | 349 . 29,581 . 27,920,807,689 ; | |
| 210 | 1,041,815,865,690,181 ; | 421 . 146,919,792,181 ; |

The following small Table gives "special divisors" ($p$) of such of Fermat's Numbers of form $(2^{4\nu+2}+1)$ as are omitted from the previous table, within Lucas's limit $n = 210$. The discovery of a few more such "special divisors" would enable the factorisation of some of these numbers to be completed.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ ... | 106 ; | 134 ; | 142 ; | 154 ; | 158 ; | 162 ; | 166 ; | 170 ; | 178 ; | 182 ; | | |
| $p$ ... | . ; | 269 ; | 569 ; | 8317 ; | 317 ; | . ; | 997 ; | 1021 . 4421 ; | 1069 ; | 1093 . 4733 ; | | |
| $n$ ... | 186 ; | 190 ; | 194 ; | 198 ; | 202 ; | 206 | | | | | | |
| $p$ ... | 373 ; | 761 . 54,721 ; | 389 . 3881 . 4657 . 5821 ; | . ; | 809 ; | . ; | | | | | | |

18. *Fermat's Numbers.* (Higher Bases).—Writing in Formulæ (3,3$a$)

$$X = 1, \quad Y = 2y^2, \quad \text{and} \quad 2n = 2\omega = 2(2\nu+1),$$

$\nu$ being any integer, then

$$\mathfrak{Y} = 1+Y^{2n} = 1+(2y^2)^{4\nu+2} = 1^4+2^2.\eta^4, \text{ where } \eta = 2^\nu.y^{2\nu+1}. \quad (57)$$

Here $\mathfrak{Y}$ is a *Bin-Aurifeuillian*, so that every such number, and its A.P.F., viz.,

$$\mathfrak{A} \div N = (1+Y^{2n}) \div (1+Y^2), \quad \text{where} \quad Y = 2y^2,$$

and also its "Reduced A.P.F." (when further reducible), admit of resolution into the two factors $L$, $M$, or $S$, $T$, of Arts. **8**, **16**.

The existing Factor-Tables (extending to 9 million) enable the *complete* factorisation (*i.e.*, into *prime* factors) of such numbers to be effected up to the following limits :—

$$2n = \quad 2 \quad ; \quad 6 \quad ; \quad 10 \quad ; \quad 14 \quad ; \quad 18 \quad ; \quad 22 \quad ; \text{ 26 to 66}$$

$$Y = 2y^2 = 2 \cdot 2121^2 \; ; \; 2 \cdot 38^2 \; ; \; 2 \cdot 5^2 \; ; \; 2 \cdot 2^2 \; ; \; 2 \cdot 2^2 \; ; \; 2 \cdot 2^2 \; ; \quad 2 \cdot 1^2$$

$$= 8{,}997{,}282 \; ; \; 2888 \; ; \; 50 \; ; \; 18 \; ; \; 8 \; ; \; 8 \; ; \quad 2$$

and, by the aid of "special divisors," *† it can be carried far beyond these limits.

*Examples.*—The following Table ‡ shows the *complete* factorisation (*i.e.* into *prime* factors) of the following ($Y = 2y^2$ throughout) :

$N = 1 + Y^2$ to $Y = 2 \cdot 40^2 = 3200$, §

$N = (1 + Y^6) \div (1 + Y^2)$ to $Y = 2 \cdot 39^2 = 3042$,

$N = (1 + Y_{2\omega}) \div (1 + Y^2)$ when $2\omega = 10, 14, 22, 26$ to various high limits,

$N = (1 + Y^{18}) \div (1 + Y^6)$.

The factorisation of the Aurifeuillian factors $L$, $M$ is shown separately, and *prime*¶ factors only are shown ; the semicolon (;) in the Tables of $2\omega = 10, 14, 18, 22, 26$, indicates that the factorisation is *complete*.

---

* Reuschle's *Tafeln complexer Primzahlen*, Berlin, 1875, give (p. 467 *et seq.*) the *least* values of $Y$ for which $(1 + Y^{2\omega})$ is divisible by any *given* prime $p < 1000$.

† The author has prepared MS. Tables (which it is hoped to publish hereafter) giving the least value of $Y$ for which $N = (1 + Y^n)$ is divisible by any *given* prime as follows :—

$n = 5$, 10, and 15, $p < 10{,}000$ ; $n = 7$, $p < 4{,}000$ ; $n = 11$, $p < 3{,}700$ ;

by which all prime divisors within these limits can be detected at once for those powers ($\omega$) and for the doubles thereof (*i.e.* $2\omega$). Most of the large factors ($> 1000$) of the larger numbers (*i.e.*, when $L$ or $M > 9$ million) here factorised were detected by these MS. tables.

‡ Specially computed by the author for this Paper.

§ The highest number of this kind ($N = 1 + Y^2$) factorisable by the existing Factor-Tables is

$$1 + 8{,}997{,}282^2 = 1^4 + 2^2 \cdot 2121^4 = (8{,}993{,}041)\,(25 \cdot 13 \cdot 27697),$$

and, with the aid of "special divisors," this could be pushed much further. Euler, in *Commentationes Arithmeticæ* (Petropol., 1849), p. 369 *et seq.*, gives a table of the small prime factors of $N = 1 + Y^2$, as far as $Y = 1500$ only : this shows the great advance in factorisation since Euler's time. Note that these numbers ($N = 1 + Y^2$) form a *Bin-Aurifeuillian Chain*, (Art. 20), *i.e.* the $M$ of each number = the $L$ of the next higher number.

¶ The high primes ($> 9$ million) occurring when $Y = 32$, 128, 512 are given on authority of Lucas's Table (quoted in footnote ‡ to Art. 17). Those occurring among the factors of $(1 + Y^{10})$ were detected, when $< 100$ million, by the author's MS. Tables above mentioned ; and, when $> 100$ million, by the method developed in Mr. F. W. Lawrence's paper on "Determination of certain Primes" in Vol. XXVIII. of these *Proceedings* (with some improvements communicated to the author in MS. by Mr. Bickmore) : this method is *most* effective in factorising and in detecting the prime character of numbers up to about 12 figures, when *given* as factors of a *known* number $(a^n \mp 1)$ where $a$ is *small* and $n$ large.

FACTORISATION OF $N = (1 + Y^{2\omega}) \div (1 + Y^2)$, [when $Y = 2y^2$].

| $y$ | $Y = 2y^2$ | $(1+Y^2)$ | | $(1+Y^6) \div (1+Y^2)$ | | $(1+Y^{10}) \div (1+Y^2)$ | |
|---|---|---|---|---|---|---|---|
| | | $L$ | $M$ | $L$ | $M$ | $L$ | $M$ |
| 1 | 2 | 1 | 5 | 1 | 13 | 5; | 41; |
| 2 | 8 | 5 | 13 | 37 | 109 | 41.61; | 5.1321; |
| 3 | 18 | 13 | 25 | 229 | 457 | 5.15,101; | 145,501; |
| 4 | 32 | 25 | 41 | 13.61 | 1321 | 101.8101; | 5.268,501; |
| 5 | 50 | 41 | 61 | 13.157 | 3061 | 5,122,541; | 7,622,561; |
| 6 | 72 | 61 | 5.17 | 13.337 | 6133 | 5.4,552,601; | 31,720,981; |
| 7 | 98 | 5.17 | 113 | 8317 | 13.853 | 61.1,311,341; | 5.701.30,341; |
| 8 | 128 | 113 | 5.29 | 14,449 | 13.1429 | 5.47,392,381; | 41.7,416,361; |
| 9 | 162 | 5.29 | 181 | 23,473 | 13.37.61 | 421.1,464,241; | 5.41.3,753,661; |
| 10 | 200 | 181 | 13.17 | 97.373 | 44,221 | 1447,960,181; | 41.43,120,981; |
| 11 | 242 | 13.17 | 5.53 | 193.277 | 64153 | 5.4561.137341; | 881.4,262,941; |
| 12 | 288 | 5.53 | 313 | 13.5869 | 37.2437 | | 5.181.281.29,401; |
| 13 | 338 | 313 | 5.73 | 105,769 | 123,397 | 5. | 61.881.262,261; |
| 14 | 392 | 5.73 | 421 | 143,053 | 13.12697 | | 5. |
| 15 | 450 | 421 | 13.37 | 189,421 | 216,481 | 941.40,768,781; | 61.701.1,025,021; |
| 16 | 512 | 13.37 | 5.109 | 246,241 | 279,073 | 5.181.1321.54,001; | 41.61.29,247,661; |
| 17 | 578 | 5.109 | 613 | 13.24229 | 37.61.157 | 181.581,433,521; | 5. |
| 18 | 648 | 613 | 5.137 | 13.30553 | 443,917 | 5.61.761.718,621; | 41. |
| 19 | 722 | 5.137 | 761 | 13.109.349 | 549,481 | 881.15401.19001; | 5.821.69,772,181; |
| 20 | 800 | 761 | 29.29 | 37.16453 | 13.73.709 | 41. | |
| 21 | 882 | 29.29 | 25.37 | 741,721 | 13.62761 | 5. | 41.61.253,765,901; |
| 22 | 968 | 25.37 | 1013 | 895,357 | 13.241.313 | | 5.61.8821.341,521; |
| 23 | 1058 | 1013 | 5.13.17 | 61.17569 | 1,169,137 | | |
| 24 | 1152 | 5.13.17 | 1201 | 1,272,913 | 829.1669 | | |
| 25 | 1250 | 1201 | 1301 | 13.37.3121 | 1,626,301 | | |
| 26 | 1352 | 1301 | 5.281 | 97.18133 | 1069.1777 | | |
| 27 | 1458 | 5.281 | 17.89 | 109.18793 | 13.169.693 | | |
| 28 | 1568 | 17.89 | 125.13 | 37.97.661 | 337.7561 | | |
| 29 | 1682 | 125.13 | 1741 | 73.37,441 | 2,928,421 | | |
| 30 | 1800 | 1741 | 1861 | 13.193.1249 | 3,349,861 | | |
| 31 | 1922 | 1861 | 5.397 | 13.73.3769 | 1597.2389 | | |
| 32 | 2048 | 5.397 | 2113 | 13.312,709 | 4,327,489 | | |
| 33 | 2178 | 2113 | 5.449 | 1009.4561 | 13.13.28933 | | |
| 34 | 2312 | 5.449 | 2381 | 73.97.733 | 13.423,457 | | |
| 35 | 2450 | 2381 | 2521 | 313.18,637 | 13.37.12,841 | | |
| 36 | 2592 | 2521 | 5.13.41 | 6,534,361 | 6,907,753 | | |
| 37 | 2738 | 5.13.41 | 29.97 | 7,296,769 | 7,702,069 | | |
| 38 | 2888 | 29.97 | 5.593 | 13.241.2593 | 61.229.613 | | |
| 39 | 3042 | 5.593 | 3121 | 37.243,769 | 73.130,057 | | |
| 40 | 3200 | 3121 | 3281 | 9,987,121 | 13.807,637 | | |

[*Note*, that the highest number $(722^{10} + 1)$, here *completely* factorised, contains 29 figures.]

FACTORISATION OF $N = (1 + Y^{20}) \div (1 + Y^2)$, [when $Y = 2y^2$]—*continued.*

| $y$ | $Y = 2y^2$ | $(1 + Y^{14}) \div (1 + Y^2)$ | | $(1 + Y^{18}) \div (1 + Y^6)$ | |
|---|---|---|---|---|---|
| | | $L$ | $M$ | $L$ | $M$ |
| 1 | 2 | 113 ; | 29 ; | 37 ; | 109 ; |
| 2 | 8 | 29.14449 ; | 113.1429 ; | 246241 ; | 279073 |
| 3 | 18 | 29.1623823 ; | 1373.17837 ; | | 37.37.25309 ; |
| 4 | 32 | 29.47392381 ; | 113.7416361 ; | 109.181.54001 ; | 37.29247661 ; |
| 5 | 50 | 29. | | 37.73.5761801 ; | |
| 6 | 72 | 29.1597.3550541; | | 37.181. | |
| 7 | 98 | | 29.953 | 37.73 | |
| 8 | 128 | 4981857697937 ; | 197.19707683773 ; | 37.118750098349 ; | 109.40388473189 ; |

| $y$ | $Y = 2y^2$ | $(1 + Y^{22}) \div (1 + Y^2)$ | |
|---|---|---|---|
| | | $L$ | $M$ |
| 1 | 2 | 397 ; | 2113 ; |
| 2 | 8 | 2113 . 312,709 ; | 397 . 4,327,489 ; |
| 3 | 18 | | 89 . |
| 4 | 32 | 2113 . 415,878,438,361 ; | 397 . 3,630,105,520,141 ; |
| 5 | 50 | 881 . | |

| $y$ | $Y = 2y^2$ | $(1 + Y^{26}) \div (1 + Y^2)$ | |
|---|---|---|---|
| | | $L$ | $M$ |
| 1 | 2 | 1613 ; | 53 . 157 ; |
| 2 | 8 | 13 . 53 . 157 . 313 . 1249 ; | 1613 . 3121 . 21,841 ; |
| 3 | 18 | 53 . | |
| 4 | 32 | 53 . 157 . 108,140,989,558,681 ; | 521 . 1613 . 51,481 . 34,110,701 ; |
| 5 | 50 | | 53 . |

**19.** *Common Aurifeuillian Factors of Bin-Aurifeuillians.*—It will now be shown that every number $F$ of form $(a^2 + b^2)$ is an Aurifeuillian Factor of *four* different Bin-Aurifeuillians, say $N_1, N_2, N_3, N_4$. It suffices to take

$$\left.\begin{aligned} N_1 &= (b \sim a)^4 + 4a^4, \quad N_2 = (b + a)^4 + 4a^4 \\ N_3 &= (a \sim b)^4 + 4b^4, \quad N_4 = (a + b)^4 + 4b^4 \end{aligned}\right\}, \tag{58}$$

and, resolving $N_1, N_2, N_3, N_4$ into their Aurifeuillian Factors $(L, M)$,

$$\left.\begin{aligned} N_1 &= \{(b - 2a)^2 + a^2\}(b^2 + a^2), \quad N_2 = (b^2 + a^2)\{(b + 2a)^2 + a^2\} \\ N_3 &= \{(a - 2b)^2 + b^2\}(a^2 + b^2), \quad N_4 = (a^2 + b^2)\{(a + 2b)^2 + b^2\} \end{aligned}\right\}, \tag{59}$$

the common factor $(a^2+b^2)$ is seen to be an Aurifeuillian Factor of *each* of them. Hence many simple relations may be derived between such a set of four Bin-Aurifeuillians, *e.g.*,

$$\left.\begin{aligned} N_1+N_4 &= N_2+N_3 = \quad 6\,(a^2+b^2)^2 = 6F^2 \\ N_2-N_1 &= N_4-N_3 = 8ab\,(a^2+b^2) \;= 8abF \\ N_1-N_3 &= N_2-N_4 = \quad 4\,(a^4-b^4) \end{aligned}\right\}. \qquad (60)$$

In order that the four Bin-Aurifeuillians should fulfil the limitations of Art. 8, *i.e.*, that each $x$ should be *odd* and prime to the corresponding $y$, it suffices that a, b should be relative primes, one *odd*, one *even*.

The above theorem is true whether $F$ be prime, or composite; but if $F$ be a composite number of form $(a^2+b^2)$, it may be considerably extended. For every factor of $F$ must necessarily be of form $(\alpha^2+\beta^2)$; and $F$ will be expressible in several ways as a sum of squares $(a^2+b^2)$; and the above theorem will be true for *each* of these ways: so that, if $r$ be the number of different ways in which $F$ is expressible as a sum of squares, then $F$ will be a common Aurifeuillian Factor of $4r$ different Bin-Aurifeuillians; and, if $F=a^2+b^2$ be any one of these partitions, the Results (58) give the values of the four Bin-Aurifeuillians corresponding.

Again, since every Bin-Aurifeuillian $(N)$ is the product of two factors $(L, M)$, each of which is of form $(a^2+b^2)$, it follows from the above that every Bin-Aurifeuillian $(N)$ is a common Aurifeuillian Factor of *eight* different Bin-Aurifeuillians. Thus, if

$$N = x^4+4y^4$$

be the base Bin-Aurifeuillian, then, taking its *two-fold* quadratic partition $(a'^2+b'^2)$, $(a''^2+b''^2)$ from $(19a)$, and substituting these for a, b, in the four formulæ (58), the *eight* Bin-Aurifeuillians (say $N_1$, $N_2$, $N_3$, ..., $N_8$) which contain $N$ as common Aurifeuillian Factor ($L$ or $M$) are obtained, viz.,

$$\left.\begin{aligned} N_1 &= (2y^2-x^2)^4+4\,(x^2)^4, \quad & N_2 &= (2y^2+x^2)^4+4\,(x^2)^4 \\ N_3 &= (x^2-2y^2)^4+4\,(2y^2)^4, \quad & N_4 &= (x^2+2y^2)^4+4\,(2y^2)^4 \\ N_5 &= (2xy-x^2+2y^2)^4+4\,(x^2-2y^2)^4 \\ N_6 &= (2xy+x^2-2y^2)^4+4\,(x^2-2y^2)^4 \\ N_7 &= (x^2-2y^2-2xy)^4+4\,(2xy)^4 \\ N_8 &= (x^2-2y^2+2xy)^4+4\,(2xy)^4 \end{aligned}\right\}, \qquad (61)$$

and writing, as usual, $N_r = L_r M_r$, the *common* Aurifeuillian Factor $(N)$ is $M_1 = L_2 = M_3 = L_4 = M_5 = L_6 = M_7 = L_8 = N$, (61*a*)

and the other Aurifeuillian Factors are, (see Art. 8),

$$\left.\begin{aligned} L_1 &= x^4+4(y^2\sim x^2)^2, & M_2 &= x^4+4(x^2+y^2)^2 \\ L_3 &= (x^2\sim 4y^2)^2+4y^4, & M_4 &= (x^2+4y^2)^2+4y^4 \\ L_5 &= (x^2-2y^2)^2+4(xy-x^2+2y^2)^2, & M_6 &= (x^2-2y^2)^2+(xy+x^2-2y^2)^2 \\ L_7 &= (x^2-2y^2-4xy)^2+4(xy)^2, & M_8 &= (x^2-2y^2+4xy)^2+(2xy)^2 \end{aligned}\right\}. \quad (61b)$$

Cor.—The above results (61, 61*b*) furnish an easy solution of the problem of Art. 13: for the four Bin-Aurifeuillians $N_1$, $N_2$, $N_3$, $N_4$ each contain a Bin-Aurifeuillian $(N)$ as *one* of their Aurifeuillian Factors, and the *other* Aurifeuillian Factor (*i.e.*, $L_1$, $M_2$, $L_3$, $M_4$) of any *one* of them becomes itself a Bin-Aurifeuillian by determining $x, y$ so that

one of $(y^2\sim x^2)$, $(y^2+x^2)$, $(x^2\sim 4y^2)$, $(x^2+4y^2) = z^2$ (a square); (61*c*)

and these are the conditions of some one or other of the Cases i, ii, iii, iv of Art. 13a, (the $x, y, N$ of this Article corresponding to the $x_1, y_1, N_1$ or $x_2, y_2, N_2$ of Art. 13a, with other interchanges of notation, as necessary).

*Examples.* — The following Table shows the *quartets* of Bin-Aurifeuillians $(N_1, N_2, N_3, N_4)$ which contain the common Aurifeuillian Factors $F = 5, 13, 17, 25, 37$; and also the *octet* of Bin-Aurifeuillians which contain the common Bin-Aurifeuillian Factor $F = 65$.

| Factor. | Aurifeuillian Parts. | | | | Bin-Aurifeuillians. | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ ; $b$ ; $F$ | $x_1$ ; $y_1$ | $x_2$ ; $y_2$ | $x_3$ ; $y_3$ | $x_4$ ; $y_4$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
| 1 ; 2 ; 5 | 1 ; 1 | 3 ; 1 | 1 ; 2 | 3 ; 2 | 5 | 85 | 65 | 145 |
| 3 ; 2 ; 13 | 1 ; 3 | 5 ; 3 | 1 ; 2 | 5 ; 2 | 325 | 949 | 65 | 689 |
| 1 ; 4 ; 17 | 3 ; 1 | 5 ; 1 | 3 ; 4 | 5 ; 4 | 85 | 629 | 1105 | 1649 |
| 3 ; 4 ; 25 | 1 ; 3 | 7 ; 3 | 1 ; 4 | 7 ; 4 | 325 | 2725 | 1025 | 3425 |
| 1 ; 6 ; 37 | 5 ; 1 | 7 ; 1 | 5 ; 6 | 7 ; 6 | 629 | 2405 | 5809 | 7585 |
| 1 ; 8 ; 65 | 7 ; 1 | 9 ; 1 | 7 ; 8 | 9 ; 8 | 2405 | 6565 | 18785 | 22945 |
| 7 ; 4 ; 65 | 3 ; 7 | 11 ; 7 | 3 ; 4 | 11 ; 4 | 9685 | 24245 | 1105 | 15665 |

20. *Bin-Aurifeuillian Chains.*—Let a series of Bin-Aurifeuillians $N_1, N_2, N_3, \ldots, N_r$ be taken, so that

$$N_r = L_r M_r = x_r^4+4y_r^4, \text{ (for each value of } r), \quad (62)$$

and let the parts $(x_r, y_r)$ of the successive functions be taken, so that

*Case* i.

$$y_1 = y_2 = y_3 = \ldots y_r,$$
$$x_2 = x_1+2y_1,\ x_3 = x_1+4y_1,\ x_4 = x_1+6y_1, \text{ \&c. } \ldots x_r = x_{r-1}+2y_1.$$

*Case* ii.

$x_1 = x_2 = x_3 = \ldots x_r,$

$y_2 = x_1 + y_1,\ y_3 = 2x_1 + y_1,\ y_4 = 3x_1 + y_1,$ &c. $\ldots\ y_r = x_1 + y_{r-1}.$

Then, in Case i,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = (x_{r-1} + y_1)^2 + y_1^2,$$
$$L_r \quad = \quad (x_r - y_r)^2 \quad + y_r^2 \quad = (x_{r-1} + y_1)^2 + y_1^2.$$

And, in Case ii,

$$M_{r-1} = (x_{r-1} + y_{r-1})^2 + y_{r-1}^2 = y_r^2 + y_{r-1}^2,$$
$$L_r \quad = \quad (x_r - y_r)^2 \quad + y_r^2 \quad = y_{r-1}^2 + y_r^2.$$

Then, in each Case, $$M_{r-1} = L_r, \tag{63}$$

or, in other words, every two *successive* members $N_{r-1}$, $N_r$ have a *common* Aurifeuillian Factor. Such a series may be called a *Bin-Aurifeuillian Chain*, and has many interesting properties: thus

$$N_r : N_{r-1} = L_r M_r : L_{r-1} M_{r-1} = M_r : L_{r-1}. \tag{64}$$

Similarly, $$N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3};$$

Hence in every *successive* quartet $(N_{r-3}, N_{r-2}, N_{r-1}, N_r)$

$$N_r N_{r-2} : N_{r-1} N_{r-3} = M_r : L_{r-3}. \tag{64a}$$

Similarly, giving to $r$ every integer value from 1 to $2r$,

$$\frac{N_{2r} N_{2r-2} \ldots N_6 N_4 N_2}{N_{2r-1} N_{2r-3} \ldots N_5 N_3 N_1} = \frac{M_{2r}}{L_1}. \tag{64b}$$

*Ex.*—See the Table of Factorisation of $(1 + Y^2)$, where $Y = 2y^2$, (Art. 18); these numbers $N = (1^4 + 4y^4)$ will be seen to form a *Bin-Aurifeuillian Chain.*

## Chapter III., Trin-Aurifeuillians.

21. *Trin-Aurifeuillians.*—When, in Results (4; 4*a*, *b*; 6*a*, *b*), $n_1 = 1$, $n = n_2 = 3$, $X = x^2$, $Y = 3y^2$, the function $N$ becomes the *Trin-Aurifeuillian*,

$$\left.\begin{aligned} N &= (x^6 + 3^3 y^6) \div (x^2 + 3y^2) = x^4 - 3x^2 y^2 + 9y^4 \\ x &\text{ being } \textit{prime}^* \text{ to } 3y, \text{ (see Art. 2)} \end{aligned}\right\}. \tag{65}$$

It has *four* quadratic partitions of form $(A^2 + 3B^2)$, *one* $(N^\circ)$ being

* Hereby $N$ will be of form $(6f + 1)$.

*general*, and *three* ($N'$, $N''$, $N'''$) *alternative*, see Art. **5**, viz.,

$$N^\circ = (x^2-3y^2)^2+3\,(xy)^2, \text{ in } all \text{ cases,} \tag{66°}$$

$$N' = (x^2-\tfrac{3}{2}y^2)^2+3\,(\tfrac{3}{2}y^2)^2, \text{ when } y \text{ is } even, \tag{66′}$$

$$N'' = (\tfrac{1}{2}x^2-3y^2)^2+3\,(\tfrac{1}{2}x^2)^2, \text{ when } x \text{ is } even, \tag{66″}$$

$$N''' = \left(\frac{x^2+3y^2}{2}\right)^2+3\left(\frac{x^2-3y^2}{2}\right)^2, \text{ when } x, y \text{ } both \text{ } odd. \tag{66‴}$$

Its two Aurifeuillian Factors ($L$, $M$) are

$$L = x^2-3xy+3y^2, \quad M = x^2+3xy+3y^2, \tag{67}$$

and are *each* expressible in *three* (alternative) ways

$$L' = (x-\tfrac{3}{2}y)^2+3\,(\tfrac{1}{2}y)^2; \qquad M' = (x+\tfrac{3}{2}y)^2+3\,(\tfrac{1}{2}y)^2; \tag{67′}$$

$$L'' = (\tfrac{1}{2}x)^2+3\,(\tfrac{1}{2}x-y)^2; \qquad M'' = (\tfrac{1}{2}x)^2+3\,(\tfrac{1}{2}x+y)^2; \tag{67″}$$

$$L''' = \left(\frac{x-3y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2; \quad M''' = \left(\frac{x+3y}{2}\right)^2+3\left(\frac{x+y}{2}\right)^2. \tag{67‴}$$

Here $L'$, $M'$, $N'$ apply when $y$ is *even*; $L''$, $M''$, $N''$ when $x$ is *even*;

$L'''$, $M'''$, $N'''$ apply when *both* $x$, $y$ are *odd*.

Also, by conformal multiplication,

$$L'M' \text{ yields both } N', N^\circ; \quad L''M'' \text{ yields both } N'', N^\circ;$$

$$L'''M''' \text{ yields both } N''', N^\circ. \tag{68}$$

Thus, when the Aurifeuillian Factors ($L$, $M$) of any *particular* Trin-Aurifeuillian ($N$) are *both prime*, they have *only one* quadratic partition each, of form $(A^2+3B^2)$; and the Trin-Aurifeuillian ($N$) has in that case only two such partitions (one of which is always $N^\circ$). If, however, either or both of $L$, $M$ be *composite*, all their factors must necessarily be of form $(A^2+3B^2)$; and $L$ or $M$ (or both $L$, $M$), and therefore also $N$, will then be polymorphs of form $(A^2+3B^2)$: but these partitions will, as a rule, be only arithmetically (not algebraically) possible; and are therefore not of so much interest as those given. In conformity with the remark at end of Art. **5**, the three analogous forms ($N'$, $N''$, $N'''$), or ($L'$, $L''$, $L'''$), or ($M'$, $M''$, $M'''$), although arithmetically distinct, differ from each other only by the unit-factor $\frac{1}{4}\,(1^2+3.1^2)$, so that *any two* of a trio can be derived from the third by conformal multiplication by that unit-form: *e.g.*,

$\frac{1}{4}\,(1^2+3.1^2)\,N'$ yields both $N''$, $N'''$, &c., &c.

**22.** *Trin-Ant-Aurifeuillians.*—When, in Results (4; 4*a*, *b*; 6*a*, *b*; 8*a*, *b*), $n_1 = 1$, $n = n_2 = 3$, $X = x^2$, $Y = -3y^2$, the function $N$ becomes the *Trin-Ant-Aurifeuillian.**

$$\left.\begin{array}{l} N = (x^6 - 3^3.y^6) \div (x^2 - 3y^2) = x^4 + 3x^2y^2 + 9y^4 \\ x \text{ being } \textit{prime}\dagger \text{ to } 3y, \text{ (see Art. 2)} \end{array}\right\}. \qquad (69)$$

It has the following quadratic partitions: one of form $(a^2+b^2)$, one $(N^\circ)$ of form‡ $(C^2-3D^2)$, and three $(N', N'', N''')$ *alternative* of form $A^2+3B^2$.

$$N = (x^2-3y^2)^2+(3xy)^2, \qquad (70)$$

$$N^\circ = (x^2+3y^2)^2-3\,(xy)^2, \qquad (70^\circ)$$

$$N' = (x^2+\tfrac{3}{2}y^2)^2+3\,(\tfrac{3}{2}y^2)^2, \text{ when } y \text{ is } \textit{even}, \qquad (70')$$

$$N'' = (\tfrac{1}{2}x^2+3y^2)^2+3\,(\tfrac{1}{2}x^2)^2, \text{ when } x \text{ is } \textit{even}, \qquad (70'')$$

$$N''' = \left(\frac{x^2-3y^2}{2}\right)^2+3\left(\frac{x^2+3y^2}{2}\right)^2, \text{ when } x, y \textit{ both odd}, \qquad (70''')$$

The above forms correspond to the forms $(LM)$, $N^\circ$, $N'$, $N''$, $N'''$ of the Trin-Aurifeuillian (Art. **21**) respectively; and the conjugate forms are interconvertible by changing $\pm y^2$ into $\mp y^2$.

**23.** *Fermat's Divisors.*—Under Art. **6a, b** the conditions of divisibility by 7 and 13 are

Every Trin-Aurifeuillian is divisible by 7 when (and only when) neither of $x$, $y$, $(x^2+3y^2)$ contains 7; also $N \equiv 3x^4$, (mod 7), when $x^2+3y^2 \equiv 0$, (mod 7), provided neither of $x$, $y$ contain 7. (71*a*)

Also $\left.\begin{array}{l} L \text{ contains 7 when } (x+y) \text{ or } (2x-y) = 7k \\ M \text{ contains 7 when } (x-y) \text{ or } (2x+y) = 7k \end{array}\right\}$. (71*b*)

Every Trin-Aurifeuillian is divisible by 13 when $xy$ is a quadratic non-residue of 13, and at the same time neither of $x$, $y$, $(x^2+3y^2)$ contains 13. (71*c*)

Every Trin-Ant-Aurifeuillian is divisible by 13 when $xy$ is a quadratic residue of 13, and at the same time neither of $x$, $y$, $(x^2-3y^2)$ contains 13. (71*d*)

[It is worth note that both 7 and 13 are Trin-Aurifeuillians; for $7 = (1^6+3^3.1^6) \div (1^2+3.1^2)$, and $13 = (2^6+3^3.1^6) \div (2^2+3.1^2)$; whilst 13 is also a Trin-Ant-Aurifeuillian, for $13 = (1^6-3^3.1^6) \div (1^2-3.1^2)$.]

Again, since the square and the sixth power of every number end in

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* This modification of the Trin-Aurifeuillian is due to Mr. Bickmore (see Art. **4**).

† Hereby $N$ will be of form $(12f+1)$.

‡ See footnote * to Art. **16a**.

the *same digit*, which is always one of 0, 1, 4, 5, 6, 9, it follows that

All Trin-Aurifeuillians and all Trin-Ant-Aurifeuillians end in one of the digits 1, 3, 7, 9. (71*e*)

When one of them ($N$) ends in 3 or 7, the conjugate ($N'$) ends in 7 or 3, so that $N+N' \equiv 0$, (mod 10); also, when one of them ends in 1 or 9, then either $N-N'$ or $N+N' \equiv 0$, (mod 10). (71*f*)

Again, from $66°-66'''$, and $70°-70'''$, it follows that—

Every prime factor ($F$) of a Trin-Aurifeuillian or Trin-Ant-Aurifeuillian is of form $(6f+1)$, $(12f+1)$ respectively. (71*g*)

[*Note*, that properties (71*af*) refer only to the simple functions $N = (x^6 \pm 3^3y^6) \div (x^2 \pm 3y^2)$, and not to the Quotient-Functions $(\mathfrak{N} \div N)$ treated of in Art. **26**, **26a**, **27**.]

**24.** *Trin-Aurifeuillian Dimorphs.*—The question now arises whether the Trin-Aurifeuillian partition of a number ($N$) is *unique*, or whether it may be *dimorphic*.

Let $$N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_1^2y_1^2 + 9y_2^4 \tag{72}$$

be, if possible, two *different* Trin-Aurifeuillian forms of the *same* number $N$; so that $$N = N_1 = N_2. \tag{73}$$

The *general* solution of this seems difficult, but a limited solution may be obtained by identifying two of the *different* quadratic partitions of type $(A^2+3B^2)$ of which $N_1$, $N_2$ are capable, Art. **21**; say

$$N_1 = N_1^\circ; \quad N_2 = N_2' \text{ or } N_2'' \text{ or } N_2''' \text{ (alternatively).} \tag{73a}$$

Thus *three* trial systems arise—

$$\text{i. } N_1^\circ = N_2'; \quad \text{ii. } N_1^\circ = N_2''; \quad \text{iii. } N_1^\circ = N_2'''. \tag{73b}$$

Identifying the *like* quadratic parts $(A_1, A_2)$, $(B_1, B_2)$ separately, each system gives two equations—

$$\text{i. } A_1^\circ = A_2', B_1^\circ = B_2'; \quad \text{ii. } A_1^\circ = A_2'', B_1 = B_2''; \quad \text{iii. } A_1^\circ = B_2''', B_1^\circ = B_2'''. \tag{73c}$$

Referring to Art. **21**, the *first** system gives

$$x_1^2 - 3y_1^2 = x_2^2 - \tfrac{3}{2}y_2^2, \quad x_1y_1 = \tfrac{3}{2}y_2^2, \tag{74}$$

whence $$(x_1 + \tfrac{1}{2}y_1)^2 - 13\left(\frac{y_1}{2}\right)^2 = x_2^2. \tag{74a}$$

These, together with the working conditions, involve

$y_1$, $y_2$ both even; $x_1$, $x_2$ both odd, and not multiples of 3. (74*b*)

* Systems ii, iii appear to involve impossibilities.

The Diophantine (74*a*) has two *general* solutions in terms of two arbitrary quantities $m$, $n$, (to be taken relative primes):—

$$\left.\begin{array}{ll} \text{ia.} & x_1+\tfrac{1}{2}y_1 = m^2+13n^2, \quad \tfrac{1}{2}y_1 = 2mn, \quad x_2 = m^2-13n^2 \\ \text{whence} & x_1 = m^2-2mn+13n^2 = (m-n)^2+3\,(2n)^2 \\ & \qquad \text{where } m, n \text{ are one } \textit{odd}, \text{ one } \textit{even} \end{array}\right\}. \qquad (75)$$

$$\left.\begin{array}{ll} \text{ib.} & x_1+\tfrac{1}{2}y_1 = \tfrac{1}{2}\,(m^2+13n^2), \quad \tfrac{1}{2}y_1 = mn, \quad x_2 = m^2-13n^2 \\ \text{whence} & x_1 = \tfrac{1}{2}\,(m^2-2mn+13n^2) = \tfrac{1}{2}\left\{(m-n)^2+3\,(2n)^2\right\} \\ & m, n \text{ both } \textit{odd}\,; \text{ giving } x_1 \text{ even}\,; \text{ solution } \textit{rejected} \end{array}\right\}. \qquad (75)$$

Again, the condition $x_1 y_1 = \frac{3}{2}y_2^2$, together with (74*b*), involves

$$x_1 = \xi^2, \quad y_1 = 6\eta^2, \quad y_2 = 2\xi\eta. \qquad (75a)$$

The value of $y_1$ ($= 4mn$, or $2mn$) shows that either $m$ or $n$ must be a multiple of 3, and, finally, that $m$, $n$ must have *one* of the forms

$$\text{ia.} \left\{\begin{array}{l} m = 3\mu^2,\ \mu^2,\ 2\mu^2,\ 6\mu^2 \\ n = 2\nu^2,\ 6\nu^2,\ 3\nu^2,\ \nu^2 \end{array}\right\}. \qquad (75b)$$

These give ia. $y_1 = 4mn = 6\,(2\mu\nu)^2$; (75*c*)

thus satisfying the condition $y_1 = 6\eta^2$; so that it only remains to determine $m$, $n$ so as to satisfy the condition $x_1 = \xi^2$, where $x_1$ is given by (75, ia.); but it does not seem easy to do this in a general manner.

*Examples.**—Taking $\mu = 1$, $\nu = 1$; $m = 3\mu^2 = 3$, $n = 2\nu^2 = 2$ in ia.

This gives $x_1 = 3^2-2.3.2+13.2^2 = 49$, which satisfies $x_1 = \xi^2$.

Here $x_1 = 49$, $y_1 = 24$; $x_2 = 43$, $y_2 = 28$.

Hence $\left.\begin{array}{l} N_1 = L_1 M_1 = 601.7657 = 601\,(13.19.31) \\ N_2 = L_2 M_2 = 589.7813 = (19.31)\,(13.601) \end{array}\right\}$ and $N_1 = N_2$.

**25.** *Tri - Trin - Ant - Aurifeuillians.* — To investigate whether a number $N$ can be at once a Trin-Aurifeuillian ($N_1$) and a Trin-Ant-Aurifeuillian $N_2$. Let, if possible,

$$\left.\begin{array}{l} N = N_1 = N_2, \\ \text{where} \quad N_1 = x_1^2-3x_1^2y_1^2+9y_1^4, \quad N_2 = x_2^2+3x_2^2y_2^2+9y_2^4 \end{array}\right\}. \qquad (76)$$

A general solution seems difficult, but a *limited* solution may be obtained by expressing $N_1$, $N_2$ in the various quadratic forms $N_1'$, $N_1''$, $N_1'''$; $N_2'$, $N_2''$, $N_2'''$ of type $N = A^2+3B^2$ of Art. **21**, **22**, and then

* From the general properties of Diophantines, it seems probable that any solution should yield an infinite set of solutions, (compare Art. **9**, **36**); but the author has not been able to determine this case.

equating the parts $A_1$, $B_1$ of any one of the forms $N_1'$, $N_1''$, $N_1'''$ separately to the *corresponding* parts $A_2$, $B_2$ of any one of the forms $N_2'$, $N_2''$, $N_2'''$. Thus would arise *nine* systems of equations, each of type $A_1 = A_2$, $B_1 = B_2$; or, taking into account the double sign ($\pm$) of $A_1$, $A_2$; $B_1$, $B_2$, there would be 36 systems of equations, (two in each). Most of these systems appear to be *impossible:* in fact, the only possible systems appear to be those of the *two* groups (I, II) below.

GROUP I. i. $N_1' = N_2'$; ii. $N_1'' = N_2'''$; iii. $N_1''' = N_2''$, (77)

which yield, disregarding the ($\pm$) sign of $A$, $B$,

i. $A_1' = A_2'$, $B_1' = B_2'$; ii. $A_1'' = A_2'''$, $B_1'' = B_2'''$; &c. (77*a*)

Taking the values of $A$, $B$ from Art. **21**, **22**, and introducing the $\pm$ signs, there is found to be *only one* possible case in each system i, ii, iii, and these three possible systems will be found to yield the *same results*, reducible in each case to the *two* conditions

$$x_1^2 - x_2^2 = 3y_1^2 = 3y_2^2. \tag{77b}$$

The further discussion of this case is reserved to Art. **26**.

GROUP II. iv. $N_1' = N_2'''$; v. $N_1'' = N_2''$; vi. $N_1''' = N_2'$, (78)

which yield, disregarding the ($\pm$) sign of $A$, $B$,

iv. $A_1' = A_2'''$, $B_1' = B_2'''$; &c. &c.

Reducing the $A$, $B$, and introducing the $\pm$ sign, as in Group I, there is found *only one* possible case in each system iv, v, vi, and these three possible cases lead to the same results, reducible in each case to the two conditions

$$y_1^2 - y_2^2 = \tfrac{1}{3}x_1^2 = \tfrac{1}{3}x_2^2;$$

but, as this involves $x_1$, $x_2$ being each $= 3\xi$, a case expressly excluded (65, 69), this solution is *rejected.*

**26.** *Various Trin-Aurifeuillians.*—The results of imposing *one* of the conditions

i. $x^2 - 3y^2 = z^2$; ii. $x^2 + 3y^2 = z^2$, (79)

on the Trin-Aurifeuillian $N$, and Trin-Ant-Aurifeuillian $N`$, where

$$N = x^4 - 3x^2y^2 + 9y^4, \quad N` = *z^4 + 3z^2y^2 + 9y^4, \tag{80}$$

(already partly considered in Art. **25**), are worth further study.

* Note that $z$ is here used (instead of $x$) in the formula for $N`$: this is done to distinguish it here from the $x$ used in the formula for $N$.

They give

i. $N = N` = x^4 - x^2z^2 + z^4 = \dfrac{x^6+z^6}{x^2+z^2} = N_{\text{vi}}$ (suppose), (81)

ii. $N = z^4 - 3z^2x^2 + 3x^4 = \mathfrak{L}$ of $\mathfrak{N}$, where $\mathfrak{N} = z^8 - 3z^4x^4 + 9x^8$, (82*a*)

$N` = x^4 - 3x^2z^2 + 3z^4 = \mathfrak{L}$ of $\mathfrak{N}$, where $\mathfrak{N} = x^8 - 3x^4z^4 + 9z^8$. (82*b*)

Thus, Condition ii gives the two higher Trin-Aurifeuillians $\mathfrak{N}$ whose smaller Aurifeuillian Factor $\mathfrak{L}$ is itself of one of the forms $N$ or $N`$.

The results of Condition i are of great interest, viz., that the forms $N, N`, N_{\text{vi}}$ are *interconvertible*,* thus also providing *factorisable* numbers of forms $N`$, and $N_{\text{vi}}$,† both being expressible in form $N = LM$.

There are *two* general solutions of each of the Diophantine conditions (i, ii) of (79) in terms of two arbitrary quantities $m$, $n$ (here $m \neq 3\mu$, and $m$, $n$ should be relatively prime).

i. $x = m^2 + 3n^2$, $y = 2mn$, $z = m^2 \sim 3n^2$; $m$, $n$ one *odd*, one *even*, (83)

giving $L = (m^2 \sim 3mn)^2 + (3mn \sim 3n^2)^2 = a_l^2 + b_l^2$,

$M = (m^2 + 3mn)^2 + (3mn + 3n^2)^2 = a_m^2 + b_m^2$.

Thus $a_l \sim b_l = a_m \sim b_m = m^2 - 3n^2 = z$.

i*a*. $x = \frac{1}{2}(m^2 + 3n^2)$, $y = mn$, $z = \frac{1}{2}(m^2 \sim 3n^2)$; $m$, $n$ both *odd*. (83*a*)

These give $L$, $M$ each $\frac{1}{4}$ of the $L$, $M$ of Case i.

ii. $x = m^2 \sim 3n^2$, $y = 2mn$, $z = m^2 + 3n^2$; $m$, $n$ one *odd*, one *even*. (84)

ii*a*. $x = \frac{1}{2}(m^2 \sim 3n^2)$, $y = mn$, $z = \frac{1}{2}(m^2 + 3n^2)$; $m$, $n$ both *odd*. (84*a*)

An interesting sub-case under Condition i arises when

$$x^2 - 3y^2 = z^2 \text{ (constant)}. \tag{85}$$

This gives $N = N` = N_{\text{vi}}$, and also gives the *Trin-Aurifeuillian Chains* further developed in Art. **34a**, *q.v.*

*Sub-Case* (Fermat's Numbers). — The sub-case of the last when $= 1$ is worth separate notice, as giving a factorisable case† of Fermat's Numbers $\mathfrak{N} = (x^6+1)$. Here

$$N = N` = N_{\text{vi}}, \quad \text{where} \quad N_{\text{vi}} = (x^6+1) \div (x^2+1), \tag{86}$$

$$N = x^4 - 3x^2y^2 + 9y^4, \qquad N` = 1 + 3y^2 + 9y^4. \tag{86*a*}$$

Calling the *successive* numbers $N_1, N_2, N_3, \ldots, N_r$, and applying the subscript $r$ to the *parts* $x$, $y$, of $N_r$, the parts $x$, $y$ of successive numbers $N$ are connected by the relations

$$x_1^2 - 3y_1^2 = x_2^2 - 3y_2^2 = \ldots = x_r^2 - 3y_r^2 = 1, \tag{87}$$

* The convertibility of $N$, $N`$ under Condition i was shown in Art. **25**; the $x_2$ of that Article is the $z$ of this Article.

† This factorisable form of $N_{\text{vi}}$ is due to Mr. Bickmore, and was communicated by him to the Oxford Mathematical Society in February, 1898.

whence $x_r = 2x_{r-1} + 3y_{r-1}$, $y_r = x_{r-1} + 2y_{r-1}$, $(x_1 = 2,\ y_1 = 1)$, (87*a*)

and the $m$, $n$ of the general formulæ are connected by the relations

$$\left.\begin{aligned} m_{2r} &= n_{2r} + n_{2r-1}, & m_{2r+1} &= n_{2r+1} + 2n_{2r} \\ n_{2r} &= \tfrac{1}{2}(m_{2r-1} + n_{2r-1}), & n_{2r+1} &= m_{2r} + n_{2r} \end{aligned}\right\}, \qquad (87b)$$

These give

$$\left.\begin{aligned} L_r &= (m_r^2 \sim 3m_r n_r)^2 + (3m_r n_r \sim 3n_r^2)^2 = a_{lr}^2 + b_{lr}^2 \\ M_r &= (m_r^2 + 3m_r n_r)^2 + (3m_r n_r + 3n_r^2) = a_{mr}^2 + b_{mr}^2 \end{aligned}\right\}, \qquad (87c)$$

Thus $a_{lr} \sim b_{lr} = a_{mr} \sim b_{mr} = m_r^2 \sim 3n_r^2 = z = 1$, (for all values of $r$)

so that $L_r$, $M_r$ each consist of a sum of two *consecutive* squares

These numbers $N_r$, (being only a sub-case of the last, given by $z = 1$), form a *Trin-Aurifeuillian Chain*, *i.e.*, a series such that $M_{r-1} = L_r$ for all values of $r$ (Art. **34a**).

*Examples.*—The upper part of the following table shows the factorisation of the numbers $N_{\text{vi}} = (x^6 + 1) \div (x^2 + 1)$, falling under both forms $N$, $N'$, into their *prime* factors, pushed to a very high limit; the values of $L$, $M$ show the *Trin-Aurifeuillian Chain* $(M_r = L_{r+1})$.

Below these are given a few more examples, the *lowest* of the numbers $N_{\text{vi}} = (x^6 + z^6) \div (x^2 + z^2)$—when $z \neq 1$ - also falling under both forms $N$, $N'$.

The lower part of the Table contains a few (the *lowest*) examples of Higher Trin-Aurifeuillians $\mathfrak{N}$ whose smaller Trin-Aurifeuillian Factor $\mathfrak{L}$ is itself a Trin-Aurifeuillian, so that $\mathfrak{N} = (L \,.\, M) \,.\, \mathfrak{M}$.

| Case. | $r$ | Data. $m$ ; $n$ | Parts. $x$ ; $y$ ; $z$ | $N = L \,.\, M$ $L$ ; $M$ | $N_{\text{vi}} = N = N'$ $(x^6 + z^6) \div (x^2 + z^2)$ |
|---|---|---|---|---|---|
| Sub-Case of i ($z = 1$). | 1 | 1 ; 1 | 2 ; 1 ; 1 | 1 ; 13 | 13 |
| | 2 | 2 ; 1 | 7 ; 4 ; 1 | 13 ; 181 | 2,353 |
| | 3 | 5 ; 3 | 26 ; 15 ; 1 | 181 ; 2521 | 456,301 |
| | 4 | 7 ; 4 | 97 ; 56 ; 1 | 2521 ; $13 \cdot 37 \cdot 73$ | 88,519,873 |
| | 5 | 19 ; 11 | 362 ; 209 ; 1 | $13 \cdot 37 \cdot 73$ ; 489,061 | 13,341,285,133 |
| | 6 | 26 ; 15 | 1351 ; 780 ; 1 | 489,061 ; 6,811,741 | |
| | 7 | 71 ; 41 | 5042 ; 2911 ; 1 | 6,811,741 ; $13 \cdot 61 \cdot 181 \cdot 661$ | |
| | 8 | 97 ; 56 | 18817 ; 10864 ; 1 | $13 \cdot 61 \cdot 181 \cdot 661$ ; | |
| Case i | | 1 ; 2 | 13 ; 4 ; 11 | 61 ; 373 | 22,753 |
| | | 5 ; 1 | 14 ; 5 ; 11 | 61 ; $13 \cdot 37$ | 29,341 |
| | | 1 ; 3 | 14 ; 3 ; 13 | 97 ; 349 | 33,853 |
| | | 4 ; 1 | 19 ; 8 ; 13 | 97 ; 1009 | 97,873 |

| Case ii | Data. $m$ ; $n$ | Parts. $x$ ; $y$ ; $z$ | $N = L \,.\, M = \mathfrak{L}$ $L$ ; $M$ | $\mathfrak{M}$ | $\mathfrak{N} = \mathfrak{L} \,.\, \mathfrak{M} = L \,.\, M \,.\, \mathfrak{M}$ |
|---|---|---|---|---|---|
| | 1 ; 1 | 1 ; 1 ; 2 | 1 ; 7 | 31 | 217 |
| | 2 ; 1 | 1 ; 4 ; 7 | 37 ; 61 | 2551 | 5,757,607 |
| | 1 ; 2 | 11 ; 4 ; 13 | 37 ; 301 | 133,831 | 1,490,475,847 |
| | 5 ; 1 | 11 ; 5 ; 14 | 31 ; $19 \cdot 19$ | 153,487 | |
| | 1 ; 3 | 13 ; 3 ; 14 | 79 ; 313 | $43 \cdot 5197$ | |

**27.** *Trin-Aurifeuillian Factors.*—Let $N_1$, $N_2$ be two numbers whose product $\mathfrak{N}$ is a Trin-Aurifeuillian. The conditions that $N_1$, $N_2$ should be the Aurifeuillian Factors $(L, M)$ of $N$ are

$$N_1 = L = x^2 - 3xy + 3y^2, \quad N_2 = M = x^2 + 3xy + 3y^2. \tag{88}$$

Hence $$x^2 + 3y^2 = \tfrac{1}{2}(N_2 + N_1), \quad 3xy = \tfrac{1}{2}(N_2 - N_1),$$

whence $$x^2 - 3y^2 = \tfrac{1}{6}\left\{36N_1N_2 - 3(N_2 - N_1)^2\right\}^{\frac{1}{2}}.$$

Hence *three* conditions must obtain that $N_1$, $N_2$ may be the Aurifeuillian factors of $N$, viz.,

$$\left\{36N_1N_2 - 3(N_2 - N_1)^2\right\} = \left\{6(x^2 - 3y^2)^2\right\} = (6Z)^2, \text{ a } \textit{square}, \tag{88a}$$

$$(N_1 + N_2) \pm 2Z = (2x)^2, \text{ an } \textit{even square}, \tag{88b}$$

$$\tfrac{1}{3}\left\{(N_1 + N_2) \mp 2Z\right\} = (2y)^2, \text{ an } \textit{even square}. \tag{88c}$$

When $N_1$, $N_2$ are given numbers, (satisfying the above conditions), the values of $x$, $y$, the Aurifeuillian parts of $N$, and therewith also the quadratic partitions $(A^2 + 3B^2)$ of $L$, $M$, $N$, are hereby given directly.

**28.** *Conformal Trin-Aurifeuillian Products.*—The question now arises whether the *product* of two or more Trin-Aurifeuillians can be a Trin-Aurifeuillian; and, conversely, whether a Trin-Aurifeuillian can be resolved into the product of two or more Trin-Aurifeuillians.

Let

$$N = x^4 - 3x^2y^2 + 9y^4, \quad N_1 = x_1^4 - 3x_1^2y_1^2 + 9y_1^4, \quad N_2 = x_2^4 - 3x_2^2y_2^2 + 9y_2^4, \tag{89}$$

and, if possible, let $$N = N_1 N_2, \text{ (where } N_1 < N_2\text{)}. \tag{90}$$

The *general* solution of (90) seems difficult, but a somewhat limited solution may be obtained by first equating the Aurifeuillian Factors $(L, M)$ of $N$ separately to $N_1$, $N_2$, thus obtaining the *two* equations

$$L = N_1, \quad M = N_2. \tag{90a}$$

Let $L, M$; $N_1, N_2$ be now expressed in the quadratic form $(A^2 + 3B^2)$; herein the forms of $L$, $M$ are each *trimorph*, whilst $N_1$, $N_2$ are each *tetramorph*, (Art. **21**). Observing that a corresponding pair $(L', M')$, $(L'', M'')$, $(L''', M''')$ must necessarily enter together into (90), and, therefore, into any one form of Equation (90*a*), whilst the forms of $N_1$, $N_2$ are *independent*, the pair of equations (90*a*) may be expressed in 48 ways, which are summarized below.

$L' = \textit{any}$ of $N_1^\circ, N_1', N_1'', N_1'''$; $M' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

$L'' = \textit{any}$ of $N_1^\circ, N_1', N_1'', N_1'''$; $M'' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

$L''' = \textit{any}$ of $N^\circ, N_1', N_1'', N_1'''$; $M''' = \textit{any}$ of $N_2^\circ, N_2', N_2'', N_2'''$, (16 pairs).

By now equating the parts $(A, B)$ of any corresponding pair $(L', M')$, $(L'', M'')$, $(L''', M''')$ separately to the parts $(A, B)$ of each of the four forms of $N_1$, $N_2$ respectively, each of the 48 systems above yields *one* group of *four* equations between the six quantities $x, y, x_1, y_1, x_2, y_2$. Further, as each of the parts $A$, $B$ may have either a $\pm$ sign, each of the above 48 groups may be formed in *four* ways.

The four equations in each of the 192 sets are *linear* in $x$, $y$, and *quadratic* in $x_1, y_1, x_2, y_2$. Hence $x$, $y$ can always be *linearly* eliminated, thus leaving *two* equations *quadratic* in $x_1, y_1, x_2, y_2$. From these the value of either pair $(x_1, y_1)$, $(x_2, y_2)$ can be solved (by quadratics) in terms of the other pair in a form involving (in the general case) *three* radicals. Thus *three* conditions must in general be satisfied (viz., that the expressions under the three radicals should be perfect squares) in order that this form of solution may be *possible.*

Thus, whenever either $N_1$ or $N_2$ is *given*, and also the Aurifeuillian parts $(x_1, y_1)$ or $(x_2, y_2)$ thereof, these criteria can be *directly* applied; and, if satisfied, the other pair $(x_2, y_2)$ or $(x_1, y_1)$ and thence also $N_2$ or $N_1$, and, lastly, $x$, $y$, and $N$, can be *directly* calculated, so that this problem is *direct*, and the solution is *unique.*

The solution of the problem *ab initio* is, however, much more difficult.

In a good many of the 192 sets of equations above specified it is found, on attempting the solution, that some impossibility* is involved. The following are the only cases which have been found completely solvable:—

$$\left.\begin{array}{lll}
\text{i.} & L' = N_1', \quad M' = N_2', & (y, y_1, y_2 \text{ all even}) \\
\text{ii.} & L' = N_1'', \quad M' = N_2'', & (y, x_1, x_2 \text{ all even}) \\
\text{iii.} & L''' = N_1', \quad M''' = N_2''', & (y_1 \text{ even, the rest odd}) \\
\text{iv.} & L''' = N_1''', \quad M''' = N_2', & (y_2 \text{ even, the rest odd}) \\
\text{v.} & L''' = N_1'', \quad M''' = N_2''', & (x_1 \text{ even, the rest odd}) \\
\text{vi.} & L''' = N_1''', \quad M''' = N_2'', & (x_2 \text{ even, the rest odd})
\end{array}\right\}. \quad (91)$$

**28a.** The detailed solutions are—

*System* i. ($y, y_1, y_2$ all even).—The equations are

$$(x-\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = L' = N_1' = (x_1^2-\tfrac{3}{2}y_1^2)^2+3\,(\tfrac{3}{2}y_1^2)^2,$$

$$(x+\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = M' = N_2' = (x_2^2-\tfrac{3}{2}y_2^2)+3\,(\tfrac{3}{2}y_2^2)^2.$$

* It seems unnecessary to give the detail.

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$x-\tfrac{3}{2}y = x_1^2-\tfrac{3}{2}y_1^2, \quad x+\tfrac{3}{2}y = x_2^2-\tfrac{3}{2}y_2^2, \quad y = 3y_1^2 = 3y_2^2,$$

giving $\quad x = \tfrac{1}{3}(2x_1^2+x_2^2)$

with the *two* conditions $x_2^2-x_1^2 = (3y_1)^2 = (3y_2)^2$ $\Big\}$. (92, i)

By varying the signs of the parts $(\pm A)$, $(\pm B)$ equated, another possible case is found ; but the results differ from the above only in the interchange of $x_1$, $x_2$, and change of sign of $x$.

*System* ii. ($y$, $x_1$, $x_2$ all even).—The equations are

$$(x-\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = L' = N_1'' = \left(\frac{x_1^2}{2}-3y_1^2\right)^2+3\left(\frac{x_1^2}{2}\right)^2,$$

$$(x+\tfrac{3}{2}y)^2+3\left(\frac{y}{2}\right)^2 = M' = N_2'' = \left(\frac{x_2^2}{2}-3y_2^2\right)^2+3\left(\frac{x_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A)$, $(\pm B)$ as follows :—

$$x-\tfrac{3}{2}y = \tfrac{1}{2}x_1^2-3y_1^2, \quad x+\tfrac{3}{2}y = \tfrac{1}{2}x_2^2-3y_2^2, \quad y = x_1^2 = x_2^2,$$

giving $\quad x = -(y_1^2+2y_2^2)$

with the *two* conditions $y_1^2-y_2^2 = x_1^2 = x_2^2$ $\Big\}$. (92, ii)

By varying the signs of the parts $(\pm A)$, $(\pm B)$ equated, another possible case is found ; but the results differ from the above only in the interchange of $y_1$, $y_2$, and change of sign of $x$.

*System* iii. ($y_1$ even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2 = L'' = N_1' = (x_1^2-\tfrac{3}{2}y_1^2)^2+3(\tfrac{3}{2}y_1^2)^2,$$

$$\left(\frac{x+3y}{2}\right)^2+3\left(\frac{x+y}{2}\right)^2 = M'' = N_2''' = \left(\frac{x_2^2+3y_2^2}{2}\right)^2+3\left(\frac{x_2^2-3y_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$-(x-3y) = 2x_1^2-3y_1^2, \qquad x-y = 3y_1^2,$$
$$x+3y = x_2^2+3y_2^2, \qquad -(x+y) = x_2^2-3y_2^2,$$

giving $\quad x = y_2^2+2y_1^2, \quad y = x_1^2 = x_2^2$

with *two* conditions $\quad y_2^2-y_1^2 = x_1^2 = x_2^2$ $\Big\}$. (92, iii)

*System* iv.—The results differ from System iii only in the interchange of $(x_1, y_1)$ with $(x_2, y_2)$, and change of sign of $x$.

*System* v. ($x_1$ even, the rest odd).—The equations are

$$\left(\frac{x-3y}{2}\right)^2+3\left(\frac{x-y}{2}\right)^2 = L''' = N_1''' = (\tfrac{1}{2}x_1^2-3y_1^2)+3(\tfrac{1}{2}x_1^2)^2,$$

$$\left(\frac{x+3y}{2}\right)^2+3\left(\frac{x+y}{2}\right)^2 = M''' = N_2''' = \left(\frac{x_2^2+3y_2^2}{2}\right)^2+3\left(\frac{x_2^2-3y_2^2}{2}\right)^2.$$

One of the possible cases is given by equating the parts $(\pm A, \pm B)$ as follows :—

$$x-3y = x_1^2-6y_1^2, \quad x-y = x_1^2,$$

$$x+3y = x_2^2+3y_2^2, \quad x+y = x_2^2-3y_2^2,$$

giving $x = \frac{1}{3}(2x_1^2+x_2^2), \quad y = 3y_1^2 = 3y_2^2$

with *two* conditions $x_2^2-x_1^2 = (3y_1)^2 = (3y_2)^2$ } . (92, v)

*System* vi.—The results differ from System v only in the interchange of $(x_1, y_1)$ with $(x_2, y_2)$, and change of sign of $y$.

The *two* conditions in all these systems are very similar, viz.,

One of type $y_1 = y_2$ (in i, v, vi) ; or $x_1 = x_2$ (in ii, iii, iv).

One of type $x_2^2 \sim x_1^2 = (3y_1)^2$, (in i, v, vi) ; or $y_2^2 \sim y_1^2 = x_1^2$ (in ii, iii, iv).

The latter condition is equivalent to $c^2 = a^2+b^2$. There are *two* general solutions of this Pythagorean equation in terms of two arbitrary quantities $(m, n)$ (*cf.* Art. **13a**); but *one* of the general solutions in each system is found (on actual trial) to violate some of the initial (arithmetical)* conditions as to *oddness* or *evenness* of some of the quantities $x_1, y_1, x_2, y_2$, leaving effectively *only one* general solution of each system; and also involving effectively

$m$, $n$ mutual primes, one *odd*, one *even* (in each system). (93)

The general solutions of i, ii, iii, v are

i. $x_2 = m^2+n^2, \quad x_1 = m^2 \sim n^2, \quad y_1 = y_2 = \frac{2}{3}mn$

$x = m^4-\frac{2}{3}m^2n^2+n^4 = (y+x_1^2), \quad y = \frac{4}{3}m^2n^2$

[$m$, $n$ mutual primes, one odd, one even ; and one a multiple of 3 ; giving $x$, $x_1$, $x_2$ all *odd* ; $y$, $y_1$, $y_2$ all *even*] } . (94, i)

* If these *arithmetical* restrictions be removed, the general solutions here omitted of Systems i, ii, iii, v will be found to merge into the general solutions given of Systems v, iii, ii, i respectively.

ii. $y_1 = m^2+n^2,\ y_2 = m^2 \sim n^2,\ x_1 = x_2 = 2mn$
$x = -(3m^4-2m^2n^2+3n^4) = -(y+3y_2^2),\ y = (2mn)^2$
[$m$, $n$ mutual primes, one odd, one even ; giving $x$, $y_1$, $y_2$ *odd* ; $y$, $x_1$, $x_2$ *even*] } . (94, ii)

iii. $y_2 = m^2+n^2,\ y_1 = 2mn,\ x_1 = x_2 = m^2 \sim n^2$
$x = m^4+10m^2n^2+n^4 = (y+3y_1^2),\ y = (m^2 \sim n^2)^2$
[$m$, $n$ mutual primes, one odd, one even, and one a multiple* of 3 ; giving $x$, $y$, $x_1$, $x_2$, $y_2$ *odd* ; $y_1$ *even*] } . (94, iii)

v. $x_2 = m^2+n^2,\ x_1 = 2mn,\ y_1 = y_2 = \frac{1}{3}(m^2 \sim n^2)$
$x = \frac{1}{3}(m^4+10m^2n^2+n^4) = y+x_1^2,\ y = \frac{1}{3}(m^2 \sim n^2)^2$
[$m$, $n$ mutual primes, one odd, one even ; such that one of $(m \nsim n)$ is a multiple of 3, giving $x$, $y$, $y_1$, $x_2$, $y_2$ *odd* ; $x_1$ *even*] } . (94, v)

*Examples.*—The following Table gives the elements and results of the *lowest* of such numbers $N_1 N_2 = N$ in each of the solvable Cases i, ii, iii, v. The numbers increase so very rapidly that it is not worth continuing. These high numbers ($N$) would be difficult to factorise otherwise.

| Case. | Data. | 1st Aurifeuillian. | | 2nd Aurifeuillian. | | Product-Aurifeuillian. | |
|---|---|---|---|---|---|---|---|
| | $m$ ; $n$ | $x_1$ ; $y_1$ | $N_1$ | $x_2$ ; $y_2$ | $N_2$ | $x$ ; $y$ | $N$ |
| i | 3 ; 2 | 5 ; 4 | 1729 | 13 ; 4 | 22,753 | 73 ; 48 | 39,339,937 |
| | 3 ; 4 | 7 ; 8 | 29857 | 25 ; 8 | 307,489 | 241 ; 192 | 9,180,699,073 |
| ii | 1 ; 2 | 4 ; 5 | 4681 | 4 ; 3 | 553 | − 43 ; 16 | 2,588,593 |
| | 3 ; 2 | 12 ; 13 | 204,777 | 12 ; 5 | 15,561 | −219 ; 144 | 3,186,534,897 |
| | 1 ; 4 | 8 ; 17 | 700,297 | 8 ; 15 | 416,521 | −739 ; 64 | |
| iii | 3 ; 2 | 5 ; 12 | 176,449 | 5 ; 13 | 244,999 | 457 ; 25 | 43,229,828,551 |
| iv | 1 ; 2 | 4 ; 1 | 217 | 5 ; 1 | 559 | 19 ; 3 | 121,303 |
| | 1 ; 4 | 8 ; 5 | 4921 | 17 ; 5 | 67,471 | 139 ; 75 | 332,024,791 |

29. *Conformal Continued Trin-Aurifeuillian Products.*—The conditions for these are so much more complex than those for the similar problem in Bin-Aurifeuillians that it has not been found possible to extend the above solutions to the product of three Trin-Aurifeuillians.

* Otherwise $y$, $N_2$, and $N$ will be multiples of 9.

**30.** *Quotient Trin-Aurifeuillians.*—Writing $3^\omega$, $x^\omega$, $y^\omega$ for 3, $x$, $y$ (where $\omega \neq 3\kappa$) in the expression (65), it becomes

$$\mathfrak{Y} = \frac{x^{6\omega}+3^{3\omega}y^{6\omega}}{x^{2\omega}+3^{\omega}y^{2\omega}} = x^{4\omega}-3^{\omega}x^{2\omega}y^{2\omega}+3^{2\omega}y^{4\omega} \qquad (95)$$

$$= X^4-3X^2Y^2+3^2Y^4, \text{ where } X = x^\omega,\ Y = 3^{\frac{1}{2}(\omega-1)}y^\omega. \qquad (95a)$$

Hence $\mathfrak{Y}$ is a Trin-Aurifeuillian; and, writing

$$Q = \frac{\mathfrak{Y}}{N} = \frac{x^{6\omega}+3^{3\omega}y^{6\omega}}{x^{2\omega}+3^{\omega}y^{2\omega}}\ \frac{x^2+3y^2}{x^6+3^3y^6}, \qquad (96)$$

it is seen that $\mathfrak{Y}$ contains $N$, and that $Q$ is the A.P.F. of $(x^{6\omega}+3^{3\omega}y^{6\omega})$ when $\omega$ is prime.

Hence $Q$ is a Quotient Trin-Aurifeuillian, resolvable into factors $S$, $T$; and $Q$, $S$, $T$ are expressible in quadratic forms conformal with those of $\mathfrak{Y}$, $\mathfrak{L}$, $\mathfrak{M}$, as in Art. **7–7b.**

As the chief interest of these numbers ($Q$) is for small values of $\omega$, ($= 5$, 7, &c.), it is better to pass on at once to these simple cases, (as general formulæ would be very lengthy).

**30a.** *Quintic Trin-Aurifeuillian.*—Writing $\omega = 5$ in the formula (95),

$$\mathfrak{N} = \frac{x^{30}+3^{15}y^{30}}{x^{10}+3^5y^{10}} = x^{20}-3^5x^{10}y^{10}+3^{10}y^{20}. \qquad (97)$$

Here $Q = \mathfrak{N} \div N$ can have each of the partitions $(A^2+3B^2)$, $(u^2-5v^2)$, $(V^2+15W^2)$, denoted by $Q_3$, $Q_5$, $Q_{15}$, (accented as required, Art. **5, 21**),

$$Q = \frac{\mathfrak{N}}{N} = \frac{x^{30}+3^{15}y^{30}}{x^{10}+3^5y^{10}} \cdot \frac{x^2+3y^2}{x^6+3^3y^6} = \frac{x^{20}-3^5x^{10}y^{10}+3^{10}y^{20}}{x^4-3x^2y^2+3^2y^4}, \qquad (98)$$

$$= x^{16}+3x^{14}y^2-3^3x^{10}y^6-3^4x^8y^8-3^5x^6y^{10}+3^7x^2y^{14}+3^8y^{16}, \qquad (98a)$$

$$Q_3^\circ = (x^8-3^2x^4y^4+3^4y^8)^2+3(xy)^2(x^6+3x^4y^2-3^2x^2y^4-3^3y^6)^2, \qquad (98b)$$

$$Q_5^\circ = (x^3+8\,.\,3x^6y^2+13\,.\,3^2x^4y^4+8\,.\,3^3x^2y^6+3^4y^8)^2-5(3xy)^2(x^2+3y^2)^6, \qquad (98c)$$

$$Q_{15}' = (x^8-2\,.\,3x^6y^2+3\,.\,3^2x^4y^4-2\,.\,3^3x^2y^6+3^4y^8)^2+15(xy)^2(x^6-3x^4y^2+3^2x^2y^4-3^3y^6)^2, \qquad (98d)$$

$$S = \mathfrak{L} \div M = (x^{10}-3^3x^5y^5+3^5y^{10}) \div (x^2+3xy+3y^2) \qquad (99)$$

$$= x^8-3x^7y+6x^6y^2-3^2x^5y^3+3^2x^4y^4-3^3x^3y^5+2\,.\,3^3x^2y^6-3^4xy^7+3^4y^8. \qquad (99a)$$

Here $S$ has three alternative forms $(A^2+3B^2)$ denoted by $S_3'$, $S_3''$, $S_3'''$; and *two* alternative forms of $(u^2-5v^2)$* denoted by $S_5'$, $S_5''$; and one form $(S_{15})$ of $(W^2+15Z^2)$.

$$S_3' = (x^4-\tfrac{3}{2}x^3y+\tfrac{3}{2}x^2y^2-\tfrac{9}{2}y^4)^2+3\left(\frac{y}{2}\right)^2(x^3-3x^2y+6xy^2-9y^3)^2,\ (y \textit{ even}), \qquad (99b)$$

$$S_3'' = \left\{\tfrac{1}{2}x(x^3-3xy^2+9y^3)-9y^4\right\}^2+3\left(\frac{x}{2}\right)^2(x^3-2x^2y+3xy^2-3y^3)^2,\ (x \textit{ even}), \qquad (99b)$$

$$S_3''' = \tfrac{1}{4}(x^4-3x^3y+6x^2y^2-9xy^3+9y^4)^2+3\,.\,\tfrac{1}{4}(x^4-x^3y+3xy^3-9y^4)^2,\ (x, y \textit{ both odd}), \qquad (99b)$$

$$S_5' = (x^4-\tfrac{3}{2}x^3y+\tfrac{15}{2}x^2y^2-\tfrac{9}{2}xy^3+9y^4)^2-5\left(\frac{3xy}{2}\right)^2(x^2-xy+3y^2)^2,\ (x \text{ or } y \textit{ even}), \qquad (99c)$$

$$S_5'' = (\tfrac{3}{2}x^4-6x^3y+15x^2y^2-18xy^3+\tfrac{27}{2}y^4)^2-5(\tfrac{1}{2}x^4-3x^3y+6x^2y^2-9xy^3+\tfrac{9}{2}y^4)^2, \quad (x, y \textit{ both odd}), \qquad (99c)$$

$$S_{15} = \left\{x^4-\tfrac{3}{2}xy(x^2+3y^2)+9y^4\right\}^2+15\,.\,\tfrac{1}{4}(xy)^2(x^2-3y^2)^2. \qquad (99d)$$

The values of $T = \mathfrak{M} \div L$, and of its quadratic partitions $T_3'$, $T_3''$, $T_3'''$; $T_5'$, $T_5''$; $T_{15}$ may be derived from those of $S$ by writing $-y$ for $y$, (Art. **3**).

* *See* footnote * to Art. **16b.**

The quadratic partitions of $Q$, viz., $Q_3^\circ$, $Q_3'$, $Q_3''$, $Q_3'''$; $Q_5^\circ$, $Q_5'$, $Q_5''$; $Q_{15}'$, $Q_{15}''$ may be obtained by conformal multiplication of the similarly accented quadratic forms of $S$, $T$: each such product yielding two forms of $Q$, viz., $Q^\circ$ and *one* of $Q'$, $Q''$, $Q'''$.

**31.** *Fermat's Numbers.* (Base 3). — These are numbers of form $\mathfrak{F}_n = 3^n+1$. Writing $x=1$, $y=3^\nu$ ($\nu$ any integer), the Trin-Aurifeuillian $N$ (Art. **21**) becomes

$$N = (1+3^3.3^{6\nu}) \div (1+3.3^{2\nu}) = (1+3^{6\nu+3}) \div (1+3^{2\nu+1}). \quad (100)$$

Hence the A.P.F. of all Fermat's Numbers $\mathfrak{F}_n = 3^n+1$, where $n = 6\nu+3$, is resolvable into the two factors $L$, $M$ of Art. **21**, viz.,

$$L = 3^{2\nu+1}-3^{\nu+1}+1, \quad M = 3^{2\nu+1}+3^{\nu+1}+1, \quad (100a)$$

and these ($L$, $M$) can be at once *completely* resolved into their *prime* factors as far as $n=39$.* Also the A.P.F. of $\mathfrak{F}_{45} = (3^{45}+1)$ is *completely* resolvable by the formulæ of Art. **30a**. Beyond this limit the factorisation into *prime* factors requires the aid of "special† divisors." When $\frac{1}{3}n = 2\nu+1$ is *composite*, the *Reduced A.P.F.* of $\mathfrak{F}_n$ is similarly resolvable into two factors ($S$, $T$) by Art. **7b**.

*Examples.*—The following Table shows the *complete* resolution‡ of the *Reduced A.P.F.* of these numbers into their *prime* factors, all but complete§ up to $n=105$: the factorisation of the Aurifeuillian Factors $L$, $M$ or $S$, $T$ being shown separately: prime factors only are shown; the semicolon (;) denotes that the factorisation is *complete*.

[*Note* that the highest number $(3^{105}+1)$, here *completely* factorised, contains 51 figures.]

* At this limit the *larger* Aurifeuillian Factor $M$ is $>9$ million, the present limit of the large Factor-Tables.

† Reuschle's *Neue zahlentheoretische Tabellen*, above quoted, gives (pp. 42–46) a table of "special prime divisors" ($p$) of $(3^n-1)$ up to $p<1000$.

‡ The whole of this Table has been worked out by the author for this paper, though much of it had been previously obtained by others. The initials B, C, L (in the columns of $L$ or $S$, $M$ or $T$) indicate the *original* computer (so far as known to the author) who first succeeded in *completely* factorising the Aurifeuillian Factors into their *prime* factors, viz.,

B denotes Mr. C. E. Bickmore; L denotes M. Le Lasseur;
C denotes Lt.-Col. A. Cunningham, (the present author).

(1) Mr. Bickmore gives the results when $n=3, 9, 15, 21, 27$; 87, 105 in his first paper quoted, pp. 8, 43; (except that, when $n=105$, $S$ is not factorised). The remaining results marked B in the Table were communicated to the author by Mr. Bickmore in MS.: besides this, Mr. Bickmore has considerably aided in, and to some extent verified, the author's results marked C.

(2) The results when $n=87$ are ascribed to M. Le Lasseur in Ed. Lucas's *Théorèmes d'Arithmétique*, Turin, 1878, p. 12.

(3) Most of the factorisations and detection of high primes done by the author (C) were effected by the method developed in Mr. Lawrence's Paper (see footnote § to Art. **18**).

§ Complete except that the factorisation of the smaller Aurifeuillian Factor $L$, when $n=31$, is undetermined.

FACTORS OF THE REDUCED A.P.F. OF $(3^n+1)$. $[n = 6\nu+3]$.

| $n$ | $L$ or $S$ | | $M$ or $T$ | |
|---|---|---|---|---|
| 3 | 1 ; | | 7 ; | |
| 9 | 19 ; | | 37 ; | |
| 15 | 31 ; | | 271 ; | |
| 21 | 7 . 43 ; | | 2269 ; | |
| 27 | 19,441 ; | B | 19927 ; | B |
| 33 | 176,419 ; | B | 25411 ; | B |
| 39 | 79 . 2887 ; | B | 157 . 1041 ; | B |
| 45 | 387,631 ; | B | 755,551 ; | B |
| 51 | 613 . 30,091 ; | B | 129,159,847 ; | B, C |
| 57 | 3079 . 53,923 ; | B | 1,162,320,517 ; | B, C |
| 63 | 127 . 883 . 2521 ; | B | 550,554,229 ; | C |
| 69 | 139 . 5107 . 132619 ; | C | 9109 . 1,476,463 ; | C |
| 75 | 26,251 . 119,101 ; | B | 2551 . 1,530,601 ; | C |
| 81 | 163 . 208,657 . 224,209 ; | C | 1297 . 5,879,415,781 ; | C |
| 87 | 261,697 . 37,464,463 ; | L, B | 349 . 142,159 . 1,383,301 ; | L |
| 93 | | | 373 . 541,447 . 3,058,399 ; | C |
| 99 | 199 . 4357 . 337,448,233 ; | C | 397 . 378,450,588,583 ; | C |
| 105 | 24151 . 3,369,031 ; | C | 211 . 1051 . 3,454,081 ; | B |
| 111 | 223 . | | | |
| 117 | 937 . | | 1171 . | |

*Note.*—To complete the factorisation of these numbers $\mathfrak{N}_n = (3^n+1)$, the factorisation of the *algebraic factors* of form $F = 3^\omega + 1$, where $\omega \not> 31$, is required. This factorisation, when $\omega < 19$, is given in Mr. Bickmore's first paper, above quoted, p. 43. The remaining results, still required, are*

$\frac{1}{4}(3^{19}+1)$
$= 2851 . 101,917 ;$
$\frac{1}{4}(3^{23}+1)$
$= 23,535,794,707 ;$
$\frac{1}{4}(3^{31}+1)$
$= 6883$
$. 22,434,744,889.$

**32.** *Fermat's Numbers.* (Higher Bases).—Writing in Formulæ (65) $X = 1$, $Y = 3y^2$, $n = 3\omega = 6\nu+3$, ($\nu$ being any integer), then

$$\mathfrak{N} = \frac{1+Y^{3\omega}}{1+Y^{\omega}} = \frac{1+(3y^2)^{6\nu+3}}{1+(3y^2)^{2\nu+1}} = \frac{1+3^3.\eta^6}{1+3.\eta^2}, \quad \text{where } \eta = 3^\nu . y^{2\nu+1}. \quad (101)$$

Here $\mathfrak{N}$ is a *Trin-Aurifeuillian*, so that every such number, and also its A.P.F., viz.,

$\mathfrak{N} \div N = \{(1+Y^{3\omega})(1+Y)\} \div \{(1+Y^{\omega})(1+Y^3)\}$, where $Y = 3y^2$;

and also its "Reduced A.P.F.," (when further reducible), admit of resolution into the two factors $L$, $M$, or $S$, $T$ of Art. 7a, b, 30.

The existing Factor-Tables (extending to 9 million) enable the *complete* factorisation (*i.e.*, into prime factors) of such numbers to be effected up to the following limits :—

| | | | | | |
|---|---|---|---|---|---|
| $3\omega =$ | 3 , | 9 , | 15 , | 21 , | 27 to 45 |
| $Y = 3y^2 =$ | $3 . 1731^2$†, | $3 . 8^2$, | $3 . 4^2$, | $3 . 2^2$, | $3 . 1^2$ |
| | = 8,989,083, | 192 , | 48 , | 12 , | 3 |

and, by the aid of "special divisors," ‡§ it can be carried far beyond these limits.

*Examples.*—The following table shows the resolution‖ into *prime* factors of

* The first result is due to Mr. Bickmore ; the other two are due to the author.

† Reference note (†) at foot of p. 430.

‡ Reuschle's *Tafeln complexer Primzahlen* give (p. 193 *et seq.*) the *least* values of $Y$ for which $(1+Y^{3\omega})$ is divisible by any *given* prime $p < 1000$.

§ See footnote † to Art. 18 about the author's MS. Tables of these.

‖ The whole of this Table has been worked out by the author for this paper. The initials B, C indicate the *original* computer, as in footnote ‡ to Art. 31, *q.v.*

$N$ = A.P.F. of $(1+Y^n)$, where $Y = 3y^2$ and $n = 3\omega$ to various high limits, for the index $n = 3\omega = 3$†, 9, 15, 21, 33.

The factorisation of the Aurifeuillian Factors $L$, $M$ or $S$, $T$ is shown separately, and *prime* factors only are shown: the semicolon (;) where $n = 9, 15, 21, 27, 33$ indicates that the factorisation is *complete*.

FACTORISATION OF THE A.P.F. OF $(1+Y^{3\omega})$, [where $Y = 3y^2$].

| $y$ | $Y = 3y^2$ | $(1+Y^3) \div (1+Y)$ | | $(1+Y^9) \div (1+Y^3)$ | | $\{(1+Y^{15})(1+Y)\} \div \{(1+Y^5)(1+Y^3)\}$ | | |
|---|---|---|---|---|---|---|---|---|
| | | $L$ | $M$ | $L$ | $M$ | $L$ or $S$ | | $M$ or $T$ | |
| 1 | 3 | 1 | 7 | 19; | 37; | 31; | | 271; | |
| 2 | 12 | 7 | 19 | 1657; | 1801; | 31 · 421; | B | 35,671; | B |
| 3 | 27 | 19 | 37 | 19,241; | 19,927; | 387,631; | B | 755,551; | B |
| 4 | 48 | 37 | 61 | 110,017; | 19 · 5851; | 31 · 134,731; | C | 6,887,341; | C |
| 5 | 75 | 61 | 7 · 13 | 127 · 3313; | 423,001; | 331 · 78,781; | C | 31 · 151 · 8311; | C |
| 6 | 108 | 7 · 13 | 127 | 181 · 6949; | 19 · 66403; | 115,693,471; | C | 161,466,931; | C |
| 7 | 147 | 127 | 13 · 13 | 19 · 167,023; | 3,179,611; | 151 · 241 · 11,161; | C | 12,511 · 43,201; | C |
| 8 | 192 | 13 · 13 | 7 · 31 | 7,073,281; | 19 · 372,763; | 1,202,388,841; | C | 211 · 7,317,091; | C |
| 9 | 243 | 7 · 13 | 271 | 37 · 38763; | 19 · 755,551; | 26,251 · 119,101; | B | 2551 · 1,530,601; | C |
| 10 | 300 | 271 | 331 | 19 · 487 · 2917; | 37 · 37 · 109 · 181; | 151 · 4231 · 11,491; | C | 31 · 541 · 534,661; | C |
| 11 | 363 | 331 | 397 | 19 · 37 · 68,023; | 73 · 655,399; | 31 · 512,130,481; | C | 241 · 79,011,151; | C |
| 12 | 432 | 397 | 7 · 67 | 37 · 2,178,541; | 19 · 4,244,059; | | | 61 · 61 · 211 · 48,271; | C |
| 13 | 507 | 7 · 67 | 547 | 19 · 6,858,109; | 73 · 109 · 16,381; | 31 · 61 · 32,386,261; | C | | |
| 14 | 588 | 547 | 631 | | 109 · 1,865,341; | 61 · 5821 · 313,711; | C | 151 · 9721 · 87,541; | C |
| 15 | 675 | 631 | 7 · 103 | 19 · | 73 · 163 · 25,849; | 61 · 751 · 4,242,421; | C | 31 · | |
| 16 | 768 | 7 · 103 | 19 · 43 | 37 · | 937 · 483,481; | 31 · 61 · 172,938,811; | C | | |
| 17 | 867 | 19 · 43 | 919 | 127 · 577 · 8893; | | 2221 · 12,841 · 18,691; | C | | |
| 18 | 972 | 919 | 13 · 79 | | | 541 · 1561,577,671; | C | 31 · 151 · 181 · 211 · 5281; | C |

† Reference note (†) at foot of p. 430.

[*Note* that the highest number $(1+Y^{15})$, here *completely* factorised, contains 45 figures: the factorisation of $(1+Y^5)$ is required to render these really complete; it is hoped to publish these hereafter.]

FACTORISATION OF THE A.P.F. OF $(1 + Y^{3\omega})$, [where $Y = 3y^2$]. —*Continued.*

| $y$ | $Y = 3y^2$ | $\{(1+Y^{21})(1+Y)\} \div \{(1+Y^7)(1+Y^3)\}$ | | | |
|---|---|---|---|---|---|
| | | $L$ or $S$ | | $M$ or $T$ | |
| 1 | 3 | 7 · 43 ; | | 2269 ; | |
| 2 | 12 | 1,885,339 ; | B | 7 · 43 · 17,011 ; | B |
| 3 | 27 | 127 · 883 · 2521 ; | B | 550,554,229 ; | C |
| 4 | 48 | 43 · | | | |
| 5 | 75 | 883 · | | | |
| 6 | 108 | | | | |
| 7 | 147 | 43 · | | | |
| 8 | 192 | 421 · | | 757 · | |
| 9 | 243 | 2269 · 24,151 · 3,369,031 ; | C | 7 · 43 · 211 · 1051 · 3,454,081 ; | B |
| 10 | 300 | | | 463 · | |

| $y$ | $Y = 3y^2$ | $(1 + Y^{27}) \div (1 + Y^9)$ | | | |
|---|---|---|---|---|---|
| | | $L$ | | $M$ | |
| 1 | 3 | 19,441 ; | B | 19,927 ; | B |
| 2 | 12 | 109 · 47,336,293 ; | B | 271 · 487 · 39,037 ; | B |
| 3 | 27 | 163 · 208,657 · 224,209 ; | C | 1297 · 5,879,415,781 ; | C |

| $y$ | $Y = 3y^2$ | $\{(1+Y^{33})(1+Y)\} \div \{(1+Y^{11})(1+Y^3)\}$ | | | |
|---|---|---|---|---|---|
| | | $L$ or $S$ | | $M$ or $T$ | |
| 1 | 3 | 176,419 ; | B | 25,411 ; | B |
| 2 | 12 | 67 · 199 · 1453 · 5479 ; | B | | |
| 3 | 27 | 199 · 4357 · 337,448,233 ; | C | 397 · 378,450,588,583 ; | C |

† Refer to †, pp. 428, 429. The highest number of this kind, $N = (1 + Y^3) \div (1 + Y)$, factorisable by the existing Factor Tables is

$$(1 + 8{,}989{,}083^3) \div (1 + 8{,}989{,}083) = (7 \cdot 73 \cdot 17581) \cdot (19 \cdot 473{,}383).$$

It will be noticed that these numbers form a *Trin-Aurifeuillian Chain* (Art. **34**), *i.e.*, the $M$ of each number = the $L$ of the next higher number.

**33.** *Common Aurifeuillian Factor of Trin-Aurifeuillians.*—It will now be shown that every number $F$ of form $(A^2+3B^2)$ is an Aurifeuillian Factor of *six* different Trin-Aurifeuillians.

Take $N_r$ as the type of six different Trin-Aurifeuillians, such that

$$N_r = (x_r^6+3^3y_r^6) \div (x_r^2+3y_r^2) = L_rM_r, \ [r = 1, 2, 3, 4, 5, 6]. \quad (102)$$

Then, taking $N_1<N_2$, $N_3<N_4$, $N_5<N_6$, and $L_r<M_r$ in general, $F$ will be a common factor of all the six $N$s, when

$$F = M_1 = L_2 = M_3 = L_4 = M_5 = L_6. \quad (103)$$

Next expressing the three $L$s and three $M$s by pairs in the three quadratic forms $(A_r^2+3B_r^2)$ of Art. **21**,

$$F = A^2+3B^2 = M_1' = L_2' = M_3'' = L_4'' = M_5''' = L_6'''. \quad (103a)$$

Equating each quadratic part $(A_r, B_r)$ of the $L$s and $M$s separately to the parts $(A, B)$ of $F$ in turn, these give the relations

$$\left.\begin{aligned} A &= x_1+\tfrac{3}{2}y_1 = x_2\sim\tfrac{3}{2}y_2 = \ \tfrac{1}{2}x_3 \ = \ \tfrac{1}{2}x_4 \ = \tfrac{1}{2}(x_5+3y_5) = \tfrac{1}{2}(x_6\sim 3y_6) \\ B &= \ \tfrac{1}{2}y_1 \ = \ \tfrac{1}{2}y_2 \ = \tfrac{1}{2}x_3+y_3 = \tfrac{1}{2}x_4\sim y_4 = \tfrac{1}{2}(x_5+y_5) \ = \tfrac{1}{2}(x_6\sim y_6) \end{aligned}\right\} \quad (104)$$

whence, finally,

$$\left.\begin{aligned} x_1 &= A-3B, & y_1 &= 2B; \\ x_2 &= A+3B, & y_2 &= 2B; \\ x_3 &= \ 2A, & y_3 &= B\sim A; \\ x_4 &= \ 2A, & y_4 &= B+A; \\ x_5 &= 3B\sim A, & y_5 &= A\sim B; \\ x_6 &= 3B+A, & y_6 &= A+B: \end{aligned}\right\} \quad (105)$$

$$\left.\begin{aligned} L_1' &= (A\sim 6B)^2+3B^2, & M_1' &= A^2+3B^2 \\ L_2' &= A^2+3B^2, & M_2' &= (A+6B)^2+3B^2 \\ L_3'' &= A^2+3\,(2A\sim B)^2, & M_3'' &= A^2+3B^2 \\ L_4'' &= A^2+3B^2, & M_4'' &= A^2+3\,(2A+B)^2 \\ L_5''' &= (3B\sim 2A)^2+3\,(2B\sim A)^2, & M_5''' &= A^2+3B^2 \\ L_6''' &= A^2+3B^2, & M_6''' &= (3B+2A)^2+3\,(A+2B)^2 \end{aligned}\right\}$$

Hence many simple relations may be found between such a set of

six Trin-Aurifeuillians, *e.g.*,

$$\left.\begin{aligned} N_1+N_4 = N_2+N_3 = N_5+N_6 = 14\,(A^2+3B^2) = 14F^2 \\ N_2-N_1 = N_4-N_3 = \tfrac{1}{2}(N_6-N_5) = 24ABF \end{aligned}\right\}. \quad (106)$$

In order that the six Trin-Aurifeuillians should fulfil the limitations of (65), it suffices that $A$, $B$ should be one *odd*, one *even*, and $A$ prime to $3B$; these make each $x$ prime to its own $3y$, and $N$ *odd*.

The above theorem is true whether $F$ be prime, or composite; but, if $F = A^2+3B^2$ be a *composite*, it may be considerably extended. For every factor of $F$ must in this case be of form $(\alpha^2+3\beta^2)$, and $F'$ will then be expressible in several ways in form $(A^2+3B^2)$, and the above theorem will be independently true for *each* of these different modes of expression of $F'$.

Again, since every Trin-Aurifeuillian $(N)$ is the product of *two* factors $(L, M)$, each of which is of form $(A^2+3B^2)$, it follows from the above that every Trin-Aurifeuillian $(N)$ is an Aurifeuillian factor of 12 different Trin-Aurifeuillians.

*Examples.*—The following Tables show the 6 Trin-Aurifeuillians which contain the common Aurifeuillian Factor $F=7$, or 13, and the set of 12 Trin-Aurifeuillians which contain the Trin-Aurifeuillian $F = 133$.

| Data. | | | Trin-Aurifeuillians. | | | | | |
|---|---|---|---|---|---|---|---|---|
| $F$ | $A, B$ | | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ | $N_6$ |
| 7 | 2, 1 | $x, y$ = | −1, 2 | 5, 2 | −4, 1 | 4, 3 | 1, 1 | 5, 3 |
| | | $L, M$ = | 19 . 7 | 7 . 67 | 31 . 7 | 7 . 79 | 1 . 7 | 97 . 7 |
| | | $N$ = | 133 | 469 | 217 | 553 | 7 | 679 |
| 13 | 1, 2 | $x, y$ = | −5, 4 | 7, 4 | 2, 1 | 2, 3 | 5, 1 | 7, 3 |
| | | $L, M$ = | 133 . 13 | 13 . 181 | 1 . 13 | 13 . 49 | 43 . 13 | 13 . 139 |
| | | $N$ = | 1729 | 2353 | 13 | 637 | 559 | 1807 |
| 133 | 11, 2 | $x, y$ = | −5, 4 | 17, 4 | 22, 9 | 22, 13 | −5, 9 | 17, 13 |
| | | $L, M$ = | 13 . 133 | 133 . 541 | 1321 . 133 | 133 . 1849 | 403 . 133 | 133 . 1459 |
| | | $N$ = | 1729 | 71953 | 175693 | 245917 | 53599 | 194047 |
| | 5, 6 | $x, y$ = | 13, 21 | 23, 12 | 10, 1 | 10, 11 | 13, 1 | 23, 11 |
| | | $L, M$ = | 1069 . 133 | 133 . 1789 | 73 . 133 | 133 . 793 | 211 . 133 | 133 . 1651 |
| | | $N$ = | 142177 | 237937 | 9709 | 105469 | 28063 | 199583 |

**34.** *Trin-Aurifeuillian Chains.*—Let a series of Trin-Aurifeuillians, $N_1, N_2, N_3, \ldots, N_r$, be taken, so that

$$N_r = x_r^4-3x_r^2y_r^2+9y_r^4 = L_rM_r, \text{ (for all values of } r), \quad (107)$$

and let the parts $(x_r, y_r)$ of the successive functions $(N_r)$ be taken, so that

*Case* i. $y_1 = y_2 = \ldots = y_r$, (an *even* number),

$x_2 = x_1 + 3y_1$, $x_3 = x_2 + 3y_2$, ..., $x_r = x_{r-1} + 3y_{r-1}$, (all *odd*). (108, i)

*Case* ii. $x_1 = x_2 = \ldots = x_r$, (an *even* number),

$y_2 = y_1 + x_1$, $y_3 = y_2 + x_2$, ..., $y_r = y_{r-1} + x_{r-1}$, (all *odd*). (108, ii)

*Case* iii. $x_1^2 - 3y_1^2 = x_2^2 - 3y_2^2 = \ldots = x_r^2 - 3y_r^2 = z^2$, (an *odd* square),

where $x_r$, $y_r$ are one *odd*, one *even*. (108, iii)

Now, by the formulæ for $I'$, $M'$; $I''$, $M''$, Art. **21**,

*Case* i. $M'_{r-1} = (x_{r-1} + \frac{3}{2}y_{r-1})^2 + 3\,(\frac{1}{2}y_{r-1})^2$, $L'_r = (x_r - \frac{3}{2}y_r)^2 + 3\,(\frac{1}{2}y_r)^2$,

*Case* ii. $M''_{r-1} = (\frac{1}{2}x_{r-1})^2 + 3\,(\frac{1}{2}x_{r-1} + y_{r-1})^2$, $L''_r = (\frac{1}{2}x_r)^2 + 3\,(\frac{1}{2}x_r \sim y_r)^2$.

*Case* iii. See next Article.

Hence, since $L = L' = L''$, and $M = M' = M''$, (the accents merely indicating the quadratic partition),

*Cases* i, ii, iii. $M_{r-1} = L_r$, (in each case), (109)

or, in other words, every *successive* pair $N_{r-1}$, $N_r$ have a common Aurifeuillian Factor. Such a series may be called a *Trin-Aurifeuillian Chain*, and has many interesting properties. Thus

$$N_r : N_{r-1} = L_r \,.\, M_r : L_{r-1} \,.\, M_{r-1} = M_r : L_{r-1}. \qquad (110a)$$

Similarly, $N_{r-2} : N_{r-3} = M_{r-2} : L_{r-3}$.

Hence in every *successive* quartet $(N_{r-3}, N_{r-2}, N_{r-1}, N_r)$

$$N_r \,.\, N_{r-2} : N_{r-1} \,.\, N_{r-3} = M_r : L_{r-3}. \qquad (110b)$$

Similarly, giving to $r$ every integer value from 1 to $2r$,

$$\frac{N_{2r} \,.\, N_{2r-2} \ldots N_6 \,.\, N_4 \,.\, N_2}{N_{2r-1} \,.\, N_{2r-3} \ldots N_5 \,.\, N_3 \,.\, N_1} = \frac{M_{2r}}{L_1}. \qquad (110c)$$

**34a.** *Trin-Aurifeuillian Chains.* (Case iii). — This case requires special study. It is defined by

$$x_r^2 - 3y_r^2 = z^2 \quad \text{(an } \textit{odd} \text{ square)}, \quad [r = 1, 2, 3, \&c.]. \qquad (111)$$

Here $z$ must be either an *odd prime* of form $z = 12\zeta \pm 1$, or a product

of such primes. Now, for any given *prime** value of $z$ of this form $>1$, there are *two minimum solutions*, say $(x_1, y_1)$, $(\xi_1, \eta_1)$, connected by the reciprocal relations

$$\left.\begin{array}{l} x_1 = 2\xi_1 \sim 3\eta_1,\ y_1 = \xi_1 \sim 2\eta_1\,; \quad \xi_1 = 2x_1 \sim 3y_1,\ \eta_1 = x_1 \sim 2y_1 \\ \text{one of } x_1, \xi_1 \textit{ odd}, \text{ one } \textit{even}\,;\ y_1, \eta_1 \text{ of opposite affection to } x_1, \xi_1 \end{array}\right\}. \quad (112a)$$

Also each such "base-solution"* gives rise to an infinite series of solutions, the *successive* solutions of either series being connected by the similar relations,

$$x_r = 2x_{r-1} + 3y_{r-1},\ y_r = x_{r-1} + 2y_{r-1}\,;\ \xi_r = 2\xi_{r-1} + 3\eta_{r-1},\ \eta_r = \xi_{r-1} + 2\eta_{r-1}, \quad (112b)$$

$$x_{r-1} = 2x_r - 3y_r,\ -y_{r-1} = x_r - 2y_r\,;\ \xi_{r-1} = 2\xi_r - 3\eta_r,\ -\eta_{r-1} = \xi_r - 2\eta_r. \quad (112c)$$

The last two relations are similar in form to, but not identical with, Equation (112*a*); such a pair of series will be called *Conjugate Series*.

Now let $$N_r = x_r^4 - 3x_r^2 y_r^2 + 9y_r^4 = L_r . M_r,$$

$$N_r = \xi_r^4 - 3\xi_r^2 \eta_r^2 + 9\eta_r^4 = \overline{L}_r . \overline{M}_r,$$

be the Trin-Aurifeuillians formed from $(x_r, y_r)$, $(\xi_r, \eta_r)$ respectively.

Also, for distinctness, suppose

$$x_1 \textit{ odd}\,;\ \text{then } x_1, \eta_1 \text{ are both } \textit{odd}\,;\ y_1, \xi_1 \textit{ even}\,;$$

and, by Art. 21,

$$L_1 = (x_1 \sim \tfrac{3}{2}y_1)^2 + 3\,(\tfrac{1}{2}y_1)^2, \quad \overline{L}_1 = (\tfrac{1}{2}\xi_1)^2 + 3\,(\tfrac{1}{2}\xi_1 \sim \eta_1)^2,$$

and, the relations (112*a*) now give

$$L_1 = \overline{L}_1, \quad (113)$$

so that the lowest Trin-Aurifeuillians $N_1$, $N_1$ of two conjugate series have the *same* smaller Aurifeuillian Factor (*i.e.*, $L_1 = \overline{L}_1$).

Again, the assumption $x_1$ *odd* involves

$$x_1, y_2, x_3, \ldots, x_{2r-1}, y_{2r}, x_{2r+1}, \text{ all } \textit{odd},$$
$$y_1, x_2, y_3, \ldots, y_{2r-1}, x_{2r}, y_{2r+1}, \text{ all } \textit{even}.$$

Hence, by Art. 21,

$$M'_{2r-1} = (x_{2r-1} + \tfrac{3}{2}y_{2r-1})^2 + 3\,(\tfrac{1}{2}y_{2r-1})^2\,; \quad L''_{2r} = (\tfrac{1}{2}x_{2r})^2 + 3\,(\tfrac{1}{2}x_{2r} \sim y_{2r})^2,$$
$$M''_{2r} = (\tfrac{1}{2}x_{2r})^2 + 3\,(\tfrac{1}{2}x_{2r} + y_{2r})^2\,; \quad L'_{2r+1} = (x_{2r+1} \sim \tfrac{3}{2}y_{2r+1})^2 + 3\,(\tfrac{1}{2}y_{2r+1})^2\,;$$

* When $z$ is *composite*, these "base-solutions" are more numerous: they are connected in pairs by the reciprocal relation (112*a*); and each base-solution gives rise to its own *Trin-Aurifeuillian Chain* as when $z$ is *prime*. *Example.*—When $z = 11.13$, there are *four* base solutions; see Table below.

and the relations (112*b*, *c*) now give

$$M'_{2r+1} = L''_{2r}, \quad M''_{2r} = L'_{2r+1},$$

which are equivalent to $M_{r-1} = L_r$ (for all values of $r$). (114)

By similar reasoning, in the conjugate $(\xi_r, \eta_r, N_r)$ series, $\overline{M}_{r-1} = \overline{L}_r$. Hence these series are *Trin-Aurifeuillian Chains*, as defined in last Article (Case iii). The property that each number $(N_r)$ of these series is expressible in the *three* forms $N = N` = N_{\text{vi}}$ has been already treated of in Art. 25, 26

From the above it is seen that any such Trin-Aurifeuillian $N_1$ gives rise to *two* chains, in each of which it is the lowest number, viz., one of Case iii and one of Case i or Case ii according as $y_1$ or $x_1$ is even. Also any number $L_1$ of form $(A^2+3B^2)$ gives rise to *four* chains in which it is a common Aurifeuillian Factor $(L_1)$ of the lowest number $(N_1)$ of each chain, viz., *one* of Case i, *one* of Case ii, and *two* of Case iii.

*Examples.*—The table shows the quadratic parts $(x_r, y_r)$ of a few numbers $N_1$, $N_2$, &c. of the Trin-Aurifeuillian Chains arising when $z = 11, 13,$ and $143$ $(= 11.13)$. The conjugate chains are in each case bracketed together. Each number $N_r$ can be expressed in the three forms $N$, $N`$, $N_{\text{vi}}$ by the formulæ of Art. 26 from the given $x$, $y$, $z$. The important case when $z = 1$ has been given in Art. 26 in fuller detail.

| $z$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
|---|---|---|---|---|
| | $x_1$ ; $y_1$ | $x_2$ ; $y_2$ | $x_3$ ; $y_3$ | $x_4$ ; $y_4$ |
| 11 | 13 ; 4 | 38 ; 21 | 139 ; 80 | 518 ; 299 |
| | 14 ; 5 | 43 ; 24 | 158 ; 91 | 589 ; 340 |
| 13 | 14 ; 3 | 37 ; 20 | 134 ; 77 | 499 ; 288 |
| | 19 ; 8 | 62 ; 35 | 229 ; 132 | 854 ; 493 |
| 143 | 146 ; 17 | 343 ; 180 | 1226 ; 703 | &c. |
| | 241 ; 112 | 818 ; 465 | 3031 ; 1748 | &c. |
| | 151 ; 28 | 386 ; 207 | 1393 ; 800 | &c. |
| | 218 ; 95 | 721 ; 408 | 2666 ; 1537 | &c. |

**35.** *Bin- and Trin-Aurifeuillians.*—These two functions have many properties in common, which are worth study together. A summary is given below. [Here C.F. denotes *Common Factor.*]

| Art. | *Bin-Aurifeuillians*, $N$, $\mathfrak{N}$. | Art. | *Trin-Aurifeuillians*, $N$, $\mathfrak{N}$. |
|---|---|---|---|
| 9. | Dimorphs are possible. | 24. | Dimorphs are possible. |
| 10. | $N = \mathfrak{L}$ or $\mathfrak{M}$ of $\mathfrak{N}$, when $x^2 \mp 2y^2$, or $2y^2 - x^2 = z^2$. | 26. | $N = \mathfrak{L}$ of $\mathfrak{N}$, when $x^2 - 3y^2 = z^2$. |
| 11. | $N = X^4 + Y^4$ is impossible. | 26. | $N = (x^6 + z^6) \div (x^2 + z^2)$, when $x$ or $2x = m^2 + 3n^2$. |
| 11. | $N = X^4 - Y^4$ is possible. | | |
| 13a. | $N_1 N_2 = N$, when $x_2^2 \mp x_1^2 = (2y_1)^2 = (2y_2)^2$; or $y_2^2 \mp y_1^2 = x_1^2 = x_2^2$. | 28a. | $N_1 N_2 = N$, when $x_2^2 \sim x_1^2 = y_1^2 = y_2^2$, or $y_2^2 \sim y_1^2 = x_1^2 = x_2^2$. |
| 17. | Factorisation of $(2^{4\nu+2}+1)$. | 31. | Factorisation of $(3^{6\nu+3}+1)$. |
| 18. | Factorisation of $(2y^2)^{4\nu+2}+1$. | 32. | Factorisation of $(3y^2)^{6\nu+3}+1$. |
| 19. | $F = a^2 + b^2$ is C.F. of 4 numbers $N$, $N_1 + N_4 = N_2 + N_3 = 6F^2$. Also $N$ is C.F. of 8 numbers $\mathfrak{N}$. | 33. | $F = A^2 + 3B^2$ is C.F. of 6 numbers $N$, $N_1 + N_4 = N_2 + N_3 = N_4 + N_5 = 14F^2$. Also $N$ is C.F. of 12 numbers $\mathfrak{N}$. |
| 20. | Bin-Aurifeuillian Chain $(M_{r=1} = L_r)$, when i. $y_r = y_1$, and $x_r = x_{r-1} + 2y_1$, or ii. $x_r = x_1$, and $y_r = x_1 + y_{r-1}$. | 34. | Trin-Aurifeuillian Chain $(M_{r-1} = L_r)$, when i. $y_r = y_1 = \epsilon$, and $x_r = x_{r-1} + 3y_1$, or ii. $x_r = x_1 = \epsilon$, and $y_r = x_1 + y_{r-1}$. |

**36.** *Bi-Trin-Aurifeuillians.*—The question now arises as to whether any numbers ($N$) exist, expressible at once in the Bin-Aurifeuillian ($N_2$), Trin-Aurifeuillian ($N_3$), and Trin-Ant-Aurifeuillian ($N_3'$) forms.

A *general* solution seems very difficult to obtain; but an interesting partial solution is obtainable by Art. **26**; from which it is seen that, writing

$$N_{\text{vi}} = \frac{x^6 + z^6}{x^2 + z^2}, \quad N_3 = \frac{x^6 + 3^3 y^6}{x^2 + 3y^2}, \quad N_3' = \frac{z^6 - 3^3 y^6}{z^2 - 3y^2}, \tag{115}$$

then the relation

$$x^3 - 3y^2 = z^2 \tag{116}$$

suffices that the number $N_{\text{vi}}$ should be expressible as both $N_3$ and $N_3'$.

Next, taking

$$z = \zeta^3, \quad x = 2\xi^2, \tag{116a}$$

the function $N_{\text{vi}} = \dfrac{\zeta^{12} + 2^6 \xi^{12}}{\zeta^4 + 2^2 \xi^4}$, a cubic Bin-Aurifeuillian, (say $N_2'''$).

The relation (116) thus becomes

$$4\xi^4 - \zeta^4 = 3y^2. \qquad (116b)$$

Hence, every solution of this Diophantine gives a number expressible in the *four* ways

$$N_{\text{vi}} = N_3 = N_3' = N_2'''. \qquad (117)$$

A *general* solution of (116*b*) seems difficult; but, if $\xi$, $\zeta$, $y$ be any *given* solution, then a *second* solution is given* by the following formulæ :—

$$\left.\begin{aligned} \xi_1 &= \xi\,(27y^4 \sim 64\xi^4), \quad \zeta_1 = \zeta\,(27y^4 \sim 4\zeta^8) \\ y_1 &= y\left\{324y^8 \sim 3\,(4\xi^4 + \zeta^4)^4\right\} \end{aligned}\right\}. \qquad (118)$$

Similarly a *third* solution ($\xi_2$, $\zeta_2$, $y_2$) may be derived; and, repeating the process, an infinity of solutions may be derived from any given solution ($\xi$, $\zeta$, $y$).

*Examples.*—One solution of (116*b*) is obvious, viz., $\xi = 1$, $\zeta = 1$, $y = 1$; giving $z = 1$, $x = 2$, whence

$$N = 13 = \frac{2^6 + 1^6}{2^2 + 1^2} = \frac{2^6 + 3^3 . 1^6}{2^2 + 3 . 1^2} = \frac{1^6 \sim 3^3 . 1^6}{1^2 \sim 3 . 1^2} = \frac{1^{12} + 2^6 . 1^{12}}{1^4 + 2^2 . 1^4}.$$

Then (118) give a *second* solution, $\xi_1 = 37$, $\zeta_1 = 23$, $y_1 = 1551$; giving $z = 529$ $x = 2738$, whence

$$N = \frac{2738^6 + 529^6}{2738^2 + 529^2} = \frac{2738^6 + 3^3 . 1551^6}{2738^2 + 3 . 1551^2} = \frac{529^6 \sim 3^3 . 1551^6}{529^2 \sim 3 . 1551^2} = \frac{23^{12} + 2^6 . 37^{12}}{23^4 + 2^2 . 37^4}.$$

The next solution ($\xi_2$, $\zeta_2$, $y_2$) is in too high numbers to be practically useful.

**36a.** *Factorisation of above numbers.*—These numbers, being Aurifeuillians of orders 2 and 3, admit of resolution into Aurifeuillian Factors *in each way;* this greatly helps their ultimate resolution into *prime* factors.

Thus $N = N_2''' = S_2 . T_2$, (Art. **16a**); and $N = N_3 = L_3 M_3$, (Art. **21**).

*Ex.*—Taking the above high number $N = N_2''' = N_3$,

$$N_2''' = \frac{23^{12} + 2^6 . 37^{12}}{23^4 + 2^2 . 37^4} = S_2\, T_2 = 3{,}664{,}453 \,.\, 14{,}785{,}321.$$

$$N_3 = \frac{2738^6 + 3^3 . 1551^6}{2738^2 + 3 . 1551^2} = L_3 M_3 = 1{,}973{,}533 \,.\, 27{,}453{,}361.$$

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* See M. Desboves's "Mémoire sur la résolution en nombres entiers de l'équation $aX^m + bY^m = cZ^m$," Art. 16, Results (67); published in *Nouv. Ann. de Mathém.*, 2e Sér., t. XVIII, 1879.

Here $T_2, M_3$ are both beyond the range of the Factor-Tables; but the resolution of $S_2$, $L_3$ supplies the small factors of $M_3$, $T_2$ respectively; thus

$$N = N_2''' = (13 \cdot 61 \cdot 4621)(457 \cdot 32{,}353) = N_3 = (61 \cdot 32{,}353)(13 \cdot 457 \cdot 4621).$$

This large number (14 figures) could not have been otherwise easily factorised.

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N.B.—It is hoped to publish later similar investigations on Aurifeuillians of higher orders, *e.g.*, order 5, 6, &c.

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## APPENDIX.

**37.** *High Primes.*—The following high primes* occurring in this paper (see Tables, Art. **18, 31, 32**) seem worth separate record. The initials (B, C) indicate the name of the original discoverer, as explained in the footnote ‡ to Art. **31**.

| Prime | Factor of | Initial | Prime | Factor of | Initial |
|---|---|---|---|---|---|
| 9,987,121 | $3200^6+1$ | C | 337,448,233 | $3^{99}+1$ | C |
| | | | 512,130,481 | $363^{15}+1$ | C |
| 31,720,981 | $72^{10}+1$ | C | 550,554,229 | $3^{63}+1$ | C |
| 32,386,261 | $507^{15}+1$ | C | 581,433,521 | $578^{10}+1$ | C |
| 40,768,781 | $450^{10}+1$ | C | | | |
| 43,120,981 | $200^{10}+1$ | C | 1,162,320,517 | $3^{57}+1$ | B, C |
| 47,336,293 | $12^{27}+1$ | B | 1,202,388,841 | $192^{15}+1$ | C |
| 69,772,181 | $722^{10}+1$ | C | 1,447,960,181 | $200^{10}+1$ | C |
| 79,011,151 | $363^{15}+1$ | C | 1,561,577,671 | $972^{15}+1$ | C |
| | | | 5,879,415,781 | $3^{81}+1$ | C |
| 115,693,471 | $108^{15}+1$ | C | | | |
| 129,159,847 | $3^{51}+1$ | B, C | 22,434,744,889 | $3^{31}+1$ | C |
| 161,466,931 | $108^{15}+1$ | C | 23,535,794,707 | $3^{23}+1$ | C |
| 172,938,811 | $768^{15}+1$ | C | | | |
| 253,765,901 | $882^{10}+1$ | C | 378,450,588,583 | $3^{99}+1$ | C |

* Believed to be all new. The mode of detection of these primes is explained in the footnotes †, ¶ to Art. **18** and ‡ (3) to Art. **31**.

*The Transformation of Linear Partial Differential Operators by Extended Linear Continuous Groups.* *By* E. B. Elliott.
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1. On a number of previous occasions, and more particularly in a paper "On the Interchange of the Variables in certain Linear Differential Operators" [*Phil. Trans.*, Vol. CLXXXI. (1890), A, pp. 19–51], I have applied a simple but artificial method to the transformation of linear differential operators from forms in which their arguments are the successive derivatives of a dependent with regard to one or more independent variables to the forms which they assume when dependent and independent variables are transposed. Again, in a paper "On the Reversion of Partial Differential Expressions with Two Independent and Two Dependent Variables" (*Proc. Lond. Math. Soc.*, Vol. XXII., pp. 79–104), I have somewhat extended the method, by application to a case in which there are more independent variables than one. Up to the present time I have not, nor, so far as I know, has any one else, applied the method to the transformation of operators consequent on continuous transformations of, or substitutions for, dependent and independent variables in terms of new dependent and independent variables; but it is one which admits of wide applications in this direction. The following paper deals with a class of such applications.

I.

2. There is no new difficulty of principle when the formulæ of transformation are linear, and the original variables are supposed connected by one relation only, so that a single one of them is dependent and the rest independent. Let us first consider the case of two variables $x, y$, supposed connected by one relation of quite arbitrary form and not necessarily known, and consider them, and the successive derivatives of the latter with regard to the former, to be expressed in terms of $x', y'$, and the successive derivatives of $y'$ with regard to $x'$ by the transformations of the general linear group

$$\left.\begin{aligned} x &= a_1x' + b_1y' + c_1 \\ y &= a_2x' + b_2y' + c_2 \end{aligned}\right\} \qquad (1)$$

and its extensions.

Let $y_r$ and $y'_r$ denote respectively $\frac{1}{r!}\frac{d^r y}{dx^r}$ and $\frac{1}{r!}\frac{d^r y'}{dx'^r}$ for all positive integral values of $r$. Corresponding finite increments $\xi$, $\eta$ of $x$ and $y$ are connected by the relation

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots, \tag{2}$$

and corresponding increments $\xi'$, $\eta'$ of $x'$ and $y'$ by the relation

$$\eta' = y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \ldots . \tag{3}$$

Moreover, if the increments $\xi'$, $\eta'$ are those of $x'$, $y'$ necessitated by the increments $\xi$, $\eta$, of $x$, $y$,

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1\eta' \\ \eta &= a_2\xi' + b_2\eta' \end{aligned}\right\}, \tag{4A}$$

which we may also write

$$\left.\begin{aligned} (a_1b_2 - a_2b_1)\,\xi' &= b_2\xi - b_1\eta \\ (a_1b_2 - a_2b_1)\,\eta' &= a_1\eta - a_2\xi \end{aligned}\right\}. \tag{4B}$$

We do not need for present purposes the expressions for either of the sets $y_1, y_2, y_3, \ldots$ and $y'_1, y'_2, y'_3, \ldots$ in terms of the other, but it is important to notice that the expressions in question do not involve explicitly $x'$, $y'$ or $x$, $y$, but that

$$y_1 = \frac{a_2 + b_2 y'_1}{a_1 + b_1 y'_1},$$

$$y_2 = \frac{(a_1b_2 - a_2b_1)\,y'_2}{(a_1 + b_1 y'_1)^3},$$

and generally that we pass from $y_r$ to $y_{r+1}$ by a total differentiation with regard to $x'$, *i.e.*, by operation with

$$2y'_2\frac{\partial}{\partial y'_1} + 3y'_3\frac{\partial}{\partial y'_2} + 4y'_4\frac{\partial}{\partial y'_3} + \ldots,$$

and division by $(r+1)$ times $a_1 + b_1 y'$. Herein lies the special virtue of an extended (*erweiterte*) *linear* group that its extensions alone, apart from the original linear equations, form a group—in the present case a group of the fourth order or four-parameter group, the parameters being $a_1$, $b_1$, $a_2$, $b_2$, while the complete extended group is of the sixth order, having the six parameters $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$.

The problem before us is the following. If

$$f(x, y, y_1, y_2, y_3, \ldots) \equiv F(x', y', y'_1, y'_2, y'_3, \ldots)$$

is any equivalence which holds, whatever be the relation connecting

$x$, $y$, in virtue of the transformation (1) and its extensions, it is required to express

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}, \ \ldots$$

as linear functions, with constant or variable coefficients, of

$$\frac{\partial F}{\partial x'}, \frac{\partial F}{\partial y'}, \frac{\partial F}{\partial y_1'}, \frac{\partial F}{\partial y_2'}, \frac{\partial F}{\partial y_3'}, \ \ldots.$$

The coefficients will, of course, be independent of the forms of $f$ and $F$.

We have seen that $x'$, $y'$ only enter explicitly in $F$, if at all, in virtue of the explicit occurrence of $x$, $y$ in $f$, and conversely. In fact,

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y'} &= b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} \end{aligned}\right\}, \tag{5A}$$

or, otherwise written,

$$\left.\begin{aligned} (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial x} &= b_2 \frac{\partial}{\partial x'} - a_2 \frac{\partial}{\partial y'} \\ (a_1 b_2 - a_2 b_1) \frac{\partial}{\partial y} &= a_1 \frac{\partial}{\partial y'} - b_1 \frac{\partial}{\partial x'} \end{aligned}\right\}, \tag{5B}$$

where, for shortness, $f$ is omitted as the subject on which $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ act, and the equivalent $F$ as that on which $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}$ act.

In order to transform $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \ldots$, let us regard the relations (2), (3), (4), which connect finite increments of $x$, $y$, $x'$, $y'$. Eliminating $\eta$ and $\eta'$, we have

$$\left.\begin{aligned} \xi &= a_1 \xi' + b_1 (y_1' \xi' + y_2' \xi'^2 + y_3' \xi'^3 + \ldots) \\ y_1 \xi + y_2 \xi^2 + y_3 \xi^3 + \ldots &= a_2 \xi' + b_2 (y_1' \xi' + y_2' \xi'^2 + y_3' \xi'^3 + \ldots) \end{aligned}\right\}, \tag{6A}$$

which do not involve $x$, $y$, $x'$, $y'$ explicitly. These two equalities cannot be independent, but are identical in meaning. For substitution from (4B) for $\xi'$, $\eta'$ in (3) gives a relation in $\xi$, $\eta$, and $y_1'$, $y_2'$, $y_3'$, ..., which, when these last are replaced by their expressions in terms of $y_1$, $y_2$, $y_3$, ..., must be identical with (2), as otherwise corresponding increments $\xi$, $\eta$ of two variables connected by a single relation would be connected by two relations, as is not the case. Let us consider the

equivalent relations (6A) in the slightly more convenient forms

$$\left.\begin{aligned} \xi &= a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \\ b_2\xi - b_1(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots) &= (a_1b_2 - a_2b_1)\,\xi' \end{aligned}\right\}. \qquad (6B)$$

Here, since $y$ is a perfectly arbitrary function of $x$, and $\xi$ is also perfectly arbitrary, we may, for a given $x$, regard $y_1, y_2, y_3, \ldots$ and $\xi$ as quite independent variables. We must then consider $y_1', y_2', y_3', \ldots$ and $\xi'$ as dependent variables, all but the last of them as dependent on $y_1, y_2, y_3, \ldots$ in virtue of the extensions of (1), and $\xi'$ as dependent on these and also on $\xi$ in virtue of either of the equivalent relations (6B) at present before us. Now, of the independent variables $y_1, y_2, y_3, \ldots, \xi$, let $y_r$ alone receive an infinitesimal increment $\delta y_r$, and let $\delta\xi'$ be the consequent increment of $\xi'$. The equalities (6B) yield us the two equivalent results

$$0 = \{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \ldots)\}\delta\xi' + b_1\left\{\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \ldots\right\}\delta y_r,$$

$$-b_1\xi'^r\delta y_r = (a_1b_2 - a_2b_1)\,\delta\xi'.$$

Accordingly,

$$\xi'\frac{\partial y_1'}{\partial y_r} + \xi'^2\frac{\partial y_2'}{\partial y_r} + \xi'^3\frac{\partial y_3'}{\partial y_r} + \ldots \equiv (a_1b_2 - a_2b_1)^{-1}\,\xi'^r\{a_1 + b_1(y_1' + 2y_2'\xi' + 3y_3'\xi'^2 + \ldots)\}. \qquad (7)$$

In this, on the right, substitute for $\xi$ its equivalent

$$\xi = a_1\xi' + b_1(y_1'\xi' + y_2'\xi'^2 + y_3'\xi'^3 + \ldots) \qquad (8)$$

from (4A). We then have on the right a rational integral expression in $\xi'$, which we can expand and arrange by powers of $\xi'$. We are thus given in (7) the identical equality of two expansions in powers of $\xi'$, the coefficients on the two sides being all in no way dependent on $\xi'$. Corresponding coefficients on left and right must then be equal. In other words, the coefficients of successive powers of $\xi'$ on the right are, in succession, the expressions for

$$\frac{\partial y_1'}{\partial y_r}, \quad \frac{\partial y_2'}{\partial y_r}, \quad \frac{\partial y_3'}{\partial y_r}, \quad \ldots$$

in terms of $y_1', y_2', y_3', \ldots$. It appears at once that the lowest power of $\xi'$ which occurs on the right is $\xi'^r$, so that $\dfrac{\partial y_r'}{\partial y_r}$ is the first of these derivatives which does not vanish. In other words, as is otherwise obvious, $y_r'$ is the lowest of the derivatives of $y'$ with regard to $x'$, whose expression in terms of $y_1, y_2, y_3, \ldots$ involves $y_r$.

3. But in the identity (7), when the right-hand member is, as above described, expressed in terms of $\xi'$ and expanded, we may put for $\xi'^s$ on the two sides any quantity or operator we please, and so for every other power of $\xi'$. Let us then put the corresponding $\frac{\partial}{\partial y'_s}$ for each $\xi'^s$ ($s = 1, 2, 3, \ldots, \infty$), the subject of operation being any function $F(x', y', y'_1, y'_2, y'_3, \ldots)$. We thus get on the left

$$\frac{\partial y'_1}{\partial y_r}\frac{\partial}{\partial y'_1} + \frac{\partial y'_2}{\partial y_r}\frac{\partial}{\partial y'_2} + \frac{\partial y'_3}{\partial y_r}\frac{\partial}{\partial y'_3} + \ldots,$$

*i.e.*,

$$\frac{\partial}{\partial y_r},$$

the subject of operation being the function $f(x, y, y_1, y_2, y_3, \ldots)$ of unaccented letters which is equivalent to $F$. Consequently, to obtain the equivalent of $\frac{\partial}{\partial y_r}$ in the form

$$A'_1\frac{\partial}{\partial y'_1} + A'_2\frac{\partial}{\partial y'_2} + A'_3\frac{\partial}{\partial y'_3} + \ldots,$$

which is its proper one when the subject of operation is expressed in terms of $x', y', y'_1, y'_2, y'_3, \ldots$ instead of in terms of $x, y, y_1, y_2, y_3, \ldots$, we have the following rule :—Expand

$$(a_1b_2 - a_2b_1)^{-1}\left\{a_1\xi' + b_1(y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \ldots)\right\}^r$$
$$\times\left\{a_1 + b_1(y'_1 + 2y'_2\xi' + 3y'_3\xi'^2 + \ldots)\right\}$$

in ascending powers of $\xi'$, writing the power of $\xi'$ last in each term, and then put in the expanded result, for every power $\xi'^s$ which occurs, the corresponding $\frac{\partial}{\partial y'_s}$.

Referring to (6A), this product to be expanded is seen to be briefly expressible as

$$(a_1b_2 - a_2b_1)^{-1}\xi^r\left\{\frac{\partial\xi}{\partial\xi'} + \frac{\partial\xi}{\partial\eta'}\frac{\partial\eta'}{\partial\xi'}\right\},$$

or, more shortly still, as

$$(a_1b_2 - a_2b_1)\,\xi^{-1}\frac{d\xi}{d\xi'},$$

where $\xi$ is supposed to be expressed, as if an actual finite increment

of $x$, in terms of $\xi'$ and the derivatives $y'_s$, which latter are not functions of $\xi'$.

Now, just as we have found it convenient to take $\xi'^s$ as a symbolic representative of $\dfrac{\partial}{\partial y'_s}$ for every positive integral $s$, so it occurs to take $\xi^r$ as a symbolic representative of $\dfrac{\partial}{\partial y_r}$. Our result may then be stated as follows:—If

$$f(x, y, y_1, y_2, y_3, \ldots) \equiv F(x', y', y'_1, y'_2, y'_3, \ldots)$$

be any identity consequent on the general linear transformation (1) and its extensions, then

$$\frac{\partial}{\partial y_r} f(x, y, y_1, y_2, y_3, \ldots) \quad (r = 1, 2, 3, \ldots, \infty),$$

whose symbolical form is $\xi^r f$,

has for its equivalent

$$(a_1 b_2 - a_2 b_1)^{-1} \xi^r \frac{d\xi}{d\xi'} F(x', y', y'_1, y'_2, y'_3, \ldots),$$

where $$a_1\xi' + b_1(y'_1\xi' + y'_2\xi'^2 + y'_3\xi'^3 + \ldots)$$

has to be substituted for $\xi$, as it would have were $\xi$, $\xi'$ corresponding increments of $x$, $x'$ instead of mere symbols, after which the result obtained has to be expanded in powers of $\xi'$, and then in the expansion every power $\xi'^s$ to be replaced by the corresponding differential operator $\dfrac{\partial}{\partial y'_s}$.

The rule for transforming any linear operator

$$P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + A_1\frac{\partial}{\partial y_1} + A_2\frac{\partial}{\partial y_2} + A_3\frac{\partial}{\partial y_3} + \ldots$$

is immediately deduced. As to the first two terms they become at once

$$(a_1 b_2 - a_2 b_1)^{-1} \left\{ (Pb_2 - Qb_1)\frac{\partial}{\partial x'} + (Qa_1 - Pa_2)\frac{\partial}{\partial y'} \right\},$$

by (5B). As to the rest of the operator, write it symbolically

$$A_1\xi + A_2\xi^2 + A_3\xi^3 + \ldots,$$

multiply it by $$(a_1 b_2 - a_2 b_1)^{-1}\frac{d\xi}{d\xi'},$$

then substitute for $\xi$ its expansion in terms of $\xi'$ as above, expand the result and arrange it as a series in powers of $\xi'$, writing each such power of $\xi'$ as the last factor in its term, and, finally, for every power $\xi'^s$ in the expansion, write the corresponding $\frac{\partial}{\partial y'_s}$.

If $P$, $Q$, $A_1$, $A_2$, $A_3$, ... are constants, the transformation is thus completed. If they are functions of $x$, $y$, $y_1$, $y_2$, $y_3$, ..., they need for useful purposes to be expressed in terms of $x'$, $y'$, $y'_1$, $y'_2$, $y'_3$, ... by means of (1) and its extensions. But, as we shall see, this expression is automatically effected in a wide and important class of cases.

Since
$$\frac{d\xi}{d\xi'}\,\frac{d\xi'}{d\xi} = 1,$$

and since the determinant of the coefficients of $\xi$, $\eta$ in the expressions for $\xi'$, $\eta'$ is the reciprocal of the determinant of the coefficients of $\xi'$, $\eta'$ in the expressions for $\xi$, $\eta$, we at once see that, as should be the case, the rule obtained for the expression of an unaccented operator as an accented one affords the exactly corresponding rule for the expression of an accented operator as unaccented.

4. Let us apply these conclusions to the transformation of what I have called in the *Phil. Trans.* (*loc. cit.* in § 1) MacMahon operators. Such operators (of four elements) are those included in the definition

$$\{\mu, \nu\,;\ m, n\}_y = \frac{1}{m}\,\Sigma\left\{(\mu+\nu s)\,Y_s^{(m)}\,\frac{\partial}{\partial y_{n+s}}\right\},$$

the summation being with regard to $s$, which assumes in turn all positive integral values not less than the greater of $m$ and $-n+1$, $m$ and $n$ being integral or zero, and $Y_s^{(m)}$ denoting the coefficient of $\xi^s$ in the expansion of

$$(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m.$$

This operator is the result of replacing in the $(\mu, \nu\,;\ m, n)$ of Major MacMahon's remarkable first paper on multilinear operators* by

$$a, b, c, d, \ldots \quad \text{by} \quad 0, y_1, y_2, y_3, \ldots,$$

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* "The Theory of a Multilinear Partial Differential Operator, with Applications to the Theories of Invariants and Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 61–88).

or, again, that of replacing in $(\mu+\nu m, \nu;\ m, n+m-1)$,

$$a, b, c, d, \ldots \quad \text{by} \quad y_1, y_2, y_3, y_4, \ldots .$$

We shall confine attention to cases in which $m$ is not negative, and $m+n$ not less than unity. The summation for $s$ in $\{\mu, \nu;\ m, n\}_y$ is then $\sum_{s=m}^{s=\infty}$, *i.e.*, no coefficient $Y_s^{(m)}$ which actually occurs in the expansion of the multinomial is absent from the summation. The elaboration of results for the excepted cases of $m+n<1$ could be added, much as in my paper to which reference has been made, but would unduly lengthen the present communication. The case $m=0$ has some speciality, and will be only partially included.

5. Symbolically expressed, as in § 3, it is clear that

$$\{\mu, \nu;\ m, n\}_y = \frac{\mu}{m}\xi^n\eta^m + \nu\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi},$$

where $\eta$ stands for $y_1\xi+y_2\xi^2+y_3\xi^3+\ldots$; and that in particular

$$\{1, 0;\ m, n\}_y = \frac{1}{m}\xi^n\eta^m,$$

and

$$\{0, 1;\ m, n\}_y = \xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}.$$

We can at once then apply the rule arrived at in § 3; and obtain that the results of transforming these two last operators by the extended linear scheme (1) are respectively in symbolic form

$$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\xi}{d\xi},$$

*i.e.*,

$$\frac{1}{m}(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_1+b_1\frac{d\eta'}{d\xi'}\right),$$

and

$$(a_1b_2-a_2b_1)^{-1}\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}\frac{d\xi}{d\xi'},$$

*i.e.*,

$$(a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^{n+1}(a_2\xi'+b_2\eta')^{m-1}\left(a_2+b_2\frac{d\eta'}{d\xi'}\right).$$

Each of these expressions expanded is a sum of multiples of terms of the two types $\xi'^p\eta'^{m+n-p}$, $\xi'^p\eta'^{m+n-p}\frac{d\eta'}{d\xi'}$, *i.e.*, a sum of multiples of MacMahon $y'$-operators with $m'+n'=m+n$. In fact, if, as usual, $\binom{n}{r}$ denote the number of combinations of $n$ things $r$ together, the

results of the transformation are, when $n$ as well as $m$ is not negative,

$$m\,(a_1b_2-a_2b_1)\,\{1,\,0;\;m,\,n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^{r+1} b_1^{n-r} a_2^s b_2^{m-s} \xi'^{r+s} \eta'^{m+n-r-s}$$

$$+ \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a^r b^{n-r+1} a_2^s b_2^{m-s} \xi'^{r+s} \eta'^{m+n-r-s} \frac{d\eta'}{d\xi'}$$

$$\equiv \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^{r+1} b_1^{n-r} a_2^s b_2^{m-s}\,(m+n-r-s) \times \{1,\,0;\;m+n-r-s, r+s\}_{y'}$$

$$+ \sum_{r=0,\,s=0}^{r=n,\,s=m} \binom{n}{r}\binom{m}{s} a_1^r b_1^{n-r+1} a_2^s b_2^{m-s} \{0,\,1;\;m+n-r-s+1,\, r+s-1\}_{y'},$$

and

$$(a_1b_2-a_2b_1)\,\{0,\,1;\;m,\,n\}_y$$

$$\equiv \sum_{r=0,\,s=0}^{r=n+1,\,s=m-1} \binom{n+1}{r}\binom{m-1}{s} a_1^r b_1^{n+1-r} a_2^{s+1} b_2^{m-1-s}\,(m+n-r-s) \times \{1,\,0;\;m+n-r-s,\, r+s\}_{y'}$$

$$+ \sum_{r=0,\,s=0}^{r=n+1,\,s=m-1} \binom{n+1}{r}\binom{m-1}{s} a_1^r b_1^{n+1-r} a_2^s b_2^{m-s} \times \{0,\,1;\;m+n-r-s+1,\, r+s-1\}_{y'}.$$

The transformation of $\{\mu,\,\nu;\;m,\,n\}_y$ is immediately deduced by adding $\frac{\mu}{m}$ times the former of these results to $\nu$ times the latter.

6. Such formulæ have necessarily some cumbrousness of expression; and clearness of realization is, I am sure, gained by thinking of them as before us in their symbolical forms, namely,

$$\xi^n\eta^m \equiv (a_1b_2-a_2b_1)^{-1}\,\xi^n\eta^m\frac{d\xi}{d\xi'}$$

$$= (a_1b_2-a_2b_1)^{-1}\,(a_1\xi'+b_1\eta')^n\,(a_2\xi'+b_2\eta')^m\left(a_1+b_1\frac{d\eta'}{d\xi'}\right);$$

and

$$\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi} \equiv (a_1b_2-a_2b_1)^{-1}\,\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi'}$$

$$= (a_1b_2-a_2b_1)^{-1}\,(a_1\xi'+b_1\eta')^{n+1}\,(a_2\xi'+b_2\eta')^{m-1}\left(a_2+b_2\frac{d\eta'}{d\xi'}\right),$$

where before interpretation expansion on the left has to be in powers of $\xi$, and on the right in powers of $\xi'$.

The interpretation presents no difficulty when $m$ as well as $m+n$

is a positive integer. Moreover, the value $m = 0$ has no speciality as far as the transformation of $\xi^n\eta^m$ is concerned; but this value has in connexion with the transformation of $\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}$. As it is not proposed to deal with this, the second of the two above transformations is better written

$$\xi^n\eta^m\frac{d\eta}{d\xi} \equiv (a_1b_2-a_2b_1)^{-1}\xi^n\eta^m\frac{d\eta}{d\xi'}$$
$$= (a_1b_2-a_2b_1)^{-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m\left(a_2+b_2\frac{d\eta'}{d\xi'}\right),$$

in which, as in the first, $m$ is a positive integer or zero, and $m+n$ a positive integer. We will, in fact, only attend to cases in which neither $m$ nor $n$ is negative in the last article, the cases to which the forms of equalities written at the end of that article apply.

7. Let us speak of an operator which is a linear function with constant coefficients of operators of the two types $\xi^n\eta^m$, $\xi^n\eta^m\frac{d\eta}{d\xi}$ with the same value of $m+n$ as being of the $(m+n)^{\text{th}}$ order. The results before us involve the fact that it transforms into another operator of the same order $m+n$.

Some facts with regard to the linearly independent operators

$$\xi,\ \eta,\ \xi\frac{d\eta}{d\xi},\ \eta\frac{d\eta}{d\xi},\quad i.e.,\ \tfrac{1}{2}\frac{d}{d\xi}(\eta^2),$$

of the first order will now be adduced. Written at length they are respectively

$$\xi = \frac{\partial}{\partial y_1},$$

$$\eta = y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}+\ldots,$$

$$\xi\frac{d\eta}{d\xi} = y_1\frac{\partial}{\partial y_1}+2y_2\frac{\partial}{\partial y_2}+3y_3\frac{\partial}{\partial y_3}+\ldots,$$

$$\eta\frac{d\eta}{d\xi} = 2.\tfrac{1}{2}y_1^2\frac{\partial}{\partial y_1}+3y_1y_2\frac{\partial}{\partial y_2}+4\,(y_1y_3+\tfrac{1}{2}y_2^2)\frac{\partial}{\partial y_3}+5\,(y_1y_4+y_2y_3)\frac{\partial}{\partial y_4}$$
$$+6\,(y_1y_5+y_2y_4+\tfrac{1}{2}y_3^2)\frac{\partial}{\partial y_5}+\ldots$$
$$= -y_1^2\xi+y_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)+V,$$

where $V$ is Sylvester's operator

$$4.\tfrac{1}{2}y_2^2\frac{\partial}{\partial y_3}+5y_2y_3\frac{\partial}{\partial y_4}+6\,(y_2y_4+\tfrac{1}{2}y_3^2)\,\frac{\partial}{\partial y_5}+7\,(y_2y_5+y_3y_4)\,\frac{\partial}{\partial y_6}+\dots .$$

They are recognized as being the known operators which determine the four independent infinitesimal transformations of the group which consists of the extensions of the general linear group (1). (A more general fact including this will be proved in § 16 below.) The general infinitesimal transformation of this group of extensions is determined by

$$\lambda\xi+\mu\eta+\nu\xi\frac{d\eta}{d\xi}+\varpi\eta\frac{d\eta}{d\xi}.$$

*i.e.*, by the general operator of the first order. The functions of the derivatives $y_1, y_2, y_3, \dots$ which are absolute differential invariants of the group of extensions are exactly those functions which have the four operators for annihilators. They are also the absolute differential invariants of the extended linear group (1) itself, as the two annihilators of such invariants given by infinitesimal variation of $c_1$ and $c_2$ in the first place are $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$, which show that there is no differential invariant of the group which involves $x$ or $y$.

Now, a function $f(y_1, y_2, y_3, \dots)$ annihilated by $\xi$ or $\dfrac{\partial}{\partial y_1}$ is free from $y_1$; one annihilated by $\eta$ is homogeneous of degree zero; one annihilated by $\xi\dfrac{d\eta}{d\xi}$ is isobaric of weight zero, weight being measured by sum of suffixes; and one annihilated by $\eta\dfrac{d\eta}{d\xi}$ in addition to the above, or, less exactingly, in addition to $\xi$ and $\eta+\xi\dfrac{d\eta}{d\xi}$, is annihilated by $V$.

Consider now the transformations of these operators of the first order, as given by § 6. They are

$$\xi\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1^2\xi'+a_1b_1\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+b_1^2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\eta\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1a_2\xi'+a_1b_2\eta'+a_2b_1\xi'\frac{d\eta'}{d\xi'}+b_1b_2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\xi\frac{d\eta}{d\xi}\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_1a_2\xi'+a_2b_1\eta'+a_1b_2\xi'\frac{d\eta'}{d\xi'}+b_1b_2\eta'\frac{d\eta'}{d\xi'}\right\},$$

$$\eta\frac{d\eta}{d\xi}\equiv(a_1b_2-a_2b_1)^{-1}\left\{a_2^2\xi'+a_2b_2\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+b_2^2\eta'\frac{d\eta'}{d\xi'}\right\}.$$

We notice here a difference of character between the middle pair and the extreme pair—the coefficients of $\xi'$, $\eta'$, $\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ are linearly independent in the one pair of cases and not so in the other pair. Look, as we of course may do, upon the equivalences as results of applying a general linear transformation to the accented variables instead of to the unaccented, *i.e.*, look upon $f(y_1, y_2, y_3, \ldots)$ as having been found as the equivalent of a function $F(y_1', y_2', y_3', \ldots)$ and not *vice versa*. We see that for $\eta$ to annihilate $f$, whatever the linear transformation may have been, or for $\xi\frac{d\eta}{d\xi}$ to annihilate it in like case, it is necessary for all four of $\xi'$, $\eta'$, $\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ to annihilate $F$. On the other hand, for $\xi$ or for $\eta\frac{d\eta}{d\xi}$ to annihilate $f$, whatever the linear transformation may have been, it is only necessary for $\xi'$, $\eta'+\xi'\frac{d\eta'}{d\xi'}$, $\eta'\frac{d\eta'}{d\xi'}$ to annihilate $F$.

We may therefore state as follows. In order that a function of the derivatives may become a homogeneous function of degree zero of the new derivatives after every linear transformation of the variables, or in order that it may become an isobaric function of weight zero, it is necessary that it have the four properties of being free from the first derivative $y_1$, homogeneous of degree zero, isobaric of weight zero, and annihilated by $\eta\frac{d\eta}{d\xi}$ or $V$; in other words, it must be an absolute differential invariant of the general linear group. But, in order that it may become, after every linear transformation of the variables, a function free from the first derivative, or a function annihilated by $V$, it is only necessary for it to have the three properties of being free from $y_1$, being of zero sum of degree and weight throughout, and being annihilated by $V$; in other words, it need be only an absolute pure reciprocant (or linear function of such), *i.e.*, an absolute differential invariant (or linear function of such) of a linear group whose generality is limited by the one relation $a_1b_2-a_2b_1$ among the coefficients—a sub-group of the general linear group.*

* By "absolute differential invariant" of a continuous group, I, in this paper, mean, with Lie, a function of derivatives, and it might be also of the variables—though these last do not occur in the cases of groups here considered—which is absolutely unaltered in form by the substitutions of the group and its extensions. A function which persists in form but for a factor is called a "relative or non-absolute differential invariant." By "absolute pure reciprocant," I mean, with

8. It is an interesting subject of investigation whether there are MacMahon operators which transform, by the general linear scheme, into themselves, as there are for mere interchange of the variables and some other very special schemes. Were there such, and were the coefficients in them free from the constants in the scheme, then any such operator would generate absolute differential invariants from absolute differential invariants. But the answer appears, for

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Sylvester, a function of second and higher derivatives which persists in form but for the assumption of a factor which is a power of $-1$ when dependent and independent variables are interchanged. A non-absolute pure reciprocant persists in form after such interchange except for a factor which is not a mere power of $-1$.

It is a prevalent, but mistaken, impression that absolute pure reciprocants are identical with absolute differential invariants of the general linear group. What Sylvester proved (*Amer. Jour.*, Vol. VIII., pp. 248, &c.) with regard to the differential invariancy of a pure reciprocant $R$ for the general linear substitution is that, if homogeneous (of degree $i$), and consequently isobaric (of weight $w$), it becomes $(a_1b_2-a_2b_1)^i(a_1+b_1y_1)^{-w-i}R$, with my notation as above: his $w$ is different. If it be an *absolute* reciprocant, what he has proved at an earlier stage is that $w+i=0$, but not that $i=0$, $w=0$ separately. There is absolute invariancy for the substitution only if $i=0$ as well as $w+i=0$. There are, however, absolute pure reciprocants of all degrees $i$, and even non-homogeneous ones. Sylvester calls those of degree zero *plenarily absolute* to indicate that they are absolutely invariant for any linear group. Absolute pure reciprocants which are not of degree zero are, if homogeneous, differential invariants of any linear group, but are absolute differential invariants only for linear groups in which generality of constants is limited by the one relation $a_1b_2-a_2b_1=1$.

Again, what I myself proved (*Proc. Lond. Math. Soc.*, Vol. XIX., p. 388) with regard to the differential invariancy of pure cyclicants or ternary reciprocants for the general linear substitution of $lx+my+nz+p$, &c., for $x$, &c., is that there is persistence in the case of a homogeneous one but for a factor of the form

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix}^i \left\{ lm'-l'm-(mn'-m'n)\frac{\partial z}{\partial x} - (nl'-n'l)\frac{\partial z}{\partial y} \right\}^{-i-\frac{1}{3}w}.$$

The second index here vanishes when the pure cyclicant is absolute; but the first $i$, as a rule, does not. Thus absolute pure cyclicants include, but are not only coextensive with, all absolute differential invariants of the general linear group in three variables. Linear functions of them are identical with absolute differential invariants of a sub-group of the general linear group, namely, of the *special* subgroup in which the generality of coefficients is limited by the one relation

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = 1.$$

Those who, like myself, in 1886 and following years, followed Sylvester in researches on reciprocants and allied classes of differential invariants, and showed, it must be confessed, imperfect acquaintance with Sophus Lie's grand work on Continuous Groups and Differential Invariants in general, have reason to be grateful to Herr O. A. Stöckert for a recent memoir, *Ueber die Beziehungen der Reciprokantentheorie zur allgemeinen Theorie der Differentialinvarianten* (Chemnitz, 1895), in which he elucidates very instructively Lie's theory in its bearing on Sylvester's and our own. In that memoir the error occurs (pp. 30, 32) of attributing to Sylvester the opinion that an absolute pure reciprocant is necessarily of zero degree and weight. The proof given fails on p. 32 at line 11.

practical purposes, negative. It is true that inspection of the symbolic equivalences of the last article gives us one such persistent operator of the first order

$$\xi\frac{d\eta}{d\xi}-\eta\equiv\xi'\frac{d\eta'}{d\xi'}-\eta',$$

$$i.e.,\quad y_2\frac{\partial}{\partial y_2}+2y_3\frac{\partial}{\partial y_3}+3y_4\frac{\partial}{\partial y_4}+\ldots\equiv y_2'\frac{\partial}{\partial y_2'}+2y_3'\frac{\partial}{\partial y_3'}+3y_4'\frac{\partial}{\partial y_4'}+\ldots,$$

so that the persistence (19) of my memoir in the *Phil. Trans.* (*loc. cit.* p. 25) is one which holds in general; but this has merely the effect of multiplying a homogeneous isobaric function by the excess of its weight over its degree. It expresses the persistence in value of the characteristic $w-i$, but gives, in cases of interest, nothing new by its operation (a conclusion drawn by means of it which, though of interest, is not new, is the case for $q=2$ of one in § 17 below). The more general operator $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}-\eta\right\}$ [*loc. cit.*, (17)], which is persistent in the special theory, is not so in the present general one except for $m=n=1$.

Nor do the operators $\xi\frac{d\eta}{d\xi}+\eta$, $\xi^{m-1}\eta^{m-1}\left\{\xi\frac{d\eta}{d\xi}+\eta\right\}$, which are negatively persistent in the special theory [*loc. cit.*, (20), (18)], yield directly anything of much interest in general. If we take the general operator of the first order,

$$\lambda\xi+\mu\eta+\nu\xi\frac{d\eta}{d\xi}+\varpi\eta\frac{d\eta}{d\xi},$$

and try to find all values of $\lambda$, $\mu$, $\nu$, $\varpi$ for which the form persists but for a factor, upon substitution from the last article, we obtain, besides the persistent $\xi\frac{d\eta}{d\xi}-\eta$ as above, only the equivalences

$$-2a_2\xi+(a_1-b_2)\left(\eta+\xi\frac{d\eta}{d\xi}\right)+2b_1\eta\frac{d\eta}{d\xi}$$
$$\equiv-2a_2\xi'+(a_1-b_2)\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)+2b_1\eta'\frac{d\eta'}{d\xi'}$$

and $(a_1-b_2)\left(a_2\xi-b_1\eta\frac{d\eta}{d\xi}\right)+2a_2b_1\left(\eta+\xi\frac{d\eta}{d\xi}\right)$
$$\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi+b_1\eta\frac{d\eta}{d\xi}\right)$$
$$\equiv\frac{\left\{a_1+b_2\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\right\}^2}{4(a_1b_2-a_2b_1)}\left\{(a_1-b_2)\left(a_2\xi'-b_1\eta'\frac{d\eta'}{d\xi'}\right)\right.$$
$$\left.+2a_2b_1\left(\eta'+\xi'\frac{d\eta'}{d\xi'}\right)\pm\sqrt{(a_1-b_2)^2+4a_2b_1}\left(a_2\xi'+b_1\eta'\frac{d\eta'}{d\xi'}\right)\right\},$$

which, involving as they do $a_1$, $b_1$, $a_2$, $b_2$ as coefficients, give us no information as to persistence for all linear transformations, so that they are only matters of curiosity in general. It is the latter, however, which, for the case of $a_1 = b_2 = 0$, $a_2 = b_1 = 1$, *i.e.*, for the case of mere interchange of variables $x$, $y$, gives the negative persistence of $\eta + \xi \frac{d\eta}{d\xi}$, *i.e.*,

$$2y \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots$$

[*loc. cit.*, (20)]. It also gives for that case the negative persistence of $\xi + \eta \frac{d\eta}{d\xi}$, *i.e.*, of

$$(1-y_1^2) \frac{\partial}{\partial y_1} + y_1 \left( 2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots \right) + V,$$

and the former gives the positive persistence of $\eta \frac{d\eta}{d\xi} - \xi$, *i.e.*, of

$$-(1+y_1^2) \frac{\partial}{\partial y_1} + y_1 \left( 2y_1 \frac{\partial}{\partial y_1} + 3y_2 \frac{\partial}{\partial y_2} + 4y_3 \frac{\partial}{\partial y_3} + \dots \right) + V,$$

two facts not expressly introduced in the memoir to which reference is made.

9. The digression may be pardonable if I consider for a moment some properties of operators of the types $\xi^n \eta^m$, $\xi^n \eta^m \frac{d\eta}{d\xi}$ without reference to transformations. To find their *alternants* will really be to do otherwise what MacMahon has already done (*Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 66–69), but the use of the present compact symbolic forms so simplifies the process, and exhibits the results in such suggestive shape, that I believe there to be justification.

Let us denote by $(A\xi + B\xi^2 + \dots)(A'\xi + B'\xi^2 + \dots)$ the operator which results from the performance of the operation represented by the first written factor on the back of that represented by the second. The part of the resultant operator which does not involve symbols of second partial differentiation will be obtained in symbolic form by making the left-hand operator act only on $A'$, $B'$, ..., dealing with the $\xi$ in the right-hand operator as if it were a constant. Thus, in particular,

$$(A\xi + B\xi^2 + C\xi^3 + \dots)\,\eta = (A\xi + B\xi^2 + C\xi^3 + \dots)(y_1\xi + y_2\xi^2 + y_3\xi^3 + \dots)$$
$$= A\xi + B\xi^2 + C\xi^3 + \dots + R,$$

where $R$ involves $\dfrac{\partial^2}{\partial y_1^2}$, $\dfrac{\partial^2}{\partial y_1 \partial y_2}$, .... So

$$(A\xi + B\xi^2 + C\xi^3 + \ldots)\,\xi^n\eta^m = m\xi^n\eta^{m-1}(A\xi + B\xi^2 + C\xi^3 + \ldots) + R';$$

and $(A\xi + B\xi^2 + C\xi^3 + \ldots)\,\xi^n\eta^m \dfrac{d\eta}{d\xi} = m\xi^n\eta^{m-1}\dfrac{d\eta}{d\xi}(A\xi + B\xi^2 + C\xi^3 + \ldots)$

$$+ \xi^n\eta^m \frac{d}{d\xi}(A\xi + B\xi^2 + C\xi^3 + \ldots) + R''.$$

Now, $\vartheta$, $\phi$ being any two linear operators, we call as usual $\vartheta\phi - \phi\vartheta$ the alternant of $\vartheta$ and $\phi$, and write it $(\vartheta . \phi)$. The terms $R$ in $\vartheta\phi$ and $\phi\vartheta$ are the same. Hence we at once deduce

$$\begin{aligned}
(\xi^\nu\eta^\mu . \xi^n\eta^m) &= \xi^\nu\eta^\mu . \xi^n\eta^m - \xi^n\eta^m . \xi^\nu\eta^\mu \\
&= m\xi^n\eta^{m-1}\xi^\nu\eta^\mu - \mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m \\
&= (m-\mu)\,\xi^{n+\nu}\eta^{m+\mu-1}, \\
\left(\xi^\nu\eta^\mu . \xi^n\eta^m \frac{d\eta}{d\xi}\right) &= m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu + \xi^n\eta^m \frac{d}{d\xi}(\xi^\nu\eta^\mu) - \mu\xi^\nu\eta^{\mu-1}\xi^n\eta^m \frac{d\eta}{d\xi} \\
&= m\xi^{n+\nu}\eta^{m+\mu-1}\frac{d\eta}{d\xi} + \nu\xi^{n+\nu-1}\eta^{m+\mu}, \\
\left(\xi^\nu\eta^\mu \frac{d\eta}{d\xi} . \xi^n\eta^m \frac{d\eta}{d\xi}\right) &= m\xi^n\eta^{m-1}\frac{d\eta}{d\xi}\xi^\nu\eta^\mu \frac{d\eta}{d\xi} + \xi^n\eta^m \frac{d}{d\xi}\left(\xi^\nu\eta^\mu \frac{d\eta}{d\xi}\right) \\
&\qquad - \mu\xi^\nu\eta^{\mu-1}\frac{d\eta}{d\xi}\xi^n\eta^m \frac{d\eta}{d\xi} - \xi^\nu\eta^\mu \frac{d}{d\xi}\left(\xi^n\eta^m \frac{d\eta}{d\xi}\right) \\
&= (\nu - n)\,\xi^{n+\nu-1}\eta^{m+\mu}\frac{d\eta}{d\xi}.
\end{aligned}$$

Thus the alternant of two MacMahon operators of given orders $(m+n,\ \mu+\nu)$ is a MacMahon operator whose order $(m+n+\mu+\nu-1)$ is one less than the sum of the orders of the two.

Moreover, the alternant of two operators of the same type $\left(\xi^n\eta^m \text{ or } \xi^n\eta^m \dfrac{d\eta}{d\xi}\right)$ is an operator of the same type. The alternant of two of different types $\left(\xi^\nu\eta^\mu \text{ and } \xi^n\eta^m \dfrac{d\eta}{d\xi}\right)$ is as a rule the sum of two operators, one of each type.

It is easy to see, by consideration of the indices and coefficients in the three types of alternant equalities that $\eta$ is the only operator of positive integral order of the type $\xi^n\eta^m$ which cannot be expressed as

an alternant of operators of the same type, and that $\xi\frac{d\eta}{d\xi}$ is the only one of the type $\xi^n\eta^m\frac{d\eta}{d\xi}$ which cannot be expressed as an alternant of operators of its type. It is also easy to see that the alternant $\left(\xi.\eta\frac{d\eta}{d\xi}\right)$ is the only one of the type $\left(\xi^\nu\eta^\mu.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$ which cannot be linearly expressed in terms of alternants of the two types $(\xi^\nu\eta^\mu.\xi^n\eta^m)$, $\left(\xi^\nu\eta^\mu\frac{d\eta}{d\xi}.\xi^n\eta^m\frac{d\eta}{d\xi}\right)$.

The alternants of operators of the first order are themselves of the first order, *i.e.*, are linear functions of themselves, a fact which expresses that, as mentioned earlier, they possess the group property. They are

$$(\xi.\eta)=\xi,$$

$$\left(\xi.\xi\frac{d\eta}{d\xi}\right)=\xi,$$

$$\left(\xi.\eta\frac{d\eta}{d\xi}\right)=\eta+\xi\frac{d\eta}{d\xi},$$

$$\left(\eta.\xi\frac{d\eta}{d\xi}\right)=0,$$

$$\left(\eta.\eta\frac{d\eta}{d\xi}\right)=\eta\frac{d\eta}{d\xi},$$

$$\left(\xi\frac{d\eta}{d\xi}.\eta\frac{d\eta}{d\xi}\right)=\eta\frac{d\eta}{d\xi}.$$

It is, moreover, evident hence that, as also stated earlier, $\xi$, $\eta+\xi\frac{d\eta}{d\xi}$, $\eta\frac{d\eta}{d\xi}$ form a sub-group, as their alternants in pairs are linear in themselves.

What the group and sub-group are has been already indicated (§ 7).

10. To return to transformations by the extended linear scheme (1). The transformation of MacMahon operators of positive integral order is what we have so far considered in application of the method of § 3. But the method is one which, as we saw, applies to all linear operators whatever. One simple MacMahon operator of zero order may be mentioned in passing, namely,

$$\frac{d\eta}{d\xi}-y_1\equiv 2y_2\frac{\partial}{\partial y_1}+3y_3\frac{\partial}{\partial y_2}+4y_4\frac{\partial}{\partial y_3}+\ldots,$$

whose effect is that of $\frac{d}{dx}$, the symbol of total differentiation with regard to $x$ in so far as $y_1, y_2, y_3, \ldots$ are functions of $x$ in virtue of the relation connecting $x$ and $y$. After substitution for $y_1$ in terms of $y'_1$, by § 1, its transformation becomes by the general rule

$$\frac{1}{a_1+b_1y'_1}\left\{\frac{d\eta'}{d\xi'}-y'_1\right\},$$

which accords with the facts of total differentiation.

We may transform linear operators whose symbolical expressions involve $\frac{d^2\eta}{d\xi^2}$, $\frac{d^3\eta}{d\xi^3}$, ..., in so far as we know the expressions for those derivatives in terms of $\frac{d\eta'}{d\xi'}$, $\frac{d^2\eta'}{d\xi'^2}$, $\frac{d^3\eta'}{d\xi'^3}$, ..., when $\xi, \eta, \xi', \eta'$ are actual increments of $x, y, x', y'$. The expressions in question are of the same form as those (§ 2) for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... in terms of $\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$, $\frac{d^3y'}{dx'^3}$, ... . Thus, for instance, since

$$\frac{d^2\eta}{d\xi^2}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{-3}\equiv(a_1b_2-a_2b_1)\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-3},$$

we have, by § 3, that the transformed equivalent of

$$\xi^n\frac{d^2\eta}{d\xi^2}=1.2y_2\frac{\partial}{\partial y_n}+2.3y_3\frac{\partial}{\partial y_{n+1}}+3.4y_4\frac{\partial}{\partial y_{n+2}}+\ldots\quad(n>0)$$

is, symbolically, the expansion in powers of $\xi'$ of

$$\xi^n\frac{d^2\eta'}{d\xi'^2}\left(\frac{d\xi}{d\xi'}\right)^{-2}=\left\{a_1\xi'+b_1(y'_1\xi'+y'_2\xi'^2+\ldots)\right\}^n$$

$$\times\left\{1.2y'_2+2.3y'_3\xi'+3.4y'_4\xi'^2+\ldots\right\}\left\{a_1+b_1(y'_1+2y'_2\xi'+3y'_3\xi'^2+\ldots)\right\}^{-2},$$

and, actually, the result of replacing, in the expansion,

$$\xi'^r \text{ by } \frac{\partial}{\partial y'_r}\quad(r=1, 2, 3, \ldots, \infty);$$

and more generally that the transformed equivalent of

$$\xi^n\eta^m\left(\frac{d\eta}{d\xi}\right)^p\left(\frac{d^2\eta}{d\xi^2}\right)^q$$

is, symbolically,

$$(a_1b_2-a_2b_1)^{q-1}(a_1\xi'+b_1\eta')^n(a_2\xi'+b_2\eta')^m$$

$$\times\left(a_2+b_2\frac{d\eta'}{d\xi'}\right)^p\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{1-p-3q}\left(\frac{d^2\eta'}{d\xi'^2}\right)^q.$$

An interesting class of examples of such transformations is afforded by our knowledge of pure reciprocants. If $R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right)$ be a pure reciprocant in $\xi$, $\eta$ of degree $i$ and weight (sum of indices of differentiation in each term) $w$, we know (§ 7, footnote) that its expression in terms of $\xi'$, $\eta'$ (regarded as actual and not symbolic) is

$$(a_1b_2-a_2b_1)^i\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right).$$

Hence the transformed equivalent of the operator whose symbolic expression is the expansion of

$$\xi^n R\left(\frac{d^2\eta}{d\xi^2}, \frac{d^3\eta}{d\xi^3}, \frac{d^4\eta}{d\xi^4}, \ldots\right) \quad (n>0),$$

where $\eta$ denotes $y_1\xi+y_2\xi^2+y_3\xi^3+\ldots$, is, symbolically, the expansion of

$$(a_1b_2-a_2b_1)^{i-1}(a_1\xi'+b_1\eta')^n\left(a_1+b_1\frac{d\eta'}{d\xi'}\right)^{1-w-i} R\left(\frac{d^2\eta'}{d\xi'^2}, \frac{d^3\eta'}{d\xi'^3}, \frac{d^4\eta'}{d\xi'^4}, \ldots\right),$$

where $\eta'$ denotes $y'_1\xi'+y'_2\xi'^2+y'_3\xi'^3+\ldots$.

## II.

11. We now proceed to the more general analogous theory when there are $q$ variables supposed connected by one relation. Let $q-1$ of them, chosen as the independent variables, be called $x_1, x_2, \ldots, x_{q-1}$, and the single dependent one $y$. After the preceding exposition of the first case of $q=2$, a much slighter general presentation will suffice than would have been necessary had a different order been chosen, and the inevitable imperfections of a notation for partial differential coefficients, and the necessary avoidance of extensively writing out explicit forms of operators dealt with, will, it is hoped, cause but little obscurity.

The notation $y_{rst\ldots}$ will be used as denoting

$$\frac{1}{r!\,s!\,t!\ldots r_{q-1}!}\,\frac{\partial^{r+s+t+\ldots+r_{q-1}}}{\partial x_1^r\partial x_2^s\partial x_3^t\ldots\partial x_{q-1}^{r_{q-1}}}\,y.$$

There are supposed to be always $q-1$ suffixes, each of which may be zero or any positive integer, while their sum is 1 at least.

Let $\xi_1, \xi_2, \xi_3, \ldots, \xi_{q-1}$ be independent simultaneous finite increments of the $q-1$ independent variables, and $\eta$ the corresponding increment of $y$. It is supposed that Taylor's theorem applies to $y$ as a function

of $x_1, x_2, \ldots, x_{q-1}$ for values considered, so that

$$\eta = \sum_{r+s+t+\ldots+r_{q-1}=1}^{r=\infty,\, s=\infty,\, t=\infty,\, \ldots,\, r_{q-1}=\infty} y_{rst\ldots}\, \xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}}.$$

As well as using $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in an actual sense, we shall also use them purely as symbols with a meaning now to be explained. The symbol $\eta$ will always mean the above expansion in terms of the symbols $\xi$. These latter symbols will have no individual meaning except when standing alone in the first power in an ultimate expansion. A power or product of symbols $\xi$ in an ultimate expansion will have a meaning as a whole, namely,

$$\xi_1^r \xi_2^s \xi_3^t \ldots \xi_{q-1}^{r_{q-1}} \text{ will mean } \frac{\partial}{\partial y_{rst\ldots r_{q-1}}},$$

a symbol of partial differentiation, the supposition being that it is acting on a function of the various partial derivatives of $y$, and, it may be, of the variables $y, x_1, x_2, \ldots, x_{q-1}$ in addition. Thus, for instance,

$$\xi_1, \xi_2, \xi_3, \xi_2^2, \xi_1^2 \xi_3, \ldots$$

will mean respectively

$$\frac{\partial}{\partial y_{100\ldots}}, \frac{\partial}{\partial y_{010\ldots}}, \frac{\partial}{\partial y_{001\ldots}}, \frac{\partial}{\partial y_{020\ldots}}, \frac{\partial}{\partial y_{201\ldots}}, \ldots.$$

By an operator such as

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}}\, \eta^m \frac{\partial \eta}{\partial \xi_1},$$

for instance, we shall mean the operator obtained by substituting in the product for $\eta$ its expansion in terms of the $\xi$s as above, performing the partial differentiation with regard to $\xi_1$, expanding the resulting product in powers and products of powers of $\xi_1, \xi_2, \ldots, \xi_{q-1}$, writing every such power or product of powers after its multiplier in the expansion, and finally substituting for each product $\xi_1^r \xi_2^s \xi_3^t \ldots$ as it occurs ultimately the corresponding symbol of partial differentiation as above.

12. We shall have to consider largely operators of the $q$ types

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m, \quad \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_1}, \quad \ldots, \quad \xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \frac{\partial \eta}{\partial \xi_{q-1}},$$

where $n_1, n_2, \ldots, n_{q-1}, m$ are positive, integral, or zero, and $\Sigma n + m \nless 1$. Such operators are for the case $q = 2$ the MacMahon operators of

the preceding section, and, for the next case $q = 3$ the analogous ternary operators which I have used on a previous occasion (*Phil. Trans.*, *loc. cit.*, pp. 36, &c.). We speak of an operator which is a linear function with constant coefficients of operators of these types for which $\Sigma n + m$ is constant as being of order $\Sigma n + m$.

The alternants of pairs of operators of these types are with ease written down as in § 9. There are three classes of fundamental alternant equalities: viz.,

$$(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\cdot\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m})$$
$$= (m-\mu)\,\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\cdot\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$
$$= m\xi_1^{n_1+\nu_1}\xi_2^{n_2+\nu_2}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu-1}\frac{\partial\eta}{\partial\xi_s}$$
$$+\nu_s\xi_1^{n_1+\nu_1}\dots\xi_s^{n_s+\nu_s-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu},$$

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}\cdot\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$
$$= m\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m-1}\frac{\partial\eta}{\partial\xi_s}\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}$$
$$+\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial}{\partial\xi_s}\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}\right)$$
$$-\mu\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu-1}\frac{\partial\eta}{\partial\xi_r}\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}$$
$$-\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial}{\partial\xi_r}\left(\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_s}\right)$$

$$= \nu_s\xi_1^{n_1+\nu_1}\dots\xi_s^{n_s+\nu_s-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r},$$
$$-n_r\xi_1^{n_1+\nu_1}\dots\xi_r^{n_r+\nu_r-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_s}$$

of which the last includes in particular ($r = s$)

$$\left(\xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_{q-1}^{\nu_{q-1}}\eta^{\mu}\frac{\partial\eta}{\partial\xi_r}\cdot\xi_1^{n_1}\xi_2^{n_2}\dots\xi_{q-1}^{n_{q-1}}\eta^{m}\frac{\partial\eta}{\partial\xi_r}\right)$$
$$= (\nu_r-n_r)\,\xi_1^{n_1+\nu_1}\dots\xi_r^{n_r+\nu_r-1}\dots\xi_{q-1}^{n_{q-1}+\nu_{q-1}}\eta^{m+\mu}\frac{\partial\eta}{\partial\xi_r}.$$

On inspection of these identities we at once see that the alternant of two operators of given orders $\Sigma\nu+\mu$, $\Sigma n+m$ is an operator of the same kind whose order $\Sigma\nu+\Sigma n+\mu+m-1$ is one less than the sum of those given orders.

In particular, the alternant of two operators of the first order is itself an operator of the first order. The linearly independent operators of the first order are $q^2$ in number: viz., $q-1$ operators $\xi_r$ $(r=1, 2, \ldots, q-1)$, one operator $\eta$, $q-1$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_1}$, $(q-1)(q-2)$ operators of the type $\xi_1\frac{\partial\eta}{\partial\xi_2}$, and $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi_r}$ $(r=1, 2, \ldots, q-1)$. The alternant identities for them in pairs are of the following types, the number of each type being written after every one, an alternant $(\vartheta.\phi)$ not being reckoned as distinct from its negative $(\phi.\vartheta)$;

$$(\xi_1.\xi_2)=0,\ \tfrac{1}{2}(q-1)(q-2); \qquad (\xi_1.\eta)=\xi_1,\ q-1;$$

$$\left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1,\ q-1; \qquad \left(\xi_1.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0,(q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2,\ (q-1)(q-2); \qquad \left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3); \qquad \left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\xi_1\frac{\partial\eta}{\partial\xi_1}+\eta,\ q-1;$$

$$\left(\xi_1.\eta\frac{\partial\eta}{\partial\xi_2}\right)=\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2); \qquad \left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_1}\right)=0,\ q-1;$$

$$\left(\eta.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2); \qquad \left(\eta.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1;$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_1\frac{\partial\eta}{\partial\xi_2}\right)=-\xi_1\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2\frac{\partial\eta}{\partial\xi_1},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\xi_2\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_1},\ q-1; \qquad \left(\xi_1\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_2\frac{\partial\eta}{\partial\xi_1}\right)=\xi_2\frac{\partial\eta}{\partial\xi_2}-\xi_1\frac{\partial\eta}{\partial\xi_1},\ \tfrac{1}{2}(q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_1\frac{\partial\eta}{\partial\xi_3}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_1}\right)=\xi_3\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\xi_3\frac{\partial\eta}{\partial\xi_4}\right)=0,\ \tfrac{1}{2}(q-1)(q-2)(q-3)(q-4);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_1}\right)=\eta\frac{\partial\eta}{\partial\xi_2},\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ (q-1)(q-2);$$

$$\left(\xi_1\frac{\partial\eta}{\partial\xi_2}.\eta\frac{\partial\eta}{\partial\xi_3}\right)=0,\ (q-1)(q-2)(q-3);$$

$$\left(\eta\frac{\partial\eta}{\partial\xi_1}.\eta\frac{\partial\eta}{\partial\xi_2}\right)=0,\ \tfrac{1}{2}(q-1)(q-2).$$

The whole number of these equalities is $\frac{1}{2}q^2(q^2-1)$.

As the alternants in pairs of the $q^2$ linearly independent operators of the first order are linear with constant coefficients in the operators themselves, except such of them as vanish, they are the operators appertaining to the infinitesimal transformations of a continuous substitution group of $q^2$ parameters. It will be seen later (§ 16) that the group in question is that which consists of the extensions of the general linear group of substitutions for the $q$ variables $x_1, x_2, \ldots, x_{q-1}, y$.

Moreover, we notice that the $q-1$ operators $\xi$, the $(q-1)(q-2)$ operators $\xi_r\frac{\partial\eta}{\partial\xi_s}$ $(r\neq s)$, and the $q-1$ operators $\eta\frac{\partial\eta}{\partial\xi}$ all occur as alternants of operators; but that the operator $\eta$ and the $q-1$ operators $\xi_r\frac{\partial\eta}{\partial\xi_r}$ only occur in alternants in the connexions $\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}$ and

$$\xi_r\frac{\partial\eta}{\partial\xi_r}-\xi_s\frac{\partial\eta}{\partial\xi_s}=\left(\eta+\xi_r\frac{\partial\eta}{\partial\xi_r}\right)-\left(\eta+\xi_s\frac{\partial\eta}{\partial\xi_s}\right).$$

The $q^2-1$ operators

$$\xi,\ \ \xi_r\frac{\partial\eta}{\partial\xi_s}\ (r\neq s),\ \ \eta\frac{\partial\eta}{\partial\xi},\ \ \eta+\xi_r\frac{\partial\eta}{\partial\xi_r}\quad (r=1,\,2,\,3,\,\ldots,\,q-1)$$

are then seen to be a set whose alternants are linear with constant coefficients in themselves. These are consequently the operators appertaining to the infinitesimal transformations of a sub-group of the general group of extensions. What we see in § 16 will also show us that this sub-group is the group of extensions of the group of linear substitutions in which the generality of the coefficients of $x_1, x_2, \ldots, x_{q-1}, y$ is limited by the one relation that the value of their determinant is unity. It will also be proved that this system of $q^2-1$ operators is the system of annihilators of absolute pure $q$-ary reciprocants, *i.e.*, that, as in the cases of $q=2$, $q=3$, such reciprocants are, with a reservation which will appear, identical with absolute differential invariants of the linear group in which generality of coefficients is limited by the one relation just stated.

13. We proceed to consider the transformation of operators linear in symbols of partial differentiation with regard to the partial derivatives $y_{rst\ldots}$ of $y$ with regard to $x_1, x_2, \ldots, x_{q-1}$, consequent upon the $q$ general linear equations of transformation of the $q$ variables

$$x_i=a_{i1}x_1'+a_{i2}x_2'+\ldots+a_{i.q-1}x_{q-1}'+a_{iq}y'+c_i\quad (i=1,\,2,\,\ldots,\,q-1),$$

$$y=a_{q1}x_1'+a_{q2}x_2'+\ldots+a_{q.q-1}x_{q-1}'+a_{qq}y'+c_q.$$

Let $\xi_1, \xi_2, \ldots, \xi_{q-1}$ be actual finite increments of $x_1, x_2, \ldots, x_{q-1}$, and $\eta$ the actual consequent increment of $y$, given as an expansion in terms of the $\xi$s by Taylor's theorem, as in § 11. Let $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$ be the corresponding increments of the new variables. Then

$$\xi_i=a_{i1}\xi_1'+a_{i2}\xi_2'+\ldots+a_{i.q-1}\xi_{q-1}'+a_{iq}\eta'\quad (i=1,\,2,\,\ldots,\,q-1),$$

$$\eta=a_{q1}\xi'+a_{q2}\xi_2'+\ldots+a_{q.q-1}\xi_{q-1}'+a_{qq}\eta';$$

which, if $\Delta$ denote the determinant of $q^2$ constituents $|\,a_{ij}\,|$, and if $A_{mn}$ be its first minor corresponding to $a_{mn}$, may also be written

$$\Delta\xi_i'=A_{1i}\xi_1+A_{2i}\xi_2+\ldots+A_{q-1.i}\xi_{q-1}+A_{qi}\eta\quad (i=1,\,2,\,\ldots,\,q-1),$$

$$\Delta\eta'=A_{1q}\xi_1+A_{2q}\xi_2+\ldots+A_{q-1.q}\xi_{q-1}+A_{qq}\eta.$$

We have also $\qquad \eta'=\Sigma\, y'_{rst\ldots}\xi_1'^{r}\xi_2'^{s}\xi_3'^{t}\ldots\xi_{q-1}'^{rq-1},$

a summation exactly corresponding to the already written

$$\eta=\Sigma\, y_{rst\ldots}\xi_1^{r}\xi_2^{s}\xi_3\ldots\xi_{q-1}^{rq-1}.$$

The formulæ of linear transformation of the variables lead to formulæ for the expression of all derivatives $y_{rst...}$ in terms of derivatives $y'_{rst...}$, and conversely. These formulæ express what are called the extensions of the linear transformation. It is unnecessary to wait to prove here the well-known fact that, because the scheme of transformation of the variables is linear, its extensions do not involve the variables explicitly, but only the derivatives. Thus in the accented equivalent of $\dfrac{\partial}{\partial y_{rst...}}$ in terms of symbols of partial differentiation with regard to accented letters there will be no occurrence of any $\dfrac{\partial}{\partial x'}$ or of $\dfrac{\partial}{\partial y'}$.

If, then, we are transforming an operator

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}+\Sigma\left(A_{rst...}\frac{\partial}{\partial y_{rst...}}\right),$$

supposed to act on a function $f$ of $x_1, x_2, \ldots, x_{q-1}, y$, and the derivatives $y_{rst...}$, into its equivalent in form suitable for action on the equivalent function $F$ of the accented variables and derivatives, we may substitute for the first part

$$\Sigma\left(P_r\frac{\partial}{\partial x_r}\right)+Q\frac{\partial}{\partial y}$$

from the formulæ

$$\Delta\frac{\partial}{\partial x_i}=A_{i1}\frac{\partial}{\partial x'_1}+A_{i2}\frac{\partial}{\partial x'_2}+\ldots+A_{i.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{iq}\frac{\partial}{\partial y'}$$
$$(i=1, 2, \ldots, q-1),$$

$$\Delta\frac{\partial}{\partial y}=A_{q1}\frac{\partial}{\partial x'_1}+A_{q2}\frac{\partial}{\partial x'_2}+\ldots+A_{q.q-1}\frac{\partial}{\partial x'_{q-1}}+A_{qq}\frac{\partial}{\partial y'};$$

and transform the remaining part exactly as if it acted on a function of the derivatives only.

14. Now, in the $q$ equations of transformation of the last article for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}, \eta'$, suppose $\eta$ replaced by its expansion in terms of the $\xi$s with coefficients like $y_{rst...}$, and $\eta'$ replaced by its similar expansion in terms of the $\xi'$s. We have then $q$ equations connecting the $q-1$ quantities $\xi$, the $q-1$ quantities $\xi'$, the derivatives $y_{rst...}$, and the derivatives $y'_{rst...}$. As the one relation supposed to exist among $y$ and $x_1, x_2, \ldots, x_{q-1}$ is of perfectly arbitrary

form, and may be varied as we please, we may consider the derivatives $y_{rst\ldots}$ as an infinite number of quite independent variables. Also $\xi_1, \xi_2, \ldots, \xi_{q-1}$ are quite independent of these derivatives and of one another. The other quantities involved in the equations we are considering are dependent variables; viz., $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ are functions of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and $\eta$, which is a function of the others and the derivatives given by its expansion, and the accented derivatives $y'_{rst\ldots}$ are functions of the unaccented derivatives determined by the extensions of the linear transformation. The equations which we are regarding, with $\eta$ and $\eta'$ replaced in them by their expansions, are in appearance $q$ equations for the determination of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ in terms of $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the derivatives. But $q-1$ equations suffice for the purpose. Hence any one of the $q$ must be a consequence of the rest, and the expressions for accented derivatives in terms of unaccented.

Let $y_{rst\ldots}$ alone among the quantities now regarded as independent variables receive an infinitesimal increment $\delta y_{rst\ldots}$, and let $\delta\xi'_1, \delta\xi'_2, \ldots, \delta\xi'_{q-1}$ be the consequent increments of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$.

It is most convenient to regard the $q$ linear equations connecting the $\xi$s and $\eta$ with the $\xi'$s and $\eta'$ as before us in their second forms—those which express accented letters in terms of unaccented. These give us at once

$$\Delta.\delta\xi'_i = A_{qi}\xi'^{r}_1\xi'^{s}_2\xi'^{t}_3\ldots\delta y_{rst\ldots} \quad (i = 1, 2, \ldots, q-1)$$

and

$$\Delta\left\{\frac{\partial\eta'}{\partial\xi'_1}\delta\xi'_1 + \frac{\partial\eta'}{\partial\xi'_2}\delta\xi'_2 + \ldots + \frac{\partial\eta'}{\partial\xi'_{q-1}}\delta\xi'_{q-1} + \sum_{\rho,\sigma,\tau,\ldots}\left[\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi'^{\rho}_1\xi'^{\sigma}_2\xi'^{\tau}_3\ldots\right]\delta y_{rst\ldots}\right\}$$

$$= A_{qq}\xi'^{r}_1\xi'^{s}_2\xi'^{t}_3\ldots\delta y_{rst\ldots},$$

where the summation with regard to $\rho, \sigma, \tau, \ldots$ covers all zero and positive integral values, such that $\rho+\sigma+\tau+\ldots \nless 1$. The fact that the first $q-1$ of these equations must lead to the last gives us at once the identity

$$\sum_{\rho,\sigma,\tau,\ldots}\left\{\frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}\xi'^{\rho}_1\xi'^{\sigma}_2\xi'^{\tau}_3\ldots\right\}$$

$$= \Delta^{-1}\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi'_1} - A_{q2}\frac{\partial\eta'}{\partial\xi'_2} - \ldots - A_{q,q-1}\frac{\partial\eta'}{\partial\xi'_{q-1}}\right\}\xi'^{r}_1\xi'^{s}_2\xi'^{t}_3\ldots.$$

Here $\Delta$ is the determinant $|\,a_{ij}\,|$ of the $q^2$ coefficients in the linear

expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$, and each $A_{rs}$ is a first minor $\dfrac{\partial \Delta}{\partial a_{rs}}$ of that determinant.

Had we in finding this identity proceeded by consideration of the linear expressions for unaccented in terms of accented letters instead of *vice versa*, the right-hand side would in the first place have presented itself in the form

$$\Delta^{-1} \frac{d(\xi_1, \xi_2, \ldots, \xi_{q-1})}{d(\xi'_1, \xi'_2, \ldots, \xi'_{q-1})} \xi'^{r} \xi'^{s} \xi'^{t} \ldots,$$

where, in the Jacobian functional determinant,

$$\frac{d\xi_r}{d\xi'_s} \text{ means } \frac{\partial \xi_r}{\partial \xi'_s} + \frac{\partial \xi_r}{\partial \eta'} \frac{\partial \eta'}{\partial \xi'_s}, \quad i.e., \; a_{rs} + a_{rq} \frac{\partial \eta'}{\partial \xi'_s}.$$

This latter form, whose identity with the former is easily verified, is occasionally the most convenient for the expression of results.

Now suppose the right-hand member of the identity arrived at to be expanded in terms of $\xi'_1, \xi'_2, \ldots, \xi'_{q-1}$ by means of the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}$ and the expansion for $\eta'$. The various coefficients in the development obtained must be severally equal to the corresponding coefficients in the left-hand expansion.

We thus have expressions in terms of accented derivatives of $y'$ for all partial derivatives $\dfrac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}}$; and these must be those which would be obtained from the extensions of the formulæ of linear transformation, were they actually exhibited.

But let us use otherwise the fact of the absolute identity of the two expansions before us. We may substitute for every product $\xi'^{\rho}_1 \xi'^{\sigma}_2 \xi'^{\tau}_3 \ldots$ any quantity or symbol we please in both of them, and the identity will still remain one. Let us then substitute for them according to the symbolism explained in § 11; *i.e.*, for $\xi'^{\rho}_1 \xi'^{\sigma}_2 \xi'^{\tau}_3 \ldots$ on both sides the corresponding $\dfrac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}$, the supposition being that it is acting on a function of the derivatives $y'_{\rho\sigma\tau\ldots}$ or some of them, and it may be also of $x'_1, x'_2, \ldots, x'_{q-1}, y'$. The left-hand expansion then becomes

$$\sum_{\rho, \sigma, \tau, \ldots} \left\{ \frac{\partial y'_{\rho\sigma\tau\ldots}}{\partial y_{rst\ldots}} \frac{\partial}{\partial y'_{\rho\sigma\tau\ldots}} \right\},$$

*i.e.*, simply

$$\frac{\partial}{\partial y_{rst\ldots}},$$

regarded as operating on the equivalent function of derivatives $y_{rst\ldots}$, and it may be also of $x_1, x_2, \ldots, x_{q-1}, y$.

Accordingly, the transformation of operators of partial differentiation $\dfrac{\partial}{\partial y_{rst\ldots}}$ is before us, and is expressed by the rule: Replace such an operator symbolically by the corresponding product $\xi_1^r \xi_2^s \xi_3^t \ldots$; multiply it by

$$\Delta^{-1}\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi_1'} - A_{q2}\frac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{q,q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\};$$

express the result in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$ by the linear formulæ and the expansion for $\eta'$; expand in powers and products of powers of $\xi_1', \xi_2', \ldots, \xi_{q-1}'$; and, finally, for each power or product $\xi_1'^{\rho} \xi_2'^{\sigma} \xi_3'^{\tau} \ldots$ which occurs in the expansion, write the corresponding operator of partial differentiation $\dfrac{\partial}{\partial y'_{\rho\sigma\tau\ldots}}$.

Moreover, the transformation of any linear partial differential operator (see § 13),

$$\Sigma\left(A_{rst\ldots}\frac{\partial}{\partial y_{rst\ldots}}\right),$$

is at once effected in like manner. Write it symbolically

$$\Sigma\left(A_{rst\ldots}\xi_1^r \xi_2^s \xi_3^t \ldots\right);$$

multiply it throughout by the factor just written; and complete the process of expansion and interpretation of symbolic products as before.

15. We will now consider in particular the transformation of the operators of § 12. We may deal with such operators together by considering the operator

$$\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m\left\{\lambda_q + \lambda_1\frac{\partial\eta}{\partial\xi_1} + \lambda_2\frac{\partial\eta}{\partial\xi_2} + \ldots + \lambda_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \lambda_q$ are $q$ arbitrary constants. The symbolism is that explained in § 11.

By immediate application of the rule just arrived at, the transformation of this operator is symbolically

$$\Delta^{-1}\left\{A_{qq} - A_{q1}\frac{\partial\eta'}{\partial\xi_1'} - A_{q2}\frac{\partial\eta'}{\partial\xi_2'} - \ldots - A_{q,q-1}\frac{\partial\eta'}{\partial\xi_{q-1}'}\right\}$$
$$\times\left\{\lambda_q + \lambda_1\frac{\partial\eta}{\partial\xi_1} + \lambda_2\frac{\partial\eta}{\partial\xi_2} + \ldots + \lambda_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}\right\}\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_{q-1}}\eta^m;$$

but this may be written in better form.

Identification of the Pfaffian equations

$$d\eta - \frac{\partial \eta}{\partial \xi_1} d\xi_1 - \frac{\partial \eta}{\partial \xi_2} d\xi_2 - \ldots - \frac{\partial \eta}{\partial \xi_{q-1}} d\xi_{q-1} = 0,$$

$$d\eta' - \frac{\partial \eta'}{\partial \xi'_1} d\xi'_1 - \frac{\partial \eta'}{\partial \xi'_2} d\xi'_2 - \ldots - \frac{\partial \eta'}{\partial \xi'_{q-1}} d\xi'_{q-1} = 0,$$

where the $\xi$s and $\eta$ are actual quantities linearly expressed in terms of the $\xi'$s and $\eta'$, gives

$$\frac{a_{qq} - a_{1q}\frac{\partial \eta}{\partial \xi_1} - a_{2q}\frac{\partial \eta}{\partial \xi_2} - \ldots - a_{q-1.q}\frac{\partial \eta}{\partial \xi_{q-1}}}{1}$$

$$= \frac{a_{qi} - a_{1i}\frac{\partial \eta}{\partial \xi_1} - a_{2i}\frac{\partial \eta}{\partial \xi_2} - \ldots - a_{q-1.i}\frac{\partial \eta}{\partial_{\ q-1}}}{-\frac{\partial \eta'}{\partial \xi'_i}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{-\Delta \frac{\partial \eta}{\partial \xi_i}}{A_{iq} - A_{i1}\frac{\partial \eta'}{\partial \xi'_1} - A_{i2}\frac{\partial \eta'}{\partial \xi'_2} - \ldots - A_{i.q-1}\frac{\partial \eta'}{\partial \xi'_{q\ 1}}} \quad (i = 1, 2, \ldots, q-1)$$

$$= \frac{\Delta}{A_{qq} - A_{q1}\frac{\partial \eta'}{\partial \xi'_1} - A_{q2}\frac{\partial \eta'}{\partial \xi'_2} - \ldots - A_{q.q-1}\frac{\partial \eta'}{\partial \xi'_{q-1}}}.$$

Thus

$$\left\{ A_{qq} - A_{q1}\frac{\partial \eta'}{\partial \xi'_1} - A_{q2}\frac{\partial \eta'}{\partial \xi'_2} - \ldots - A_{q.q-1}\frac{\partial \eta'}{\partial \xi'_{q-1}} \right\} \frac{\partial \eta}{\partial \xi_i}$$

$$= -\left\{ A_{iq} - A_{i1}\frac{\partial \eta'}{\partial \xi'_1} - A_{i2}\frac{\partial \eta'}{\partial \xi'_2} - \ldots - A_{i.q-1}\frac{\partial \eta'}{\partial \xi'_{q-1}} \right\}.$$

Applying this fact for each value of $i$, we see that the transformed operator above may be written

$$\xi_1^{n_1} \xi_2^{n_2} \ldots \xi_{q-1}^{n_{q-1}} \eta^m \left\{ \lambda'_q + \lambda'_1 \frac{\partial \eta'}{\partial \xi'_1} + \lambda'_2 \frac{\partial \eta'}{\partial \xi'_2} + \ldots + \lambda'_{q-1} \frac{\partial \eta'}{\partial \xi'_{q-1}} \right\},$$

where

$$\Delta \lambda'_i = A_{1i}\lambda_1 + A_{2i}\lambda_2 + \ldots + A_{q-1.i}\lambda_{q-1} + A_{qi}(-\lambda_q) \quad (i = 1, 2, \ldots, q-1),$$

$$\Delta(-\lambda'_q) = A_{1q}\lambda_1 + A_{2q}\lambda_2 + \ldots + A_{q-1.q}\lambda_{q-1} + A_{qq}(-\lambda_q),$$

or, more simply,

$$\lambda_i = a_{i1}\lambda_1' + a_{i2}\lambda_2' + \ldots + a_{i.q-1}\lambda_{q-1}' + a_{iq}(-\lambda_q') \quad (i = 1,2,\ldots,q-1),$$

$$-\lambda_q = a_{q1}\lambda_1' + a_{q2}\lambda_2' + \ldots + a_{q.q-1}\lambda_{q-1}' + a_{qq}(-\lambda_q'),$$

so that $\lambda_1, \lambda_2, \ldots, \lambda_{q-1}, -\lambda_q$ express in terms of $\lambda_1', \lambda_2', \ldots, \lambda_{q-1}', \lambda_q'$, as do quantities cogredient with $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$. We shall exemplify the use of this fact later.

For the present our object is to draw a conclusion from the form of the transformed operator without paying special attention to this connexion between the $\lambda$s and $\lambda'$s. We have, in particular, that the transformations of

$$\ldots$$

$$\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_q-1}\eta^m \quad \text{and} \quad \xi_1^{n_1}\xi_2^{n_2}\ldots\xi_{q-1}^{n_q-1}\eta^m\frac{\partial\eta}{\partial\xi_r}$$

are of the above form, *i.e.*, since we can express $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ linearly in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$, are linear functions of operators of the same two forms as themselves with different values of $n_1, n_2, \ldots, n_{q-1}, m$, and $r$. The sum $\Sigma n+m$ is, however, the same for each part of the transformed operators as in the operators subjected to transformation, in virtue of the linearity of the expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$. Consequently, an operator of any order $\Sigma n+m$ transforms into a linear function with constant coefficients of operators of the same order $\Sigma n+m$.

In particular, the complete system of operators of the first order (§ 12) transforms into the complete system of operators of the first order, a fact in accord with the next article.

16. I will now utilize the method of transformation before us to prove that, as stated earlier, the $q^2$ independent operators of the first order are the operators appertaining to the $q^2$ independent infinitesimal substitutions of the group with $q^2$ parameters, which is constituted by the extensions of the general linear group with $q(q+1)$ parameters; so that the writing down of explicit forms of these operators of infinitesimal substitution to any extent is reduced to mere multiplication of multinomials, and, indeed, to the mere squaring of one multinomial $\eta$.

Consider, first, a substitution which leaves $x_1, x_2, \ldots, x_{q-1}$ unaltered, but replaces $y$ by

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{q-1}x_{q-1} + (1+\alpha_q)\,y + \gamma,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_q, \gamma$ are infinitesimal constants. Here $\gamma$ does not

affect the derivatives $y_{rst\ldots}$. The increment which it gives to a function $f$ of variables and derivatives is only $\gamma \dfrac{\partial}{\partial y} f$.

Again, $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ affect each only one, a first, derivative. Thus, for instance, $\alpha_1$ alters only $y_{100\ldots}$, changing it into $y_{100\ldots} + \alpha_1$. Thus the $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ increments of $f$ are respectively

$$\alpha_1\left(x_1 \frac{\partial}{\partial y} + \frac{\partial}{\partial y_{100\ldots}}\right) f, \quad \alpha_2\left(x_2 \frac{\partial}{\partial y} + \frac{\partial}{\partial y_{010\ldots}}\right) f, \quad \alpha_3\left(x_3 \frac{\partial}{\partial y} + \frac{\partial}{\partial y_{001\ldots}}\right), \quad \ldots,$$

*i.e.*, $$\alpha_1\left(x_1 \frac{\partial}{\partial y} + \xi_1\right) f, \quad \alpha_2\left(x_2 \frac{\partial}{\partial y} + \xi_2\right) f, \quad \ldots, \quad \alpha_{q-1}\left(x_{q-1} \frac{\partial}{\partial y} + \xi_{q-1}\right) f.$$

Once more, the existence of $\alpha_q$ changes $y_{rst\ldots}$ into $(1 + \alpha_q)\, y_{rst\ldots}$. Its effect is then to give $f$ the increment

$$\alpha_q\left(y \frac{\partial}{\partial y} + \Sigma y_{rst\ldots} \frac{\partial}{\partial y_{rst\ldots}}\right) f,$$

*i.e.*, $$\alpha_q\left(y \frac{\partial}{\partial y} + \eta\right) f.$$

Again, consider a substitution which leaves $y$ and $x_2, x_3, \ldots, x_{q-1}$ unaltered, but replaces $x_1$ by

$$(1 + \beta_1)\, x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_{q-1} x_{q-1} + \beta_q y + \gamma',$$

where the $\beta$s and $\gamma'$ are infinitesimal constants. The consequent increment of $f$ is in just the same way what, in a notation which regards $y, x_2, x_3, \ldots, x_{q-1}$ as independent variables and $x_1$ as dependent, would be written

$$\left\{ \gamma' \frac{\partial}{\partial x_1} + \beta_2\left(x_2 \frac{\partial}{\partial x_1} + \xi_2\right) + \beta_3\left(x_3 \frac{\partial}{\partial x_1} + \xi_3\right) + \ldots \right.$$

$$\left. \ldots + \beta_{q-1}\left(x_{q-1} \frac{\partial}{\partial x_1} + \xi_{q-1}\right) + \beta_q\left(y \frac{\partial}{\partial x_1} + \eta\right) + \beta_1\left(x_1 \frac{\partial}{\partial x_1} + \xi_1\right) \right\} f;$$

but we have here to express the symbolic operators by their equivalents when, as in fact, $y$ is dependent and $x_1, x_2, \ldots, x_{q-1}$ independent. In other words, we have to transform the operator as in the last two articles, taking

$$x_2 = x_2', \quad x_3 = x_3', \quad \ldots, \quad x_{q-1} = x_{q-1}', \quad y = y', \quad x_1 = x_1',$$

where, on the left, $x_1$ is dependent, and, on the right, $y'$; and then remove accents.

Now, here, $\Delta = \begin{vmatrix} 1, & 0, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & 0, & \ldots, & 0, & 0 \\ 0, & 0, & 1, & \ldots, & 0, & 0 \\ \ldots & & \ldots & & \ldots & \\ 0, & 0, & 0, & \ldots, & 0, & 1 \\ 0, & 0, & 0, & \ldots, & 1, & 0 \end{vmatrix} = -1,$

and the other factor which has to be introduced as well as $\Delta^{-1}$ may, as we saw in § 14, be calculated as the Jacobian of the old independent Greek letters (in this case $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta$) with regard to the new independent ones (in this case $\xi'_2, \xi'_3, \ldots, \xi'_{q-1}, \xi'_1$). It is then

$$\begin{vmatrix} 1, & 0, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & 0, & \ldots, & 0, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & 0, & \ldots, & 1, & 0 \\ \dfrac{\partial \eta'}{\partial \xi'_2}, & \dfrac{\partial \eta'}{\partial \xi'_3}, & \dfrac{\partial \eta'}{\partial \xi'_4}, & \ldots, & \dfrac{\partial \eta'}{\partial \xi'_{q-1}}, & \dfrac{\partial \eta'}{\partial \xi'_1} \end{vmatrix} = \frac{\partial \eta'}{\partial \xi'_1}.$$

Thus we have to substitute for $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta, \xi_1$, their equivalents with $y$ dependent $-\xi_2 \dfrac{\partial \eta'}{\partial \xi'_1}, \quad -\xi_3 \dfrac{\partial \eta'}{\partial \xi'_1}, \ \ldots, \ -\xi_{q-1} \dfrac{\partial \eta'}{\partial \xi'_1}, \quad -\eta \dfrac{\partial \eta'}{\partial \xi'_1}, \quad -\xi_1 \dfrac{\partial \eta'}{\partial \xi'_1}$, where $\xi_2, \xi_3, \ldots, \xi_{q-1}, \eta, \xi_1$ are the same as $\xi'_2, \xi'_3, \ldots, \xi'_{q-1}, \eta', \xi'_1$. Doing so, and removing the accents which have been used only for temporary convenience, the aggregate of the second set of infinitesimal increments of $f$ becomes

$$\left\{ \gamma' \frac{\partial}{\partial x_1} + \beta_2 \left( x_2 \frac{\partial}{\partial x_1} - \xi_2 \frac{\partial \eta}{\partial \xi_1} \right) + \beta_3 \left( x_3 \frac{\partial}{\partial x_1} - \xi_3 \frac{\partial \eta}{\partial \xi_1} \right) + \ldots \right.$$

$$\left. \ldots + \beta_{q-1} \left( x_{q-1} \frac{\partial}{\partial x_1} - \xi_{q-1} \frac{\partial \eta}{\partial \xi_1} \right) + \beta_q \left( y \frac{\partial}{\partial x_1} - \eta \frac{\partial \eta}{\partial \xi_1} \right) + \beta_1 \left( x_1 \frac{\partial}{\partial x_1} - \xi_1 \frac{\partial \eta}{\partial \xi_1} \right) \right\} f.$$

In these two partial aggregates of infinitesimal substitutions, symbolic operators of all types $\xi_r, \ \eta, \ \xi_r \dfrac{\partial \eta}{\partial \xi_r}, \ \xi_r \dfrac{\partial \eta}{\partial \xi_s} (r \neq s), \ \eta \dfrac{\partial \eta}{\partial \xi_s}$ have been introduced, every one which has occurred, occurring once only; and no operators, having reference to derivatives $y_{rst\ldots}$, of any other kind are present. It is clear that we complete the entire aggregate of infinitesimal substitutions by giving next $x_2$ its most general infinitesimal increment, keeping $x_1, x_3, \ldots, x_{q-1}, y$ unaltered; then $x_3$

in like manner, and so on. No new type of operator is thus introduced; but every one of each type, as above, is just once presented. The proof for the case of $x_1$, which has been given, applies to all the others with alterations of suffixes.

We thus have it clearly before us that the $q(q+1)$ operators appertaining to the independent infinitesimal substitutions of the general linear group and its extensions are

(i.) one of the type $\dfrac{\partial}{\partial y}$,

(ii.) $q-1$ of the type $\dfrac{\partial}{\partial x_r}$,

(iii.) $q-1$ of the type $x_r \dfrac{\partial}{\partial y} + \xi_r$,

(iv.) one of the type $y \dfrac{\partial}{\partial y} + \eta$,

(v.) $q-1$ of the type $x_r \dfrac{\partial}{\partial x_r} - \xi_r \dfrac{\partial \eta}{\partial \xi_r}$,

(vi.) $(q-1)(q-2)$ of the type $x_r \dfrac{\partial}{\partial x_s} - \xi_r \dfrac{\partial \eta}{\partial \xi_s}$ $(r \neq s)$,

(vii.) $q-1$ of the type $y \dfrac{\partial}{\partial x_s} - \eta \dfrac{\partial \eta}{\partial \xi_s}$.

The $q^2$ operators appertaining to the group which consists of the extensions only are given by (iii.) to (vii.) inclusive with the $\dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial x_r}$, $\dfrac{\partial}{\partial x_s}$ parts omitted.

As to the sub-group of the general linear group in which the generality of the constants is limited by the one condition

$$\Delta \equiv | a_{ij} | = 1,$$

the fact of its having one infinitesimal substitution fewer is expressed by the vanishing of the sum of the infinitesimal multipliers of the $q$ operators (iv.) and (v.). The conclusions which follow have been stated in the latter part of § 12.

17. Let us now think of differential invariants, with a view to arriving at a proof that the absolute differential invariants of this

last-mentioned sub-group—an invariant sub-group (*Lie*) or self-conjugate sub-group (*Burnside*) of the general linear group—are identical with absolute pure $q$-ary reciprocants; or, more precisely, that absolute pure $q$-ary reciprocants are coextensive with these absolute differential invariants of the sub-group, which are homogeneous or entirely of odd or entirely of even degree.

Notice that it is not a necessity, but a convention, to regard the differential invariants of the sub-group, or the pure reciprocants, as homogeneous,* just as it is not a necessity, but a convention, to regard ordinary invariants,† except those of a single quantic, as homogeneous. The convention is reasonable; for the separate homogeneous parts of a non-homogeneous differential invariant or pure reciprocant are themselves differential invariants or pure reciprocants, just as in the case of ordinary invariants. It will illustrate the usefulness of our present method to prove this for absolute differential invariants of the self-conjugate sub-group.

The following is an equivalence of symbolic operators for the general linear scheme :—

$$\xi_1\frac{\partial\eta}{\partial\xi_1}+\xi_2\frac{\partial\eta}{\partial\xi_2}+\ldots+\xi_{q-1}\frac{\partial\eta}{\partial\xi_{q-1}}-\eta\equiv\xi_1'\frac{\partial\eta'}{\partial\xi_1'}+\xi_2'\frac{\partial\eta'}{\partial\xi_2'}+\ldots+\xi_{q-1}'\frac{\partial\eta'}{\partial\xi_{q-1}'}-\eta'.$$

To prove it we apply § 15. The operator which transforms into

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* In the second of my papers on ternary reciprocants the unjustifiable remark was made (*Proc. Lond. Math. Soc.*, Vol. XVIII., p.157) that a ternary reciprocant is *necessarily* homogeneous, and was based on the unjustifiable statement that this is so for ordinary invariants of a system of binary quantics.

† An ordinary $q$-ary invariant or covariant (non-absolute) is an absolute invariant of a group holohedrically isomorphous with a linear homogeneous group of $q$ variables in which the generality of coefficients is limited by the one relation $\Delta = 1$, the special sub-group of the general linear homogeneous group. For instance, when the variables in $(a_0, a_1, \ldots, a_p)(x, y)^p$ are substituted for, according to this special group, the group of substitutions for the coefficients involves the same independent parameters, and the infinitesimal transformations, namely, in the one case $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$, and, in the other,

$$a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+\ldots,\quad pa_1\frac{\partial}{\partial a_0}+(p-1)\,a_2\frac{\partial}{\partial a_1}+\ldots,\quad pa_0\frac{\partial}{\partial a_0}+(p-2)\,a_1\frac{\partial}{\partial a_1}+\ldots,$$

have the same composition. An absolute invariant or covariant is an absolute invariant of a group holohedrically isomorphous with the general linear homogeneous group.

$\xi_r' \dfrac{\partial \eta'}{\partial \xi_r'}$ is, by immediate application of that article,

$$\xi_r' \left\{ a_{1r} \frac{\partial \eta}{\partial \xi_1} + a_{2r} \frac{\partial \eta}{\partial \xi_2} + \ldots + a_{q-1,r} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qr} \right\}$$

$$(r = 1, 2, \ldots, q-1),$$

and that which transforms into $-\eta'$ is

$$\eta' \left\{ a_{1q} \frac{\partial \eta}{\partial \xi_1} + a_{2q} \frac{\partial \eta}{\partial \xi_2} + \ldots + a_{q-1,q} \frac{\partial \eta}{\partial \xi_{q-1}} - a_{qq} \right\};$$

addition of which gives, by means of the linear expressions for $\xi_1, \xi_2, \ldots, \xi_{q-1}, \eta$ in terms of $\xi_1', \xi_2', \ldots, \xi_{q-1}', \eta'$, the identity which has been written.

From this equivalence, a persistence in form of an operator, it follows that

$$\xi_1 \frac{\partial \eta}{\partial \xi_1} + \xi_2 \frac{\partial \eta}{\partial \xi_2} + \ldots + \xi_{q-1} \frac{\partial \eta}{\partial \xi_{q-1}} - \eta$$

is an operator which, acting on an absolute differential invariant of a linear group, produces another absolute differential invariant. It annihilates one of the general linear group, but not, as a rule, one of the sub-group.

Now absolute differential invariants of the sub-group we are considering have $\eta + \xi_1 \dfrac{\partial \eta}{\partial \xi_1}$, $\eta + \xi_2 \dfrac{\partial \eta}{\partial \xi_2}$, ... for annihilators (§ 12). Hence the effect on any such of $\xi_r \dfrac{\partial \eta}{\partial \xi_r}$ (for any $r$) is the same as that of $-\eta$. It follows that $-q\eta$, and consequently $\eta$, is an operator which generates absolute differential invariants of the sub-group from others. Now the effect of $\eta$ is to multiply any homogeneous function by its degree. If, then,

$$H_i + H_j + H_k + \ldots$$

is an absolute differential invariant with parts of degrees $i, j, k, \ldots$, it follows in succession that

$$iH_i + jH_j + kH_k + \ldots,$$

$$i^2 H_i + j^2 H_j + k^2 H_k + \ldots,$$

&c., &c.,

are also absolute differential invariants; and, consequently, that $H_i$, $H_j$, $H_k$, ... separately are. We lose, then, no generality, as to complete systems, if we confine attention to differential invariants which are homogeneous.

Let us now consider what a homogeneous absolute differential invariant of our sub-group becomes when a linear substitution is applied for which $\Delta = -1$ instead of $+1$. Such a substitution may be effected by two stages: (1) a change of sign in $y$, and (2) a substitution of determinant $+1$. The first stage does not alter the differential invariant if its degree be even, but changes its sign if its degree be odd. The second stage makes no alteration in it, being merely the performance of a substitution for which it is invariant. Thus a function of derivatives which does not contain parts of both odd and even degrees, and which is an absolute differential invariant for a linear group for which $\Delta = +1$, is, at most, changed in sign by a substitution for which $\Delta = -1$.

18. In my paper "On Ternary and $n$-ary Reciprocants" (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 191), the only one in which cases of $n$ (*i.e.*, $q$) $>3$ have been introduced, I gave only a provisional definition of a $q$-ary reciprocant, expressing no confidence that the best form of presentation was given. It now seems better to adopt the following definition for the class of $q$-ary reciprocants which is of importance in the present connexion. A function of the derivatives of $y$ with regard to $x_1, x_2, \ldots, x_r, \ldots, x_{q-1}$ is called an absolute $q$-ary reciprocant when it is equal, but for a constant factor, to the same function of the derivatives of $x_r$ with regard to $x_1, x_2, \ldots, y, \ldots, x_{q-1}$, for each of the $q-1$ values of $r$. Here $y$ occupies the old position of $x_r$ in the series of independent variables. The reciprocant is called *pure* when first derivatives do not occur in it. All this is in strict accord with Sylvester's definition of ordinary reciprocants. The constant factor is either $\pm 1$, as the conclusions which follow will establish.

We notice, first, that any absolute differential invariant, which does not contain parts of both odd and even degrees, of the linear group with generality of coefficients limited by the one relation $\Delta = 1$, is an absolute $q$-ary reciprocant. For the substitution which effects a change of independent variable as above is (as in § 16) one for which $\Delta = -1$, and the effect of this on the differential invariant is, at most, to change its sign, by the preceding article. One which contains parts both of odd and of even degrees is not a $q$-ary reciprocant, but the sum of two such of opposite characters.

We have, further, to prove conversely that any absolute pure $q$-ary reciprocant is an absolute differential invariant for the special linear group. We shall see this by showing that it has the full system of annihilators of absolute differential invariants of the group.

The property of absolute pure $q$-ary reciprocants which suffices to give the entire conclusion is that any one, and all its equivalents arising from interchanges of the dependent with one of the independent variables, are free from first derivatives.

That the reciprocant $R_y$ itself ($y$ dependent) is free from first derivatives is expressed by saying that it has the $q-1$ annihilators

$$\xi_1,\ \xi_2,\ \xi_3,\ \ldots,\ \xi_{q-1}.$$

That its equivalent with $x_1$ dependent and $y$ occupying the place of $x_1$ is free from first derivatives is, in like manner, expressed by saying that it has $q-1$ annihilators which, were $x_1$ dependent variable, would be

$$\eta,\ \xi_2,\ \xi_3,\ \ldots,\ \xi_{q-1}.$$

Now, as in § 16, these annihilators, expressed in the notation which has meaning when $y$ is dependent, are, respectively,

$$-\eta\frac{\partial\eta}{\partial\xi_1},\quad -\xi_2\frac{\partial\eta}{\partial\xi_1},\quad -\xi_3\frac{\partial\eta}{\partial\xi_1},\quad \ldots,\quad -\xi_{q-1}\frac{\partial\eta}{\partial\xi_1}.$$

In like manner, expressing that the equivalents of $R_y$ with $x_2, x_3, \ldots, x_{q-1}$ in succession as the one dependent variable are free from first derivatives, we get $q-2$ other sets of $q-1$ annihilators of $R_y$, but no new types are introduced. So far, then, the information is that $R_y$ has

(i.) $q-1$ annihilators $\xi_1, \xi_2, \ldots, \xi_{q-1}$,

(ii.) $q-1$ annihilators $\eta\dfrac{\partial\eta}{\partial\xi_r}$ $(r = 1, 2, \ldots, q-1)$,

(iii.) $(q-1)(q-2)$ annihilators $\xi_r\dfrac{\partial\eta}{\partial\xi_s}$ $(r\neq s;\ r, s = 1, 2, \ldots, q-1)$,

*i.e.*, together $q^2-q$ annihilators.

But each of the facts (ii.) really gives two facts of annihilation independent of one another and of (i.) and (iii.), in consequence of $R_y$ being free from first derivatives. Consider, for instance, the annihilation by $\eta\dfrac{\partial\eta}{\partial\xi_1}$. The terms in the expansion of $\eta\dfrac{\partial\eta}{\partial\xi_1}$ which

involve the first derivative $y_{100\ldots}$ in their coefficients must by themselves annihilate $R_y$, as otherwise operation with them would produce terms in $\eta \dfrac{\partial \eta}{\partial \xi_1} R_y$ against which no other terms could cancel. Now, as $\eta$ only involves $y_{100\ldots}$ in the term $y_{100\ldots}\xi_1$, these terms in $\eta \dfrac{\partial \eta}{\partial \xi_1}$ are $y_{100\ldots}\left(\xi_1 \dfrac{\partial \eta}{\partial \xi_1} + \eta\right)$. Similarly as to $\eta \dfrac{\partial \eta}{\partial \xi_r}$ in general. Thus, besides the $q^2 - q$ annihilators above classified, $R_y$ must also have

(iv.) the $q-1$ annihilators $\xi_r \dfrac{\partial \eta}{\partial \xi_r} + \eta$, $(r = 1, 2, \ldots, q-1)$.

Consequently, our supposed absolute pure $q$-ary reciprocant $R_y$ has for annihilators the full system of $q^2 - 1$ operators which belong to the infinitesimal substitutions of the group of extensions of the linear group with generality of coefficients limited by the one relation $\Delta = 1$. $R_y$ is then an absolute differential invariant of that special linear group.

Accordingly, the statement that absolute pure $q$-ary reciprocants are identical with those absolute differential invariants which do not contain parts of both odd and even degrees of the special linear group is correct.

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*Thursday, April 7th,* 1898.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

There being only five members present, and so no quorum, no meeting could be held.

The Chairman (provisionally) communicated the following papers:—

An Essay towards the Generating Functions of Ternariants: Professor Forsyth.

On Systems of One-Vectors in Space of $n$ Dimensions: Mr. W. H. Young.

Zeroes of the Bessel Functions: Mr. H. M. Macdonald.

The following presents were received for the Library :—

"Proceedings of the Royal Society," Vol. LXII , Nos. 386, 387.

"Year-Book of the Royal Society," 1896–7, No. 1.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," V., VI ; 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 3 ; Leipzig, 1898.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. IV., No. 6 ; New York, March, 1898.

"Science Abstracts," Vol. I., Pts. 1, 2 ; London, Taylor & Francis, 1898.

Rayet, G.—"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde" de Juin 1894 à Mai 1895 ; Juin 1895 à Mai 1896, 8vo ; Bordeaux, 1895–6.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 2 ; Napoli, 1898.

"Mémoires de la Société des Sciences Physiques et Natureles de Bordeaux," Tome I., Cahier 1, 2, 1895–6 ; Tome II , Cahier 1, 2, 1896. "Procès-Verbaux de Séances de la Société des Sciences Physiques et Naturelles de Bordeaux," Année 1894–5 ; 1895–6.

"Comptes Rendus de la Section de Physique et de Chimie de Varsovie," 1895–6 ; 1897.

"Comptes Rendus de la Section Biologique, 1895–6," et "Procès-Verbaux des Réunions Générales," Année 7 ; Varsovie.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 5 ; Roma, 1898.

"Educational Times," April, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XII., Fasc. 1, 2 ; 1898.

"Rendiconti del Circolo Matematico di Palermo," Tome XII., Fasc. 1, 2 ; February, 1898.

"Annals of Mathematics," Vol. XII., No. 1 ; Virginia, February, 1898.

"Indian Engineering," Vol. XXIII., Nos. 8, 9, 10, Feb. 19–March 5, 1898.

Nova Acta :—

"Beiträge zur Geschichte der Trigonometrie," von A. v. Braunmühl, Bd. LXXI., 1 ; Halle, 1897.

"Nassîr Eddîn Tûsi und Regiomontan," Bd. LXXI., 2, von A. v. Braunmühl ; Halle, 1897.

"Zur Geschichte der Geometrie mit constanter Zirkelöffnung," Bd. LXXI., 3, von W. M. Kotta.

("Abh. der Kaiserl. Leop.-Carol. Deutschen Akad. der Naturforschen.")

"On the Definite Integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt$, with extended Tables of Values," by Jas. Burgess, Edinburgh (Offprint from the "Transactions of the Royal Society of Edinburgh," Vol. XXXIX., Pt. II., No. 9).

*On Systems of One-Vectors in Space of n Dimensions.* *By* W. H. Young. Received March 26, 1898. Provisionally communicated April 7, 1898.

In the present paper a system of one-vectors* in space of $n$ dimensions is reduced to a normal form.

The space in question is supposed to be flat, *i.e.*, of zero curvature.

I use $S_k$ to denote a flat space of $k$ dimensions contained in the whole space $S_n$.

The normal form in $S_{2m-1}$ consists of $m$ one-vectors, the line of action of one being arbitrary. In $S_{2m}$ the normal form again consists of $m$ one-vectors, lying, however, in a covariant $S_{2m-1}$.

In the course of the discussion certain other properties of the system are obtained. The whole subject is closely connected with the theory of linear complexes in space of $n$ dimensions. I here confine myself to giving a geometrical definition of line coordinates in such a space, and a simple geometrical deduction of the known quadratic equations between them.

It is well known that in $S_n$ we can construct, in an infinite number of ways, a fundamental $(n+1)$-pyramid which does not lie in any space of lower dimensions.

Theorem I. *Any one-vector through an arbitrary point $O$ may be replaced by $n$ one-vectors along lines joining $O$ to the $n$ points of any fundamental $n$-pyramid lying in any $S_{n-1}$ not passing through $O$.*

These $n$ points together with $O$ define a fundamental $(n+1)$-pyramid in the $S_n$.

Take the plane through the line of action of the one-vector and one of the edges meeting at $O$. This plane meets the $S_{n-1}$ determined by the remaining $(n-1)$ edges through $O$ in a straight line. We can then, by the parallelogram law, replace our one-vector by two components, one along the first edge, and one in the $S_{n-1}$. In this

* The reader may, if he please, substitute the word "force" for "one-vector," throughout the present paper, which is, in part, introductory to one on vectors of a more general nature. In a paper entitled "Sulla Statica dei Corpi Rigidi nello Spazio a Quattro Dimensioni," *Giornale di Battaglini*, xxxiv., 1896, some of the properties here given are discussed for four dimensions by De Francesco.

$S_{n-1}$ the $n$ vertices of the original $(n+1)$-pyramid determine a fundamental $n$-pyramid, and the component of our one-vector in this $S_{n-1}$ goes through one of the vertices, viz., $O$. Thus, by induction, the theorem follows, it being certainly true when $n = 2$.

The above tacitly assumes that the one-vector does not lie in any $S_k$ of the fundamental $(n+1)$-pyramid. If this be the case, we only have to effect the reduction for the corresponding $(k+1)$-pyramid, the components corresponding to the remaining edges through $O$ vanishing.

THEOREM II. *Any one-vector may be replaced by one-vectors along the edges of the fundamental $(n+1)$-pyramid.*

Let us assume that the line of action of the one-vector does not lie in any $S_k$ of the fundamental $(n+1)$-pyramid. It must intersect at least one of the $S_{n-1}$'s of the pyramid in a point at a finite distance. Join this point of intersection to the vertex $S_0$ of the pyramid opposite to this $S_{n-1}$. The plane through the one-vector and this line will meet the $S_{n-1}$ in question in a straight line through the same point of intersection. Hence, by the parallelogram law, the one-vector may be replaced by two one-vectors, one through the $S_0$ and the other in the $S_{n-1}$. By Theorem I. the former component may be replaced by components along the edges through the $S_0$. The possibility of the reduction then depends on replacing the one-vectors in the $S_{n-1}$ by components along the edges of the fundamental $(n+1)$-pyramid which lie in that $S_{n-1}$. But these are the edges of an $n$-pyramid which is a fundamental one in that $S_{n-1}$. The required result follows by induction, for it obviously holds when $n = 1$.

THEOREM III. *A given system of one-vectors may be replaced in one and only one way by one-vectors along the edges of a given fundamental $(n+1)$-pyramid.*

The possibility of this reduction follows from Theorem II.

To prove the uniqueness, we remark that, if two modes of reduction are possible, reversing one system so obtained and combining with the other, we obtain a system in equilibrium.* This system of one-vectors along the edges may be divided into two classes, viz., those through an arbitrarily chosen vertex and those in the opposite $S_{n-1}$ of the fundamental $(n+1)$-pyramid. Unless the former all vanish, they can, by repeated application of the parallelogram law, be reduced

* A system is said to balance, or to be in equilibrium, when, by introducing suitable pairs of equal and opposite one-vectors and by repeated application of the parallelogram law, it can be reduced to two equal and opposite one-vectors.

to a single one-vector through the chosen vertex. This one-vector, lying outside the $S_{n-1}$, cannot be replaced by, and therefore also cannot balance, a system of one-vectors in it. The theorem now follows by induction.

Cor. I. *A system of one-vectors has* $\frac{1}{2}(n+1)\,n$ *"coordinates," viz., the equivalent one-vectors along the edges of the fundamental* $(n+1)$*-pyramid. There are, moreover,* $\frac{1}{2}(n+1)\,n$ *necessary and sufficient conditions of equilibrium, viz., each coordinate must vanish.*

Cor. II. *There are* $\frac{1}{2}(n-1)\,(n-2)$ *necessary and sufficient conditions that a given system of one-vectors should be equivalent to a single one-vector.*

This is evident if we reflect that there are $\infty^{2n-2}$ straight lines in $S_n$, and that therefore a straight line has $(2n-2)$ independent coordinates; a single one-vector therefore $(2n-1)$. The required number is got by subtracting this number from the number of coordinates of a general system.

Cor. III. *Any system of one-vectors in an* $S_n$ *may be replaced by a one-vector through any arbitrary point* $O$*, and a system of one-vectors in an arbitrary* $S_{n-1}$ *which does not pass through* $O$*.*

In a special case the one-vector may of course vanish, or the system in the $S_{n-1}$ be in equilibrium.

We now proceed to generalize certain known theorems about systems of one-vectors in $S_3$. It will be convenient to consider first the cases of $S_4$ and $S_5$.

Theorem IV. *Any system of one-vectors in* $S_4$ *is equivalent to two one-vectors, and has as covariant the* $S_3$ *containing them.*

By Cor. III. to Theorem III. such a system is equivalent to a one-vector through an arbitrary point $O$ and a system of one-vectors in an $S_3$. This latter system, if not reducible to a single one-vector, is, as is well known, equivalent to two one-vectors, one of which may be chosen to act along any line we please in the $S_3$. Take it to act along a line through the point in which the one-vector through $O$ meets the $S_3$. Then it may be compounded with the latter one-vector, and thus the system is reducible to two one-vectors, which, in general, do not intersect.

These two one-vectors may be replaced by any other suitable pair in the $S_3$ determined by them; viz., the line of action of one of the two one-vectors may be chosen arbitrarily in the $S_3$, the line of action of the other and the magnitude of both one-vectors being then determined. The $S_3$ itself is a covariant of the system.

In fact, if it were possible by reducing in another way to get a second pair lying in a different $S_3$, we could reverse this pair and obtain with the first pair a system in equilibrium. Since, however, two $S_3$'s intersect in an $S_2$, we can arrange that one one-vector of our second pair should lie in this $S_2$, and the other accordingly not. We should then have a single one-vector not in the first $S_3$, balancing one-vectors lying in it, which is obviously absurd.

THEOREM V. *a. Any system of one-vectors in $S_5$ is equivalent to three one-vectors, one of which may be chosen to act through any point we choose.*

The reduction to three one-vectors follows from the previous theorem by means of Cor. III. to Theorem III. One of the one-vectors passes by construction through an arbitrary point $O$. If the $S_3$ determined by any two one-vectors be intersected by the line of action of the third, the three one-vectors would lie in an $S_4$. That this would be a special case is obvious, since we might choose as our system three straight lines not lying in an $S_4$. Omitting for the present the discussion of special cases, we may assume in what follows that the three one-vectors do not lie in an $S_4$.

THEOREM V. *b. Any arbitrary line being chosen, one of the three one-vectors can be made to act along it.*

We already have one arbitrary point $O$ on the first one-vector. Take any other point $P$. The plane through $P$ and the first one-vector meets the $S_3$ determined by the other two in a point $Q$. The first one-vector may now be replaced by two components, along $OP$ and $OQ$. The other two one-vectors may be replaced by two one-vectors in their $S_3$, one of which passes through $Q$. Compounding the two one-vectors meeting at $Q$, we are left with three one-vectors, one of which acts along the arbitrary line $OP$.

THEOREM V. *c. Any arbitrary $S_3$ being chosen, two of the one-vectors can be made to act in it.*

By the above we may take any straight line in the $S_3$ as line of action of one of the one-vectors. The other two one-vectors lie in a second $S_3$ which will intersect the first in a line. One of the two one-vectors may then be taken to act along this line and the result follows.

THEOREM V. *d. The arbitrary straight line being chosen* (V. *b*) *the $S_3$ containing the other two one-vectors is determined, also the magnitude of the first one-vector and the system in the $S_3$.*

Suppose first, if possible, that a second reduction could lead to a

different $S_3$. Reverse the three one-vectors so found, and we have with the first three a system in equilibrium. The two $S_3$'s have a common straight line, and one one-vector of each pair may be taken to act along it. We have now two one-vectors in one of the original $S_3$'s, one one-vector along the arbitrary line and another one-vector. These latter two one-vectors may be replaced by two others in the $S_3$ determined by them, one of which may, as before, be taken in the original $S_3$ in which the other pair of one-vectors lies. We are left with three one-vectors in this $S_3$ and one outside it. This system can evidently not be in equilibrium unless the last one-vector vanishes. This involves, however, the one-vector along the arbitrary line meeting one of the two $S_3$'s, the possibility of which has been already excluded.

The first part of the theorem is therefore proved by a *reductio ad absurdum*. A moment's consideration shows that two different reductions to the same straight line and $S_3$ are impossible, for it would lead to a one-vector along the straight line balancing a system in the $S_3$. Thus the whole theorem is proved.

Theorem V.*e*. *The arbitrary $S_3$ being chosen* (V.*c*.), *the line of action of the third one-vector is determined, also the magnitude of the third one-vector and the system in $S_3$.*

The proof, being similar to the last, may be omitted.

Theorem V.*f*. *If the line of action* ($S_1$) *of one of the one-vectors pass through an arbitrary point* ($S_0$), *the two other one-vectors lie in a fixed $S_4$ passing through the $S_0$.*

Let $O$ be the point. Choose any other straight line through $O$. The plane through this line and the line of action of the original one-vector meets the conjugate $S_3$ of this latter line in a point $P$. The one-vector may now be replaced by one along the new line and one along $OP$. Make one of the pair of one-vectors in the $S_3$ pass through $P$, and we have reduced to a one-vector along the new line and two one-vectors, which, by construction, lie in the $S_4$ determined by $O$ and the first $S_3$. Hence by (V.*d*.) the result follows.

Theorem V.*g*. *If the $S_3$ of two of the one-vectors pass through an arbitrary plane* ($S_2$), *the remaining one-vector lies in another plane which intersects the first, and therefore lies with it in an $S_4$.*

Choose any other $S_3$ through the plane. The $S_4$ containing the two $S_3$'s meets the third one-vector in a point $O$. The first two one-vectors may be replaced by one in the common plane, and a second which, by a known theorem in $S_3$, however we choose the first one-vector, passes

through a fixed point $P$ of that plane. We may replace this second one-vector by a one-vector in the plane and one along $OP$. Compounding this with the one-vector already passing through $P$, we are left, outside the chosen $S_3$, with a one-vector in the plane determined by $P$ and the original one-vector through $O$.

Hence, by V. *e.*, the theorem follows.

Similarly, the converse may be proved, and the theorem that, if the $S_3$ containing two of the one-vectors turn round a fixed point, the line of action of the third one-vector lies in a fixed $S_4$ passing through the point.

All these theorems for $S_4$ and $S_5$ will be found summed up in the following four theorems for the general case of $S_n$.

THEOREM VI. *Any system of one-vectors in "even space" $S_n$, where $n = 2m$, is reducible to $m\left(=\frac{n}{2}\right)$ one-vectors, lying in a covariant $S_{n-1}$.*

THEOREM VII. *a. Any system of one-vectors in "odd space" $S_n$, where $n = 2m-1$, is reducible to $m\left(=\frac{n+1}{2}\right)$ one-vectors, the line of action of one of which may be chosen arbitrarily. The $S_{n-2}$ containing the remaining $(m-1)\left(=\frac{n-1}{2}\right)$ one-vectors is then determined, and so are the magnitude of the first one-vector and the system in the $S_{n-2}$.*

THEOREM VII. *b. Any $S_{2k-1}$ in the $S_{2m-1}$ being chosen, $k$ of the one-vectors can be made to act in it. The $S_{2m-1-2k}$ containing the remaining $(m-k)$ one-vectors is then determined, and so are the systems in the $S_{2k-1}$, $S_{2m-1-2k}$ respectively.*

THEOREM VII. *c. Any $S_{2p}$ in the $S_{2m-1}$ being chosen, the $S_{2k-1}$ containing $k$ of the one-vectors can be made to pass through it, if $k > p$ (or to lie in it if $k < p$). The conjugate $S_{2m-1-2k}$ then lies in a fixed $S_{2m-2p-2}$ (or passes through a fixed $S_{2m-2p-2}$), intersecting the $S_{2p}$ and therefore lying with it in an $S_{2m-2}$.*

Assuming these theorems to have been demonstrated for all values of $n$ up to $(2m-2)$ inclusive, we can prove them generally by induction.

It is evident that, VII. being proved, VI. follows at once from Theorem III., Cor. III.

To prove Theorem VII. By Theorem III., Cor. III., the system may be replaced by a one-vector through an arbitrary point $O$, and a system in an $S_{n-1}$ not passing through $O$. Since, therefore,

$n-1 = 2m-2$ the system is equivalent to $m$ one-vectors, one of which passes through an arbitrary point $O$.

Choose any other point $P$. The plane through $P$ and the one-vector through $O$ meet the $S_{n-1}$ of the other one-vectors in a point $Q$. Replace the one-vector through $O$ by one-vectors along $OP$, $OQ$, and choose one of the remaining one-vectors to pass through $Q$. Compounding the two one-vectors through $Q$, the system is reduced to a one-vector along an arbitrary line $OP$ and $(m-1)$ other one-vectors.

It is now obvious that, instead of a straight line, we may choose an $S_3$, and make two one-vectors act in it. Because, one one-vector being chosen in the $S_3$, the $S_3$ and the $S_{n-2}$ containing the other $(m-2)$ one-vectors intersect in a straight line, which may be taken as line of action of one one-vector.

Similarly, an $S_5$, $S_7$, or any odd space may be arbitrarily chosen.

Next, to prove that the choice of one arbitrary line determines the conjugate $S_{n-2}$.

Suppose two reductions lead to two different $S_{n-2}$'s; reverse one system so obtained, and we have with the other a system in equilibrium. Choosing $(m-2)$ one-vectors of each system in the $S_{n-4}$ common to both $S_{n-2}$'s, we have left three one-vectors in a perfectly determinate $S_5$; one of these three may, of course, be chosen in the $S_{n-4}$, since an $S_5$ and $S_{n-4}$ have a common line. We are left with

$$\left.\begin{array}{c}\text{two one-vectors in an } S_3\\ \text{and}\\ \text{a system in an } S_{n-4}\end{array}\right\}.$$

In the general case, the $S_3$ and $S_{n-4}$ have no common point, and therefore the whole system can only be in equilibrium if each partial system is in equilibrium by itself. This, however, leads at once to a special system in which the arbitrarily chosen line intersects its polar $S_{n-2}$, and hence, by a *reductio ad absurdum*, the theorem is proved.

It is now obvious that the magnitude of the one-vector along the arbitrary line and the system in the polar $S_{n-2}$ are determined; otherwise we could obtain a system of one-vectors in an $S_{n-2}$, balanced by a single one-vector whose line of action lies entirely outside that $S_{n-2}$.

The proof of Theorems *b* and *c*, being very simple and identical in principle with the proofs of the corresponding theorems for five dimensions, may be omitted.

*Counting up the Constants.*—A system of one-vectors in $S_n$ has $\frac{1}{2}n(n+1)$ constants. If we divide this up into an equivalent system in an $S_k$ and one in an $S_{n-k-1}$, we have

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}k(k+1) - \tfrac{1}{2}(n-k-1)(n-k) = (k+1)(n-k)$$

constants still at our disposal; this is exactly the number of coordinates of an $S_k$, and we should therefore expect to be able to choose the $S_k$ arbitrarily, as has been shown by the foregoing discussion to be the case.

THEOREM VIII. *When n is odd, and equal to* $2m-1$, *the m one-vectors to which a given (general) system can be reduced determines a fundamental* $(n+1)$*-pyramid of constant volume.*

We know that the theorem is true for space of three dimensions. We may therefore assume it to hold for $S_{n-2}$. It then follows for $S_n$. For, any arbitrary line being chosen, we have a determinate length along it representing the one-vector along it, and, corresponding to the other $(m-1)$ one-vectors, a fundamental $(n-1)$-pyramid of constant volume in a determinate $S_{n-2}$ (polar $S_{n-2}$), not intersecting the line. We have, therefore, a fundamental $(n+1)$-pyramid whose volume remains constant, however we choose the $(n-1)$ one-vectors, as long as one line of action remains unaltered. Now take any other arbitrary line. Its polar $S_{n-2}$ has an $S_{n-4}$ common with the first $S_{n-2}$. Take any line in this $S_{n-4}$. We may arrange that a one-vector should act along this line in each reduction. It follows that the volume of the pyramid is the same for the two reductions.

COROLLARY. *When n is even, and equal to* $2m$, *the m one-vectors to which a given general system can be reduced determine an n-pyramid which is fundamental in the* $S_{n-1}$ *containing it, and whose volume is constant.*

This last theorem puts us in a position to discuss the special cases in odd spaces. If the arbitrary line meets its polar $S_{n-2}$, the volume of the pyramid is zero, and the system can be reduced to one in space of $(n-2)$ dimensions, and therefore to $(m-1)$ one-vectors. Conversely, unless the volume of the pyramid is zero, the system cannot be reduced to $(m-1)$ one-vectors, or to a system in space of $(n-2)$ dimensions. Such a system we may call "once specialized." If the new fundamental $(n-1)$-pyramid collapse, we have a further reduction—a system twice specialized. In general, we may have a system specialized any number of times, up to $(m-1)$ times specialized, which would be a system equivalent to a single one-vector, or $m$ times specialized, which would be a system in equilibrium.

THEOREM IX. *The necessary and sufficient conditions that a system should be equivalent to a single one-vector may be expressed in a single statement. The one-vectors along the edges of every tetrahedron of the fundamental* $(n+1)$*-pyramid must be equivalent to a single one-vector.*

That the conditions are necessary follows from the manner in

which we replace a one-vector by one-vectors along the fundamental $(n+1)$-pyramid. We have, therefore, only to show that the conditions are sufficient. Consider the case $n = 4$. Evidently, if the system be not equivalent to a single one-vector, when the one-vectors in every $S_3$ of the 5-pyramid are equivalent to a single one-vector, the two one-vectors to which the system is equivalent must be such that one passes through any arbitrary vertex of the pyramid; that is, every vertex of the fundamental pyramid lies in the characteristic $S_3$ defined by the two one-vectors, which is, of course, impossible.

When $n = 5$, by what has just been proved, if the one-vectors in every $S_3$ of the pyramid are equivalent to a single one-vector, this is true for every $S_4$ of the pyramid. Hence, if the whole system be not equivalent to a single one-vector, we may replace the system by two one-vectors—one through any vertex of the fundamental pyramid, the other in the opposite $S_4$. But the $S_3$ defined by these two one-vectors cannot pass through all the vertices of the fundamental pyramid. Proceeding in this way we may obviously extend the theorem to $n$ dimensions.

*Absolute Line Coordinates.*—Let us choose definite directions along the edges of the fundamental $(n+1)$-pyramid as positive (viz., 12, 23, 31, 41, 42, 43, 51, 52, 53, 54, and so on). Now let us suppose a one-vector of unit magnitude to act in a definite sense along any straight line, and let us denote the ratio of the component one-vectors along any edge $\overline{rs}$ to the one-vector represented by that edge, attention being paid to sign, by $p_{rs}$. We have then, in the case when the system is equivalent to a single one-vector, a convenient system of absolute coordinates for the line of action including its sense. The construction of a line whose coordinates are thus defined follows by repeated application of the parallelogram law, and the same is true of the determination of the coordinates of a line whose position and sense are given. The identical relations satisfied by these $\frac{1}{2}(n+1)n$ coordinates, of which only $2(n-1)$ are independent, will be obtained by applying Theorem IX.

They are the well known equations*:—

$$p_{12}p_{34} + p_{23}p_{14} + p_{31}p_{24} = 0,$$
$$p_{12}p_{3r} + p_{23}p_{1r} + p_{31}p_{2r} = 0 \qquad (r = 4, 5, \ldots n),$$

* Each of these equations merely expresses the fact that the moment of the system constituted by the one-vectors acting along the edges of the corresponding tetrahedron about itself vanishes.

and similar equations for every four indices. There are evidently $\frac{1}{4!}(n+1)\,n\,(n-1)\,(n-2)$ such equations. Though necessary and sufficient, they are not all independent; there are syzygies between them. We have also a quadratic non-homogeneous equation, expressing the fact that the resultant of all the component one-vectors is of unit magnitude, viz.,

$$\Sigma\, a_{rs}^2 p_{rs}^2 + 2\Sigma\, a_{rs}a_{ij}\cos(rs, ij)\,p_{rs}p_{ij} = 1$$

with an obvious notation.

We may, of course, if we please, avoid this equation by using only the ratios of the $p$'s.

It will be noted that this method of defining line coordinates is independent of point or plane coordinates; and the equations between them are obtained without any analysis, being immediate consequences of the definition.

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## *An Essay towards the Generating Functions of Ternariants.*

*By* Professor A. R. Forsyth. Received March 17th, 1898. Communicated April 7th, 1898.

The present paper contains an attempt to apply the principle of the method devised by Cayley for the construction of generating functions for binary forms to the corresponding problem for ternary forms. In the case of ternariants, regarded as determined by their leading coefficients, there is the difficulty that the universal concomitant $u_x$ has unity for its leading coefficient, and that therefore no change is made in the leading coefficient of a concomitant on multiplying the concomitant by any power of $u_x$. It is necessary to take account of this consideration in discussing the problem from the side of the leading coefficients; accordingly, instead of dealing solely with the asyzygetic concomitants for the construction of a generating function, I have rejected those which are reducible by means of a power of $u_x$. Those which remain I have called a complete set of *hyposyzygetic* concomitants: in terms of $u_x$ and of the members of such a set, every

concomitant, multiplied possibly by some power of $u_x$, can be expressed as a rational integral function.

The investigation aims at the construction of a numerical generating function for a quantic: as the concomitants considered for this purpose are hyposyzygetic, the numerical function is called sub-generating. The simple case of the ternary quadratic is completely solved, the finite form of the numerical sub-generating function being obtained. In the case of the ternary cubic, the expression in finite form of the sub-generating function has not been obtained; the expansion, however, is carried well beyond the limit of the degrees to which, as is known, the asyzygetic concomitants belong. By means of the sub-generating function, it appears that, among the asyzygetic concomitants, there are two syzygies (non-linear, of course) of degree 6: both of these are obtained.

*Isobarism of Concomitants.*

**1.** The bipartite ternary quantic of order $n$ and class $\nu$ can be symbolically represented by $a_x^n u_a^\nu$, or by

$$(a_1x_1+a_2x_2+a_3x_3)^n\,(u_1a_1+u_2a_2+u_3a_3)^\nu,$$

where the coefficients $a$ and $\alpha$ are umbral. To secure the isobarism of the quantic and to establish the isobarism of its concomitants, we assign* weights as follows:—

| | |
|---|---|
| $\rho$ to $x_1$ and $\alpha_1$, | 0 to $u_1$ and $a_1$, |
| 1 to $x_2$ and $\alpha_2$, | $\rho-1$ to $u_2$ and $a_2$, |
| 0 to $x_3$ and $\alpha_3$, | $\rho$ to $u_3$ and $a_3$, |

$\rho$ being an arbitrary quantity. Using $a_{k,l,\kappa,\lambda}$ to denote the coefficient of

$$\frac{n!}{i!\,k!\,l!}\,\frac{\nu!}{\iota!\,\kappa!\,\lambda!}\,x_1^i x_2^k x_3^l u_1^\iota u_2^\kappa u_3^\lambda$$

in $a_x^n u_a^\nu$, where $i+k+l=n$ and $\iota+\kappa+\lambda=\nu$, we have

$$a_{k,l,\kappa,\lambda} = a_1^{n-k-l} a_2^k a_3^l a_1^{\nu-\kappa-\lambda} \alpha_2^\kappa \alpha_3^\lambda;$$

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* *American Journal of Mathematics*, Vol. XII. (1890), pp. 1–60, 115–160; in particular, p. 7. Other assignments, apparently more general, are possible; thus, we might assign weights $\sigma$, $\sigma+\rho-1$, $\sigma+\rho$ to $u_1$ and $a_1$, $u_2$ and $a_2$, $u_3$ and $a_3$, respectively; and weights $\tau$, $\tau-\rho+1$, $\tau-\rho$ to $\alpha_1$, $\alpha_2$, $\alpha_3$, respectively: where $\sigma$ and $\tau$ are arbitrary. The effect is to add to the weight in the text a weight $n\sigma+\nu(\tau-\rho)$ to each coefficient; no discriminating advantage is gained.

Many of the statements made in the following investigation are proved in the paper just quoted.

consequently the weight of $a_{k,l,\kappa,\lambda}$ is

$$k(\rho-1)+l\rho+(\nu-\kappa-\lambda)\rho+\kappa$$
$$=\nu\rho+(k-\kappa)(\rho-1)+(l-\lambda)\rho.$$

Let there be a mixed concomitant $\Phi$, of degree $r$ in the coefficients, of order $m$ in the point-variables $x$, and of class $p$ in the line-variables $u$; and let its leading term, taken to be that which involves the highest power of $x_1$ and the highest power of $u_1$, be

$$(\Sigma N a^{r_1}_{k_1 l_1 \kappa_1 \lambda_1} a^{r_2}_{k_2 l_2 \kappa_2 \lambda_2} \dots a^{r_s}_{k_s l_s \kappa_s \lambda_s}) x_1^m u_1^p,$$

where $N$ is numerical and changes from term to term in the coefficient of this leading term, and where

$$r_1+r_2+\dots+r_s=r.$$

It is known that, solely by means of direct differential operations, the concomitant is completely derivable from this leading term.

The weight of the leading term, say $w_1$, is

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(k_q-\kappa_q)(\rho-1)+(l_q-\lambda_q)\rho\right\}\right]+m\rho.$$

The concomitant $\Phi$ is covariantive for interchange of $x_2$ and $x_3$, which changes $a_{kl\kappa\lambda}$ into $a_{lk\lambda\kappa}$; consequently, the weight of the leading term is also

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(l_q-\lambda_q)(\rho-1)+(k_q-\kappa_q)\rho\right\}\right]+m\rho.$$

The concomitant is covariantive for interchange of $x_1$ and $x_2$, which changes $a_{k,l,\kappa,\lambda}$ into $a_{n-k-l,l,\nu-\kappa-\lambda,\lambda}$ and so changes the leading term into that which involves $x_2^m u_2^p$; consequently, as the concomitant is isobaric, the weight of this term is also $w_1$, and therefore

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(n-k_q-l_q-\nu+\kappa_q+\lambda_q)(\rho-1)+(l_q-\lambda_q)\rho\right\}\right]$$
$$+m+p(\rho-1).$$

Again, the concomitant is covariantive for interchange of $x_1$ and $x_3$; using the isobaric property as in the last case, we have

$$w_1=\Sigma\left[r_q\left\{\nu\rho+(k_q-\kappa_q)(\rho-1)+(n-k_q-l_q-\nu+\kappa_q+\lambda_q)\rho\right\}\right]+p\rho.$$

Adding the last three expressions obtained for $w_1$, we have

$$3w_1=\Sigma\left[r_q\left\{3\nu\rho+(n-\nu)(2\rho-1)\right\}\right]+m(1+\rho)+p(2\rho-1)$$
$$=3m\rho+3r\left\{\nu\rho+\tfrac{1}{3}(n-\nu)(2\rho-1)\right\}-(m-p)(2\rho-1).$$

Hence the weight of the literal coefficient, say $w$, of the leading term is

$$w=w_1-m\rho$$
$$=r\nu\rho+\tfrac{1}{3}(2\rho-1)\left\{r(n-\nu)-m+p\right\}.$$

Moreover, equating the first two expressions given for $w_1$, we have

$$\Sigma \{r_q (k_q - \kappa_q)\} = \Sigma \{r_q (l_q - \lambda_q)\}.$$

*Numerical Sub-Generating Functions for Unipartite Quantics.*

2. Taking more particularly the case of unipartite ternary quantics of order $n$, represented symbolically by $a_x^n$, we write

$$a_x^n = \Sigma\Sigma \frac{n!}{i!\,k!\,l!} x_1^i x_2^k x_3^l a_{kl},$$

where $i+k+l=n$. Then the weight of $a_{kl}$ is $k(\rho-1)+l\rho$. As before, let there be a concomitant $\Phi$ appertaining to $a_x^n$, of degree $r$ in the coefficients, of order $m$ in the point-variables $x$, and of class $p$ in the line-variables $u$; and let its leading term be

$$(\Sigma N a_{k_1 l_1}^{r_1} a_{k_2 l_2}^{r_2} \dots a_{k_s l_s}^{r_s}) x_1^m u_1^p,$$

where $N$ is numerical. Then the weight of the leading coefficient is

$$\tfrac{1}{3}(2\rho-1)(n\nu-m+p);$$

and also

$$\Sigma r_q (k_q - l_q) = 0.$$

Now, it is part of the general theory of ternariants that the leading coefficient of a concomitant—that is, the coefficient of the term which involves the highest power of $x_1$ and the highest power of $u_1$—satisfies the two partial differential equations

$$D_1 = 0, \quad D_0 = 0,$$

where

$$\left.\begin{aligned} D_1 &= \Sigma k a_{k-1,l+1} \frac{\partial a}{\partial a_{k,l}} \\ D_0 &= \Sigma l a_{k+1,l-1} \frac{\partial a}{\partial a_{k,l}} \end{aligned}\right\}.$$

These equations show that every such leading coefficient is a simultaneous invariant of the system of binary forms

$$(a_{0,0}\mathfrak{X} * )^0,$$

$$(a_{1,0}, a_{0,1}\mathfrak{X} * )^1,$$

$$(a_{2,0}, a_{1,1}, a_{0,2}\mathfrak{X} * )^2,$$

$$(a_{3,0}, a_{2,1}, a_{1,2}, a_{0,3}\mathfrak{X} * )^3,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots,$$

$$(a_{n,0}, a_{n-1,1}, a_{n-2,2}, \dots, a_{2,n-2}, a_{1,n-1}, a_{0,n}\mathfrak{X} * )^n.$$

Moreover, interchange of the binary variables (which need not be specified explicitly) changes a coefficient $a_{k,l}$ into $a_{l,k}$: hence it changes $D_0$ into $D_1$, and *vice versa*. We know from the theory of binariants that, instead of solving $D_1 = 0$ and $D_0 = 0$ as a simultaneous Jacobian system of partial differential equations, it is sufficient to take solutions of either of the equations, provided those solutions are of the weight proper to their degree; for, when they possess this appropriate weight, they satisfy the other of the equations.

3. Construct now a homogeneous function of degree $r$ in the foregoing coefficients, say

$$\Sigma N_{1,2,\dots} a_{k_1 l_1}^{r_1} a_{k_2 l_2}^{r_2} \dots a_{k_s l_s}^{r_s},$$

where $r_1 + r_2 + \dots + r_s = r$, retaining such combinations of degree $r$

as are of weight $\quad \frac{1}{3}(2\rho - 1)(nr - m + p),$

say of weight $\quad \frac{1}{3}(2\rho - 1)(nr - \alpha),$

where $\alpha$ is some integer, not necessarily positive. Then we have

$$\tfrac{1}{3}(2\rho-1)(nr-\alpha) = \Sigma r_i \{k_i(\rho-1) + l_i\rho\};$$

so that, as $\rho$ is arbitrary,

$$\tfrac{2}{3}(nr-\alpha) = \Sigma r_i (k_i + l_i),$$

$$\tfrac{1}{3}(nr-\alpha) = \Sigma r_i k_i;$$

and therefore $\quad \Sigma r_i k_i = \Sigma r_i l_i,$

and $\quad \frac{1}{3}(2\rho-1)(nr-\alpha) = \Sigma r_i \{l_i(\rho-1) + k_i\rho\},$

the last equation showing that, corresponding to any term, there exists in this general function the term obtained from it by interchanging the coefficients equidistant from the beginning and the end of any of the binary forms.

Let the function, thus constructed of the weight appropriate to the degree, be limited so as to satisfy one of the equations $D_1 = 0$, $D_0 = 0$; then, from what has been pointed out as regards binariants, it satisfies the other of the equations. Consequently, it appears that the imposition of the properties of homogeneity and isobarism upon the function enables us to consider only a single one of the two equations, the other of them (when that one is satisfied) being satisfied of itself.

For instance, in the case of the ternary cubic,

$$a_{00}(a_{30}a_{01}^2-2a_{21}a_{01}a_{10}+a_{12}a_{10}^2)$$

is a solution of $D_1=0$ but not of $D_6=0$; while

$$a_{30}a_{01}^3-3a_{21}a_{01}^2a_{10}+3a_{12}a_{01}a_{10}^2-a_{03}a_{10}^3$$

is a solution of $D_1=0$ and of $D_6=0$. Both functions are of degree 4; the weight of the former is

$$3(\rho-1)+2\rho=5\rho-3,$$

while the weight of the latter is

$$\begin{aligned}3(\rho-1)+3\rho &= 6\rho-3\\ &= \tfrac{1}{3}(2\rho-1)\{4.3-3\}.\end{aligned}$$

The latter is thus of a weight appropriate to the degree 4 in the coefficients of the ternary cubic; the former is not. In fact, the effective test is that *the weight should be divisible by* $2\rho-1$; the quotient, $3(nr-m+p)$, is known to be an integer because it is the index of the concomitant.

4. At this stage, it should be noticed that, so far as determinations are effected solely by the differential equations $D_1=0$ and $D_6=0$, the concomitant is determinate save as to a power of $u_x$, for the leading coefficient of a concomitant $\Theta$ is the same as that of $\Theta u_x^i$, where $i$ is any integer. Further, so far as determinations are effected solely by the weight of the leading coefficient, it is the integer $m-p$ which is determined; as the association of any power of $u_x$ with $\Theta$, or the removal of any power of $u_x$ from $\Theta$, makes the same change in $m$ and $p$, the integer $m-p$ is the same for $\Theta$ as for $\Theta u_x^i$. Consequently, if we limit ourselves to the two differential equations and to the consideration of the weight of the leading coefficient, the concomitant is determined save as to a power of $u_x$.

Suppose that a complete set of linearly independent solutions of the equations is known, that is, a set such that, in terms of its members, every other solution of the desired type can be expressed as a rational integral function. Each member of the set determines a concomitant, subject to a power of $u_x$; it will be assumed that such power is removed from the expression, so that the leading coefficient, taken solely in connexion with the two equations,* is associated with the

* The actual determination of $m$ and $p$ can be effected in other ways without ambiguity; in the present investigation, only the two equations $D_1=0$ and $D_6=0$ are being used.

simplest form of the concomitant. Now let any other solution of the equations of the desired type be given; it is expressible as a rational integral function of members of the preceding set, and therefore the corresponding concomitant in its simplest form is (save as to a possible power of $u_x$) expressible as a rational integral function of the members in the set of concomitants together with $u_x$. If the possible power of $u_x$ indicated be $u_x^0$, the corresponding concomitant is reducible, and would, therefore, be excluded from a complete asyzygetic system of the quantic. If the possible power of $u_x$ indicated be a positive power, the corresponding concomitant is irreducible in connexion with the set retained, but is reducible by a power of $u_x$; such a concomitant will be called *hyposyzygetic*. A system of concomitants of a quantic will be called a complete hyposyzygetic system of the quantic when every concomitant can be represented, except as to a power of $u_x$, as a rational integral function of $u_x$, and of the members of the system. In fact, if we use $u_x$ as a reducing factor so that, if $\Theta u_x^i$ is expressible in terms of other concomitants, we do not retain $\Theta$, then an asyzygetic system would be reduced, by means of $u_x$, to a hyposyzygetic system. Manifestly, a hyposyzygetic system is intermediate between an algebraically complete system and a complete asyzygetic system.

In order, therefore, to obtain a complete hyposyzygetic system, it is necessary to obtain a complete system of linearly independent solutions of the equations $D_1 = 0$, $D_6 = 0$.

5. Let $C_{n,r,\alpha}$ be the number of different products that can be formed from the coefficients

$$a_{k,l} \quad \begin{pmatrix} k, l = 0, 1, \ldots, n \\ k+l \leqslant n \end{pmatrix},$$

which are of degree $r$ in those coefficients, and are of weight $\frac{1}{3}(2\rho-1)(nr-\alpha)$; and, associating an arbitrary constant with each of these products, let the sum of all these quantities be forced to satisfy one or other of the equations $D_1 = 0$, $D_6 = 0$.

Consider, first, the effect of the operation $D_6$ upon the function thus constructed. It leaves the degree unaltered, and it decreases the weight by unity; so that, if $D_{n,r,\alpha}$ is the number of different products that can be formed from the coefficients $a_{k,l}$ which are of degree $r$ and of weight $\frac{1}{3}(2\rho-1)(nr-\alpha)-1$, there must be $D_{n,r,\alpha}$ relations among the arbitrary constants in order that the function may satisfy the equation $D_6 = 0$. Hence there are $C_{n,r,\alpha} - D_{n,r,\alpha}$ linearly

independent solutions of $D_0 = 0$ which are of degree $r$ and weight $\frac{1}{8}(2\rho-1)(nr-\alpha)$; and each of these is a solution of $D_1 = 0$.

Consider, secondly, the effect of the operation $D_1$ upon the function which has been constructed. It leaves the degree unaltered, and it increases the weight by unity; so that, if $E_{n,r,\alpha}$ is the number of different products that can be formed from the coefficients $a_{k,l}$ which are of degree $r$ and of weight $\frac{1}{8}(2\rho-1)(nr-\alpha)+1$, there must be $E_{n,r,\alpha}$ relations among the arbitrary constants in order that the function may satisfy the equation $D_1 = 0$. Hence there are

$$C_{n,r,\alpha} - E_{n,r,\alpha}$$

linearly independent solutions of $D_1 = 0$ which are of degree $r$ and weight $\frac{1}{8}(2\rho-1)(nr-\alpha)$; and each of these is a solution of $D_0 = 0$.

6. Since the weight of $a_{k,l}$ is

$$k(\rho-1)+l\rho,$$

the number of different products of degree $r$ and weight $w$ that can be formed from these quantities is the coefficient of $z^r x^w$ in the expansion of

$$\Pi\Pi \frac{1}{1-zx^{k(\rho-1)+l\rho}}$$

in ascending powers of $z$ and $x$, the double product being taken for values of $k$ and $l$, such that

$$k, l = 0, 1, \ldots, n,$$
$$k+l \leqslant n.$$

Let $$x^{\rho-1} = a, \quad x^{\rho} = b,$$

so that, as $\rho$ is arbitrary, $a$ and $b$ can be regarded as replacing $x$ and $\rho$; then

$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)} = (ab)^{\frac{1}{8}(nr-\alpha)},$$
$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)-1} = a^{\frac{1}{8}(nr-\alpha)+1} b^{\frac{1}{8}(nr-\alpha)-1},$$
$$x^{\frac{1}{8}(2\rho-1)(nr-\alpha)+1} = a^{\frac{1}{8}(nr-\alpha)-1} b^{\frac{1}{8}(nr-\alpha)+1}.$$

Hence $C_{n,r,\alpha}$ is the coefficient of $z^r a^{\frac{1}{8}(nr-\alpha)} b^{\frac{1}{8}(nr-\alpha)}$ in

$$\Pi\Pi \frac{1}{1-za^k b^l};$$

$D_{n,r,\alpha}$ is the coefficient of $z^r a^{\frac{1}{8}(nr-\alpha)+1} b^{\frac{1}{8}(nr-\alpha)-1}$ in the same expansion;

and $E_{n,r,a}$ is the coefficient of $z^r a^{\frac{1}{2}(nr-a)-1} b^{\frac{1}{2}(nr-a)+1}$ in the same expansion. Now

$$\begin{aligned}
&\Pi\Pi(1-za^kb^l)\\
&= (1-z)\\
&\quad\times(1-za)(1-zb)\\
&\quad\times(1-za^3)(1-zab)(1-zb^2)\\
&\quad\times \ldots\ldots\\
&\quad\vdots\\
&\quad\times(1-za^n)(1-za^{n-1}b)(1-za^{n-2}b^2)\ldots(1-za^2b^{n-2})(1-zab^{n-1})(1-zb^n),
\end{aligned}$$

which is symmetrical in $a$ and $b$; consequently $D_{n,r,a} = E_{n,r,a}$, and the two numbers $C_{n,r,a} - D_{n,r,a}$ and $C_{n,r,a} - E_{n,r,a}$ are the same, as might have been expected from their significance. Denoting each of them by $N_{n,r,a}$, and taking

$$\theta_n = \tfrac{1}{3}(nr-a),$$

we have $N_{n,r,a}$, the number of linearly independent solutions of $D_1 = 0$ and $D_6 = 0$, as equal to

$$\text{coefficient of } z^r a^{\theta_n} b^{\theta_n} - \text{coefficient of } z^r a^{\theta_n+1} b^{\theta_n-1},$$

in the expansion of $\{\Pi\Pi(1-za^kb^l)\}^{-1}$,

in ascending powers of $z$, $a$, $b$.

7. That the solutions are linearly independent, so that $N_{n,r,a}$ is their exact number, follows as in the case of binariants; in fact, we are dealing with the two equations characteristic of invariants of a set of binary forms, and it is known* that the process gives the exact number.

Further, we know that each solution, common to $D_1 = 0$ and $D_6 = 0$, which is homogeneous and isobaric in the coefficients, determines a concomitant. Hence $N_{n,r,a}$ *is the number of hyposyzygetic concomitants appertaining to a ternary quantic of order n which are of degree r in the coefficients, and for which the order m, the class p, and*

* See, in particular, Elliott, "Some Properties of Homogeneous Isobaric Functions," *Proc. Lond. Math. Soc.*, Vol. xxiv. (1893), pp. 21–36.

*the number* $\alpha$ *are connected by the relation*

$$m-p=\alpha.$$

This number may be denoted by*

$$N_{n,r,\alpha}\left\{\Pi\Pi\,(1-za^kb^l)\right\}^{-1}.$$

8. Before proceeding to the consideration of special cases, two remarks should be noted.

First, the leading coefficient, in and by itself (that is, through its degree and its weight), only determines the quantity $m-p$ of the corresponding concomitant $\Phi$; this is to be expected, because (as already remarked) the leading coefficient of $\Phi u_x^\rho$ is the same as that of $\Phi$. The discrimination can be made by the literal operators $D_2, D_4$; $D_3$, $D_5$; in the other differential equations satisfied by the concomitant; and therefore the leading coefficient, by means of the other equations, determines uniquely a concomitant.

Secondly, if $\Theta$ be a concomitant of degree $r$ having a value $\alpha$ of $\theta_n$, and, if $\Phi$ be a concomitant of degree $s$ having a value $\beta$ of $\theta_n$, then $\Theta\Phi$ is a concomitant of degree $r+s$ having a value $\alpha+\beta$ of $\theta_n$. This simple property is useful in the identification of concomitants of successive degrees.

*Application to the Ternary Quadratic.*

9. Taking the case of the ternary quadratic, so that $n=2$, and writing

$$\kappa=\tfrac{1}{3}(2r-\alpha),$$

it follows from the preceding investigation that the number of its hyposyzygetic concomitants, determined by degree $r$ and by $\alpha$, as the excess of order over class (not necessarily a positive excess), is

$$\text{coefficient of } z^r a^\kappa b^\kappa - \text{coefficient of } z^r a^{\kappa\mp1} b^{\kappa\pm1},$$

in the expansion of

$$(1-z\,.\,1-za\,.\,1-zb\,.\,1-za^2\,.\,1-zab\,.\,1-zb^2)^{-1},$$

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* In connexion with this result and the remark at the end of § 2, it is well to refer to the papers—

Sylvester and Franklin, *Amer. Jour.*, II. (1879), pp. 295, 296;
Franklin, ib., III. (1880), pp. 151–153;
Hammond, ib., VIII. (1886), pp. 138–155.

For these references, as well as for some suggestive criticism of this paper, I am indebted to Professor Elliott.

which is the same thing as

$$\text{co. } z^r a^\kappa b^\kappa \text{ in } \frac{1-\frac{b}{a}}{1-z\,.\,1-za\,.\,1-zb\,.\,1-za^2\,.\,1-zab\,.\,1-zb^2}$$

$$= \text{co. } z^r a^\kappa b^\kappa \text{ in } \frac{1}{1-z\,.\,1-zab}\,V,$$

say, where $$V = \frac{1-\frac{b}{a}}{1-za\,.\,1-zb\,.\,1-za^2\,.\,1-zb^2}.$$

For the present purpose, we need retain in $V$ only exact powers of $ab$; these can be selected as follows. We have

$$\frac{1-\frac{b}{a}}{1-za\,.\,1-zb} = \frac{1}{a}\left[a-b+z\,(a^2-b^2)+z^2\,(a^3-b^3)+z^3\,(a^4-b^4)+\ldots\right],$$

and

$$\frac{1}{1-za^2\,.\,1-zb^2} = \left[1+z\,(a^2+b^2)+z^2\,(a^4+a^2b^2+b^4)+\ldots\right];$$

taking the product of the two expressions in square brackets, we require terms of the form $z^s a^{\sigma+1} b^\sigma$. To select these, we take, in the expansion of $(1-za^2\,.\,1-zb^2)^{-1}$, the successive powers of $z$, and we write down the proper combinations of the various terms with the terms in the upper expression; the aggregate thus is

$$\begin{array}{l} \quad a \\ +z\,\{a^2\,(-b)+b^2\,(z^2a^3)\} \\ +z^2\{a^4\,(-z^2b^3)+a^2b^2\,(a)+b^4\,(z^4a^5)\} \\ +z^3\{a^6\,(-z^4b^5)+a^4b^2\,(-b)+a^2b^4\,(z^2a^3)+b^6\,(z^6a^7)\} \\ +z^4\{a^8\,(-z^6b^7)+a^6b^2\,(-z^2b^3)+a^4b^4\,(a)+a^2b^6\,(z^4a^5)+b^8\,(z^8a^9)\} \\ +\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}$$

Rearranging these, we have

$$a\,(1+z^3b^2a^2+z^6b^4a^4+z^9b^6a^6+\ldots),$$

by taking all the end terms beginning with the first line;

$$-azab\,(1+z^3a^2b^2+z^6a^4b^4+z^9a^6b^6+\ldots),$$

by taking all the first terms beginning with the second line;

$$+az^2a^2b^2\,(1+z^3b^2a^2+z^6b^4a^4+z^9b^6a^6+\ldots),$$

by taking the terms next the end, beginning with the second line ;

$$-az^3a^3b^3(1+z^3a^2b^2+z^6a^4b^4+z^9a^6b^6+\ldots),$$

by taking all the second terms, beginning with the third line; and so on. The sum therefore is

$$a(1-zab+z^2a^2b^2-z^3a^3b^3+\ldots)(1+z^3a^2b^2+z^6a^4b^4+z^9a^6b^6+\ldots)$$

$$=\frac{a}{(1+zab)(1-z^3a^2b^2)}.$$

Hence for our purpose $V$ effectively is

$$\frac{1}{(1+zab)(1-z^3a^2b^2)};$$

and therefore the number of hyposyzygetic concomitants of degree $r$ and order-class excess $\kappa$ belonging to the ternary quadratic is

$$\text{co. } z^r a^\kappa b^\kappa \text{ in } \frac{1}{(1-z)(1-zab)(1+zab)(1-z^3a^2b^2)}$$

$$= \text{co. } z^r a^\kappa b^\kappa \text{ in } \frac{1}{1-z\,.\,1-z^2a^2b^2\,.\,1-z^3a^2b^2}.$$

The corresponding concomitants are therefore three. First, $r=1$, $\kappa=0$, $\alpha=m-p=2$; using the other differential equations of the concomitants, we find $p=0$, $m=2$; the concomitant is the ternary quadratic $U$. Secondly, $r=2$, $\kappa=2$, $\alpha=m-p=-2$; using the other differential equations, we find $m=0$, $p=2$; the concomitant is the reciprocant $F$. Thirdly, $r=3$, $\kappa=2$, $\alpha=0$; using the other differential equations, we find $m=0$, $p=0$; the concomitant is the discriminant $\Delta$.

The complete hyposyzygetic system for the ternary quadratic is composed of $U$, $F$, $\Delta$; with which must be associated the universal concomitant $u_x$. This system happens to be co-extensive with the asyzygetic system for the quadratic.

10. Another method of proceeding is as follows. The required number of concomitants is the excess of the coefficient of $z^r a^\kappa b^\kappa$ (say $P_{r,\kappa}$) over the coefficient of $z^r a^{\kappa+1} b^{\kappa-1}$ (say $Q_{r,\kappa}$) in the expansion of

$$\left\{(1-z)(1-za)(1-zb)(1-za^2)(1-zab)(1-zb^2)\right\}^{-1}.$$

Then $P_{r,\kappa}$ is the number of distinct sets of solutions of the equations

$$\left.\begin{aligned} p_1+p_2+p_3+\ p_4+p_5+\ p_6 &= r \\ p_2 \quad +2p_4+p_5 \quad\ &= \kappa \\ p_3 \quad\ +p_5+2p_6 &= \kappa \end{aligned}\right\};$$

and $Q_{r,\kappa}$ is the number of distinct sets of solutions of the equations

$$\left.\begin{aligned} q_1+q_2+q_3+\ q_4+q_5+\ q_6 &= r \\ q_2 \quad\ +2q_4+q_5 \quad\ &= \kappa+1 \\ q_3 \quad\ +q_5+2q_6 &= \kappa-1 \end{aligned}\right\}.$$

Moreover, we have $2\kappa \leqslant nr$

in general, so that in this case we have

$$\kappa \leqslant r;$$

and $\kappa, = \frac{1}{3}(2r-\alpha)$, cannot be negative; so that in taking the coefficient of $z^r$, the values of $\kappa$ to be considered are 0, 1, ..., $r$.

When $r=0$, $\kappa=0$; then $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is $u_x$.

When $r=1$, $\kappa=0$ or 1. When $\kappa=0$, then $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is $U$. When $\kappa=1$, then $P_{r,\kappa}=1$, $Q_{r,\kappa}=1$; there is no concomitant.

When $r=2$, $\kappa=0, 1, 2$. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is $U^2$. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=1$; the single concomitant is $F$.

When $r=3$, $\kappa=0, 1, 2, 3$. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is $U^3$. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=4$, $Q_{r,\kappa}=2$; the two concomitants are $UF$, $\Delta$. When $\kappa=3$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant.

When $r=4$, $\kappa=0, 1, 2, 3, 4$. When $\kappa=0$, $P_{r,\kappa}=1$, $Q_{r,\kappa}=0$; the single concomitant is $U^4$. When $\kappa=1$, $P_{r,\kappa}=2$, $Q_{r,\kappa}=2$; there is no concomitant. When $\kappa=2$, $P_{r,\kappa}=6$, $Q_{r,\kappa}=4$; the two concomitants are $U^2F$, $U\Delta$. When $\kappa=3$, $P_{r,\kappa}=6$, $Q_{r,\kappa}=6$; there is no concomitant. When $\kappa=4$, $P_{r,\kappa}=3$, $Q_{r,\kappa}=2$; the single concomitant is $F^2$.

It thus appears that, in the fourth degree, there is no new irreducible concomitant; and there is thus suggested the inference that the complete hyposyzygetic system of the ternary quadratic is made up of $u_x$, $U$, $F$, $\Delta$.

11. The preceding methods, effective for the quadratic, do not appear to be effective for the cubic without considerable labour; the following method, however, is effective for the quadratic and promises to be possible for the cubic.

Let $ab = u^2$, $\frac{b}{a} = v^2$, so that $a = \frac{u}{v}$, $b = uv$. Then

$$V = \frac{1-v^2}{1-z\frac{u}{v}\,.\,1-zuv\,.\,1-z\frac{u^2}{v^2}\,.\,1-zu^2v^2}.$$

Now in $V$ the terms required are of the form $z^i(ab)^\kappa$, that is, of the form $z^i u^{2\kappa}$; in other words, the aggregate of terms independent of $v$ in the expansion of $V$ is required. This aggregate manifestly is given by those terms of

$$\Sigma z^t u^{p+q+2r+2s} v^{-p+q-2r+2s} - \Sigma z^t u^{p+q+2r+2s} v^{2-p+q-2r+2s}$$

which are free from $v$; the index $t$ being $p+q+r+s$. For appropriate terms from the first sum, we have

$$p+2r = q+2s,$$
$$t = p+q+r+s$$
$$= 2q-r+3s;$$

they lead to a sum $\Sigma_1$, where

$$\Sigma_1 = \Sigma\Sigma\Sigma z^{2q-r+3s} u^{2q+4s},$$

the summation extending over values of $q$ and $s$, and of values of $r$ consistent with

$$p+2r = q+2s.$$

The terms in the second set lead similarly to a sum $\Sigma_2$, where

$$\Sigma_2 = \Sigma\Sigma\Sigma z^{2q-r+3s+2} u^{2q+4s+2}.$$

the summation extending over values of $q$ and $s$, and of values of $r$ consistent with

$$p+2r = q+2s+2.$$

The full aggregate of terms then is

$$\Sigma_1 - \Sigma_2.$$

The possible indices of $u$, of the forms $2q+4s$ and $2q+4s+2$, can be arranged in two sets, those evenly even and those oddly even.

Consider, first, those evenly even, in particular, the terms giving $u^{4m}$. In $\Sigma_1$, we therefore take

$$2q+4s = 4m,$$

so that

$$q = 2\theta, \quad s = m-\theta,$$

and $\theta$ ranges from 0 to $m$. Also

$$p+2r = q+2s = 2m,$$

so that $r$ ranges from 0 to $m$; and

$$2q+3s = 3m+\theta.$$

Hence, in $\Sigma_1$, the coefficient of $u^{4m}$ is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}.$$

In $\Sigma_2$, we take $2q+4s+2 = 4m,$

so that $q = 2\theta-1, \quad s = m-\theta,$

and $\theta$ ranges from 1 to $m$. Also

$$p+2r = q+2s+2 = 2m+1,$$

so that $r$ ranges from 0 to $m$; and

$$2q+3s+2 = 3m+\theta.$$

Hence, in $\Sigma_2$, the coefficient of $u^{4m}$ is

$$\sum_{\theta=1}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}.$$

Consequently, in $\Sigma_1-\Sigma_2$, the coefficient of $u^{4m}$ is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta-r} - \sum_{\theta=1}^{m} \sum_{r=0}^{m} z^{3m+\theta-r}$$

$$= \sum_{r=0}^{m} z^{3m-r}$$

$$= \sum_{\theta=0}^{m} z^{3m-\theta} = \sum_{\theta=0}^{m} z^{q+3s},$$

and $2q+4s = 4m$; so that the aggregate of the evenly even terms can be represented in the form

$$\Sigma\,(zu^2)^q\,(z^3u^4)^s,$$

where $q$ is even for this aggregate.

Consider, secondly, the terms oddly even, in particular, the terms giving $u^{4m+2}$. In $\Sigma_1$, we therefore take

$$2q+4s = 4m+2,$$

so that $q = 2\theta+1, \quad s = m-\theta,$

and $\theta$ ranges from 0 to $m$. Also

$$p+2r = q+2s = 2m+1,$$

so that $r$ ranges from 0 to $m$; and

$$2q+3s = 3m+\theta+2.$$

Hence, in $\Sigma_1$, the coefficient of $u^{4m+2}$ is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta+2-r}.$$

In $\Sigma_2$, we take $2q+4s+2 = 4m+2,$

so that $q = 2\vartheta, \quad s = m-\theta,$

and $\theta$ ranges from 0 to $m$. Also

$$p+2r = q+2s+2 = 2m+2,$$

so that $r$ ranges from 0 to $m+1$ ; and

$$2q+3s+2 = 3m+\theta+2.$$

Hence, in $\Sigma_2$, the coefficient of $u^{4m+2}$ is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m+1} z^{3m+\theta+2-r}.$$

Consequently, in $\Sigma_1-\Sigma_2$, the coefficient of $u^{4m+2}$ is

$$\sum_{\theta=0}^{m} \sum_{r=0}^{m} z^{3m+\theta+2-r} - \sum_{\theta=0}^{m} \sum_{r=0}^{m+1} z^{3m+\theta+2-r}$$

$$= -\sum_{\theta=0}^{m} z^{2m+\theta+1}$$

$$= -\sum_{\theta=0}^{m} z^{3m-\theta+1} = -\sum_{\theta=0}^{m} z^{q+3s},$$

and $2q+4s = 4m+2$ ; so that the aggregate of the oddly even terms can be represented in the form

$$-\Sigma\,(zu^2)^q\,(z^3u^4)^s,$$

where $q$ is odd for this aggregate.

Hence the total aggregate of terms required is

$$\sum_{q \text{ even}} (zu^2)^q\,(z^3u^4)^s - \sum_{q \text{ odd}} (zu^2)^q\,(z^3u^4)^s$$

$$= \Sigma\,(-zu^2)^q\,(z^3u^4)^s,$$

for all values of $q$ and $s$. The initial values for the even sum are $q = 0,\ s = 0$ ; and for the odd sum are $q = 1,\ s = 0$ ; hence this aggregate is

$$\sum_{q=0}^{\infty} (-zu^2)^q \sum_{s=0}^{\infty} (z^3u^4)^s$$

$$= \frac{1}{1+zu^2 \,.\, 1-z^3u^4},$$

and therefore the part of $V$ required is

$$\frac{1}{1+zu^2 \,.\, 1-z^3u^4}$$

$$= \frac{1}{1+zab \,.\, 1-z^3a^2b^2}.$$

This result leads to the same sub-generating function for the quadratic as before.

*Application to Ternary Cubic.*

12. We now consider the unipartite ternary cubic. Here $n = 3$, so that, writing

$$\kappa = \tfrac{1}{3}(3r-m+p)$$

$$= r-\tfrac{1}{3}(m-p),$$

we require a number represented by

$$\text{coefficient of } z^r a^\kappa b^\kappa - \text{coefficient of } z^r a^{\kappa-1} b^{\kappa+1},$$

in the expansion of $\Delta^{-1}$, where

$$\begin{aligned}\Delta = {} & (1-z) \\ & \times (1-za)(1-zb) \\ & \times (1-za^2)(1-zab)(1-zb^2) \\ & \times (1-za^3)(1-za^2b)(1-zab^2)(1-zb^3)\,;\end{aligned}$$

or, what is the same thing, the coefficient of $z^r a^\kappa b^\kappa$ in the expansion of

$$\frac{1-\dfrac{b}{a}}{\Delta}.$$

Write $\dfrac{b}{a} = v^2$, $ab = u^2$; then $z^r a^\kappa b^\kappa = z^r u^{2\kappa}$, and therefore what is required is the aggregate of terms independent of $v$ in the expansion of

$$\frac{1}{1-z \,.\, 1-zu^2}\, V,$$

where

$$V = \frac{1-v^2}{1-z\dfrac{u}{v} \,.\, 1-zuv \,.\, 1-z\dfrac{u^2}{v^2} \,.\, 1-zu^2v^2 \,.\, 1-z\dfrac{u^3}{v^3} \,.\, 1-z\dfrac{u^3}{v} \,.\, 1-zu^3v \,.\, 1-zu^3v^3}.$$

This aggregate manifestly is given by the terms of

$$\Sigma z^t u^{\lambda+\mu+2(\pi+\rho)+3(\alpha+\beta+\gamma+\delta)} v^{-\lambda+\mu+2(-\pi+\rho)+(-3\alpha-\beta+\gamma+3\delta)}$$

$$-\Sigma z^t u^{\lambda+\mu+2(\pi+\rho)+3(\alpha+\beta+\gamma+\delta)} v^{2-\lambda+\mu+2(-\pi+\rho)+(-3\alpha-\beta+\gamma+3\delta)}$$

which are free from $v$; and the index of $z$ is

$$t = \lambda+\mu+\pi+\rho+\alpha+\beta+\gamma+\delta.$$

For the appropriate terms of the first sum, we have

$$\lambda+2\pi+3\alpha+\beta = \mu+2\rho+3\delta+\gamma;$$

and then $$t = 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha = T,$$

say, and the index of $u$ is

$$2(\mu+2\rho+\beta+2\gamma+3\delta) = 2\Theta,$$

say; so that the appropriate terms of the first sum are

$$\Sigma_1 = \Sigma z^T u^{2\Theta},$$

the summation extending over values of $\mu$, $\rho$, $\beta$, $\gamma$, $\delta$, and over values of $\lambda$, $\pi$, $\alpha$, such that

$$\lambda+2\pi+3\alpha+2\beta+\gamma = \mu+2\rho+\beta+2\gamma+3\delta = \Theta,$$

that is, values of $\lambda$, $\pi$, $\alpha$ such that

$$\lambda+2\pi+3\alpha = \Theta-2\beta-\gamma.$$

The terms in the second set lead similarly to an aggregate

$$\Sigma_2 = \Sigma z^{T+2} u^{2\Theta+2},$$

with the same symbolic signification of $T$ and $\Theta$; the summation extends over values of $\mu$, $\rho$, $\beta$, $\gamma$, $\delta$, and over values of $\lambda$, $\pi$, $\alpha$, such that

$$\lambda+2\pi+3\alpha = \Theta+2-2\beta-\gamma.$$

The full aggregate of the terms is consequently

$$\Sigma_1 z^T u^{2\Theta} - \Sigma_2 z^{T+2} u^{2\Theta+2},$$

where

$$\Theta = \mu+2\rho+\beta+2\gamma+3\delta,$$

$$T = 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha;$$

the summation in both cases extends over values of $\mu$, $\rho$, $\beta$, $\gamma$, $\delta$; and further over values of $\pi$ and $\alpha$ which, in the first sum, are consistent with

$$\lambda+2\pi+3\alpha = \Theta-2\beta-\gamma,$$

and, in the second sum, are consistent with

$$\lambda+2\pi+3\alpha = \Theta+2-2\beta-\gamma.$$

13. It will be convenient to describe the terms in $\Sigma_1$ as the positive terms, and those in $\Sigma_2$ as the negative terms.

The positive terms that involve $u^{2\Phi}$ are given by values of the integers such that

$$\left.\begin{aligned}\mu+2\rho+3\delta &= \Phi- \beta-2\gamma\\ \lambda+2\pi+3\alpha &= \Phi-2\beta- \gamma\end{aligned}\right\};$$

so that sets of values of $\mu, \rho, \delta$; $\lambda, \pi, \alpha$; are obtainable for particular values of $\beta$ and $\gamma$. For each set of simultaneous values, the index of $z$ is $T$, where

$$\begin{aligned}T &= 2\mu+3\rho+2\gamma+4\delta-\pi-2\alpha\\ &= \lambda+\mu+\pi+\rho+\alpha+\delta+\beta+\gamma.\end{aligned}$$

And the range of values of $\beta$ and $\gamma$ giving rise to possible terms is determined by the relations

$$\left.\begin{aligned}\beta+2\gamma &\leqslant \Phi\\ 2\beta+ \gamma &\leqslant \Phi\end{aligned}\right\}.$$

The negative terms that involve $u^{2\Phi}$ are given by values of the integers such that

$$\left.\begin{aligned}\mu'+2\rho'+3\delta' &= \Phi-1- \beta'-2\gamma'\\ \lambda'+2\pi'+3\alpha' &= \Phi+1-2\beta'- \gamma'\end{aligned}\right\};$$

the range of values of $\beta'$ and $\gamma'$ is determined by the relations

$$\left.\begin{aligned}\beta'+2\gamma' &\leqslant \Phi-1\\ 2\beta'+ \gamma' &\leqslant \Phi+1\end{aligned}\right\};$$

and the index of $z$ for any set of simultaneous values is $T'$, where

$$T' = \lambda'+\mu'+\pi'+\rho'+\alpha'+\delta'+\beta'+\gamma'.$$

Now we have

$$\left.\begin{aligned}(\beta'-1)+2(\gamma'+1) &\leqslant \Phi\\ 2(\beta'-1)+ (\gamma'+1) &\leqslant \Phi\end{aligned}\right\},$$

so that we may take, in general,

$$\beta'-1 = \beta, \quad \gamma'+1 = \gamma,$$

leading to

$$\beta'+\gamma' = \beta+\gamma;$$

and then

$$\begin{aligned}\mu'+2\rho'+3\delta' &= \Phi-(\beta'-1)-2(\gamma'+1)\\ &= \Phi-\beta-2\gamma,\\ \lambda'+2\pi'+3\alpha' &= \Phi-2(\beta'-1)-(\gamma'+1)\\ &= \Phi-2\beta-\gamma,\end{aligned}$$

so that, for all the cases in which the change from $\beta'$ and $\gamma'$ to $\beta$ and $\gamma$ is possible, we may take

$$\mu' = \mu, \quad \rho' = \rho, \quad \delta' = \delta,$$
$$\lambda' = \lambda, \quad \pi' = \pi, \quad \alpha' = \alpha.$$

In each such case, $$T' = T;$$

and, consequently, for each such case, the positive and the negative term cancel each other.

There therefore survive only such terms in which the change from $\beta'$ and $\gamma'$ to $\beta$ and $\gamma$ is not possible. Of these there are two sets, viz., those given by $\gamma = 0$ in the positive series, and those given by $\beta' = 0$ in the negative series.

When $\gamma = 0$, the range of values of $\beta$ is from 0 to $[\frac{1}{2}\Phi]$, where $[m]$ denotes the positive integral part of $m$. For these, we have

$$T = \lambda+\mu+\pi+\rho+\alpha+\delta+\beta$$
$$= 2\Phi-2\beta-\pi-2\alpha-\rho-2\delta;$$

the values of $\pi$, $\alpha$, $\rho$, $\delta$ are such that

$$\left.\begin{aligned} 2\rho+3\delta &\leqslant \Phi-\beta \\ 2\pi+3\alpha &\leqslant \Phi-2\beta \end{aligned}\right\};$$

and thus the aggregate of the positive terms in the coefficient of $u^{2\Phi}$ is

$$\sum_{\beta=0}^{[\frac{1}{2}\Phi]} \sum_{\pi} \sum_{\alpha} \sum_{\rho} \sum_{\delta} z^T.$$

When $\beta' = 0$, the range of values of $\gamma'$ is from 0 to $[\frac{1}{2}(\Phi-1)]$. For these we have

$$T' = \lambda'+\mu+\pi'+\rho'+\alpha'+\delta'+\gamma'$$
$$= 2\Phi-2\gamma'-\pi'-2\alpha'-\rho'-2\delta';$$

the values of $\pi'$, $\alpha'$, $\rho'$, $\delta'$ are such that

$$\left.\begin{aligned} 2\pi'+3\alpha' &\leqslant \Phi+1-\gamma' \\ 2\rho'+3\delta' &\leqslant \Phi-1-2\gamma' \end{aligned}\right\};$$

and thus the aggregate of the negative terms in the coefficient of $u^{2\Phi}$ is

$$-\sum_{\gamma'=0}^{[\frac{1}{2}(\Phi-1)]} \sum_{\pi'} \sum_{\alpha'} \sum_{\rho'} \sum_{\delta'} z^{T'}.$$

As the range of $\gamma'$ is the same as that of $\beta$, except only for the value $\beta = \frac{1}{2}\Phi$ when $\Phi$ is an even integer, we may write $\gamma' = \beta$; for

the value $2\gamma' = \Phi$ admits of no solution of

$$2\rho' + 3\delta' \leqslant \Phi - 1 - 2\gamma',$$

and, consequently, no terms corresponding to this value are thus introduced into the negative aggregate. Also, for purposes of comparison, we write

$$\pi' = \rho_1, \quad \alpha' = \delta_1, \quad \rho' = \pi_1, \quad \delta' = \alpha_1;$$

so that the negative terms arise through values such that

$$\left.\begin{aligned} 2\rho_1 + 3\delta_1 &\leqslant \Phi + 1 - \beta \\ 2\pi_1 + 3\alpha_1 &\leqslant \Phi - 1 - 2\beta \end{aligned}\right\},$$

and $T'$ is the same function of $\rho_1, \delta_1, \pi_1, \alpha_1$ as $T$ is of $\rho, \delta, \pi, \alpha$.

Now the tableau of equations giving rise to positive terms is

$$\begin{array}{llll} 2\rho + 3\delta = \Phi, \Phi-1, \Phi-2, \Phi-3, & & 2\pi + 3\alpha = \Phi, \Phi-1, \Phi-2, \Phi-3, \Phi-4, & \\ \Phi-1, \Phi-2, \Phi-3, & & \Phi-2, \Phi-3, \Phi-4, & \\ \Phi-2, \Phi-3, & & \Phi-4, & \\ \vdots & & \vdots & \end{array}$$

and the corresponding tableau for negative terms is

$$\begin{array}{llll} 2\rho_1 + 3\delta_1 = \Phi+1, \Phi, \Phi-1, \Phi-2, & & 2\pi_1 + 3\alpha_1 = \Phi-1, \Phi-2, \Phi-3, \Phi-4, \Phi-5, & \\ \Phi, \Phi-1, \Phi-2, & & \Phi-3, \Phi-4, \Phi-5, & \\ \Phi-1, \Phi-2, & & \Phi-5, & \\ \vdots & & \vdots & \end{array}$$

When $2\rho + 3\delta$ and $2\rho_1 + 3\delta_1$ have the same value, and $2\pi + 3\alpha$ and $2\pi_1 + 3\alpha_1$ have the same value, the various forms of $T$ are the same in number and value as the various forms of $T'$; and these corresponding terms cancel. When account is taken of all the sets that thus cancel, it appears that the surviving positive terms are given by

$$2\pi + 3\alpha = \Phi - 2\beta, \quad 2\rho + 3\delta \leqslant \Phi - \beta,$$

and the surviving negative terms by

$$2_1\pi + 3\alpha_1 \leqslant \Phi - 1 - 2\beta, \quad 2\rho_1 + 3\delta_1 = \Phi + 1 - \beta.$$

Some further removal of cancelling terms is possible, but the resulting conditions are more complicated, and do not prove so easy of application.

Accordingly, the required expansion of the function can be expressed in the form $\sum_{\Phi=0} u^{2\Phi}\{\Sigma z^T - \Sigma' z^{T'}\}$

where $$T = 2\Phi - 2\beta - \pi - 2\alpha - \rho - 2\delta,$$

the summation of $z^T$ being for values of $\pi$, $\alpha$, $\rho$, $\delta$ such that

$$2\pi + 3\alpha = \Phi - 2\beta, \quad 2\rho + 3\delta \leqslant \Phi - \beta;$$

and $$T' = 2\Phi - 2\beta - \pi_1 - 2\alpha_1 - \rho_1 - 2\delta_1,$$

the summation of $z^{T'}$ being for values of $\pi_1$, $\alpha_1$, $\rho_1$, $\delta_1$ such that

$$2\pi_1 + 3\alpha_1 \leqslant \Phi - 1 - 2\beta, \quad 2\rho_1 + 3\delta_1 = \Phi + 1 - \beta.$$

14. It has been established by Gordan* that the ternary cubic has 34 concomitants in its asyzygetic system, including itself and $u_x$; and that these concomitants are of various degrees from 0 (that of $u_x$) up to 12. It is thus of importance to possess the expansion of the preceding function, certainly complete as far as $z^{12}$ inclusive; as a matter of fact, I have calculated it as far as $z^{20}$ inclusive. Denoting this part by $V_1$, I find that the expression for $V_1$ is that given in Table I. (p. 509): the column down the page gives the powers of $z$, and the row across the page gives the powers of $u$; thus the coefficient of $z^6$ is

$$u^8 + 4u^{12} - u^{14} + 2u^{16},$$

and so for others.

The calculations proved rather long, but otherwise not difficult; they were effected as follows. It is easy to prove that, for any value of $\Phi$, the lowest power of $z$ that can occur in the coefficient of $u^{2\Phi}$ is $[\frac{1}{3}(2\Phi+2)]$; in order, therefore, to have the complete coefficient of $z^{20}$, it is sufficient to consider the values $\Phi = 0, 1, \ldots, 30$.

In the first place, a subsidiary table was formed of the values of $x+2y$ for values of $x$ and $y$ such that $2x+3y \leqslant m$, where $m$ has the values 0, 1, ..., 31; and in each entry the values of $x+2y$ corresponding to $2x+3y = m$ were underlined. For instance, consider

$$2x + 3y \leqslant 9;$$

we have $$y = 3;\ 2\ ;\ 1\ ;\ 0$$

$$x = \underset{*}{0};\ 1, 0;\ \underset{*}{3}, 2, 1, 0;\ 4, 3, 2, 1, 0;$$

giving $$x+2y = \underset{*}{6};\ 5, 4;\ \underset{*}{5}, 4, 3, 2;\ 4, 3, 2, 1, 0.$$

* *Math Ann.*, Bd. I. (1869), pp. 90-128.

TABLE I. ($V_1$).

| | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 | 52 | 54 | 56 | 58 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 1 | .. | −1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 2 | ... | ... | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 3 | ... | ... | 1 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 4 | ... | ... | ... | ... | 2 | −1 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 5 | ... | ... | ... | ... | 1 | −1 | 3 | −1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | ... | ... | ... | ... | 1 | 0 | 4 | −1 | 2 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | ... | ... | ... | ... | ... | ... | 3 | −1 | 4 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | ... | ... | ... | ... | ... | ... | 2 | 0 | 5 | 0 | 4 | −1 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 9 | ... | ... | ... | ... | ... | ... | 1 | 0 | 4 | 0 | 6 | −1 | 4 | −1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | ... | ... | ... | ... | ... | ... | ... | ... | 3 | 1 | 7 | −1 | 8 | −1 | 2 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 1 | 6 | 0 | 10 | −1 | 6 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 12 | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 1 | 5 | 1 | 11 | 0 | 10 | 1 | 5 | −1 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 13 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 3 | 1 | 9 | 0 | 13 | 1 | 9 | −1 | 4 | −1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 14 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 1 | 8 | 2 | 13 | 2 | 14 | 0 | 10 | −1 | 2 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | ... | ... | ... | ... | ... | ... | ... | ... | ... | .. | 1 | 1 | 6 | 2 | 12 | 3 | 16 | 3 | 16 | 0 | 7 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 16 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 4 | 2 | 10 | 4 | 17 | 5 | 20 | 1 | 13 | 1 | 5 | −1 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 17 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 1 | 8 | 4 | 15 | 8 | 22 | 2 | 19 | 3 | 11 | −1 | 4 | −1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 18 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 1 | 6 | 4 | 14 | 9 | 23 | 4 | 24 | 5 | 21 | 1 | 11 | −1 | 2 | ⋮ | ⋮ | ⋮ | ⋮ |
| 19 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 4 | 3 | 11 | 9 | 21 | 5 | 26 | 5 | 25 | 3 | 19 | 0 | 7 | 0 | 1 | ⋮ | ⋮ |
| 20 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 2 | 9 | 8 | 19 | 7 | 27 | 8 | 30 | 5 | 28 | 3 | 15 | 1 | 5 | −1 | 1 |

The entry was made in the form

$$x+2y = \underline{6, 5^2}, 4^3, 3^2, 2^2, 1, 0.$$

When the values for $2x+3y \leqslant 10$

are required, they are the preceding set of values, together with those for which

$$2x+3y = 10,$$

and the result is $x+2y = \underline{6^3, 5^3}, 4^3, 3^2, 2^2, 1, 0.$

The values of $\Phi$ were then taken in succession; for each such value of $\Phi$, positive terms are given by $\beta = 0$ to $[\frac{1}{2}\Phi]$, and negative terms by $\beta = 0$ to $[\frac{1}{2}(\Phi-1)]$. The values of $\pi+2\alpha$ and $\rho+2\delta$, such that

$$2\pi+3\alpha = \Phi-2\beta, \quad 2\rho+3\delta \leqslant \Phi-\beta,$$

for each of the values of $\beta$ in succession, were extracted from the subsidiary table, and the corresponding values of $T$ were obtained, all values greater than 20 being neglected. Similarly, for the negative terms. The quantity

$$\Sigma z^T - \Sigma z^{T'}$$

was then formed; this gave the expression for the coefficient of $u^{2\Phi}$. This was done for each of the values of $\Phi$ in succession, and the final result was arranged in powers of $z$, in the form given above.

15. The function which gives the number of hyposyzygetic concomitants of any assigned degree, and any assigned difference of order and class, by means of the coefficients of the appropriate power of $z$ and power of $u$, is

$$\frac{V_1}{(1-z)(1-zu^2)}.$$

It is convenient to multiply numerator and denominator by $1+zu^2$; and, when the new form of the function is denoted by

$$\frac{N}{(1-z)(1-z^2u^4)},$$

the expression for $N$ is given in Table II. (p. 511), the terms in $z^1$ and $z^2$ having disappeared. I have not, however, succeeded in reducing it to such terms that it will exhibit, in finite form, the expression for this sub-generating function of the ternary cubic. Various simplifications are possible, but they are not given, as they do not appear to be necessarily in the direct line leading to the function.

TABLE II. (*N*).

| | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 50 | 52 | 54 | 56 | 58 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 3 | ... | ... | 1 | 1 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 4 | ... | ... | ... | 1 | 2 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 5 | ... | ... | ... | ... | 1 | 1 | 2 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 6 | ... | ... | ... | ... | 1 | 1 | 3 | 2 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 7 | ... | ... | ... | ... | ... | 1 | 3 | 3 | 3 | 2 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 8 | ... | ... | ... | ... | ... | ... | 2 | 3 | 4 | 4 | 4 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 9 | ... | ... | ... | ... | ... | ... | 1 | 2 | 4 | 5 | 6 | 3 | 3 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | ... | ... | ... | ... | ... | ... | ... | 1 | 3 | 5 | 7 | 5 | 7 | 3 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 11 | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 4 | 7 | 7 | 9 | 7 | 5 | 2 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 12 | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 3 | 6 | 7 | 11 | 10 | 9 | 7 | 5 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 13 | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 4 | 6 | 10 | 11 | 13 | 11 | 10 | 4 | 3 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 14 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 4 | 9 | 11 | 13 | 15 | 15 | 9 | 9 | 3 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 3 | 7 | 10 | 14 | 16 | 18 | 17 | 16 | 10 | 6 | 2 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 16 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 5 | 8 | 12 | 16 | 20 | 21 | 23 | 17 | 13 | 8 | 5 | 0 | 1 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 17 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 5 | 10 | 14 | 19 | 25 | 27 | 22 | 20 | 16 | 12 | 4 | 3 | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 18 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 3 | 7 | 12 | 18 | 24 | 31 | 26 | 26 | 24 | 24 | 12 | 10 | 3 | 1 | ⋮ | ⋮ | ⋮ | ⋮ |
| 19 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 1 | 5 | 9 | 15 | 23 | 30 | 28 | 30 | 29 | 30 | 24 | 20 | 11 | 6 | 2 | 1 | ⋮ | ⋮ |
| 20 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | 2 | 6 | 12 | 19 | 28 | 28 | 32 | 34 | 35 | 30 | 31 | 22 | 15 | 8 | 5 | 0 | 1 |

16. As regards the 34 asyzygetic concomitants, their explicit expressions, when the cubic is taken in its canonical form, are given by Cayley.* Adopting his symbols, the combinations of the variables $z$ and $u$ corresponding to the respective concomitants are as follows:—

$$z = U;$$

$$z^2u^4 = \Theta;$$

$$z^3u^4 = H \;:\; z^3u^6 = B \;:\; z^3u^8 = P;$$

$$z^4u^6 = J \;:\; z^4u^8 = S, \Theta' : z^4u^{12} = F;$$

$$z^5u^8 = K \;:\; z^5u^{10} = B' \;:\; z^5u^{12} = Q, \overline{K};$$

$$z^6u^{10} = E \;:\; z^6u^{12} = T, \Theta'' : z^6u^{14} = \overline{E};$$

$$z^7u^{12} = K' \;:\; z^7u^{14} = B'' \;:\; z^7u^{16} = \overline{K}' : z^7u^{18} = \overline{M};$$

$$z^8u^{12} = \Psi \;:\; z^8u^{14} = E' \;:\; z^8u^{18} = \overline{J}, \overline{E}';$$

$$z^9u^{14} = M \;:\; z^9u^{18} = B''' \;:\; z^9u^{22} = \overline{M}';$$

$$z^{10}u^{18} = E'' : z^{10}u^{22} = \overline{E}'';$$

$$z^{11}u^{18} = M';$$

$$z^{12}u^{18} = \Omega \;:\; z^{12}u^{30} = \Pi;$$

unity corresponds to $\Lambda$, which represents $u_1x_1 + u_2x_2 + u_3x_3$. If, therefore, corresponding to each form $z^ru^{2\kappa}$, we construct an expression $1 - z^ru^{2\kappa}$, take the product of all these, and expand its reciprocal in powers of $z$ and $u$, then the coefficient of $z^{r'}u^{2\kappa'}$ is the number of concomitants, say $f(r', k')$, determined by the numbers $r'$ and $k'$. When full expansion of the sub-generating function is effected, let $f_1(r', k')$ be the coefficient of $z^{r'}u^{2\kappa'}$, which accordingly is the number of hyposyzygetic concomitants determined by the numbers $r'$ and $k'$; consequently

$$f(r', k') - f_1(r', k')$$

is the number of syzygies among the concomitants, of degree $r'$ and having their difference of order and class determined by $k'$, which can be constructed out of the complete asyzygetic system of the cubic.

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* *American Journal of Mathematics*, Vol. IV. (1881), pp. 1–15; *Collected Mathematical Papers*, Vol. XI., pp. 342–356.

Retaining only terms involving powers of $z$ not higher than the sixth, the preceding constructed function gives rise to an expression

$$\frac{L}{(1-z)(1-z^2u^4)},$$

where $L = (1+z^3u^4+z^6u^8+\ldots)(1+z^3u^6+z^6u^{12}+\ldots)(1+z^3u^8+z^6u^{16}+\ldots)$
$(1+z^4u^6+\ldots)(1+2z^4u^8+\ldots)(1+z^4u^{12}+\ldots)$
$(1+z^5u^8+\ldots)(1+z^5u^{10}+\ldots)(1+2z^5u^{12}+\ldots)$
$(1+z^6u^{10}+\ldots)(1+2z^6u^{12}+\ldots)(1+z^6u^{14}+\ldots)$

$$\begin{aligned}= 1 &+ z^3(u^4+u^6+u^8)\\ &+ z^4(u^6+2u^8+u^{12})\\ &+ z^5(u^8+u^{10}+2u^{12})\\ &+ z^6(u^8+2u^{10}+4u^{12}+2u^{14}+u^{16})+\ldots.\end{aligned}$$

Now the sub-generating function is

$$\frac{N}{(1-z)(1-z^2u^4)},$$

where $$\begin{aligned}N = 1 &+ z^3(u^4+u^6+u^8)\\ &+ z^4(u^6+2u^8+u^{12})\\ &+ z^5(u^8+u^{10}+2u^{12})\\ &+ z^6(u^8+u^{10}+3u^{12}+2u^{14}+u^{16})+\ldots.\end{aligned}$$

We therefore infer that there is one syzygy among the concomitants determined by the type $z^6u^{10}$, and that there is one syzygy among those determined by the type $z^6u^{12}$.

17. To obtain the former of these syzygies, we construct the concomitants determined by the type $z^6u^{10}$; they are

$E$, of class 2, order 5,
$UB'$, „ „ 3, „ 6,
$\Theta J$, „ „ 3, „ 6,
$HB$, „ „ 3, „ 6,
$U\Theta B$, „ „ 5, „ 8;

and there is therefore a linear relation, with only numerical

coefficients, between

$$E\Lambda^3,\ UB'\Lambda^2,\ \Theta J\Lambda^2,\ HB\Lambda^2,\ U\Theta B.$$

This linear relation cannot involve $U\Theta B$; for every other term involves $\Lambda^2$, and the existence of such a relation would then imply the reducibility of $\Theta$ or $B$ by the factor $\Lambda$, which is known not to be possible. The factor $\Lambda^2$ may therefore be removed; and the linear relation has the form

$$E\Lambda = \alpha UB' + \beta\Theta J + \gamma HB,$$

where $\alpha$, $\beta$, $\gamma$ are numerical. The coefficients of the highest power of $u_1$, viz., of $u_1^3$, must be the same on the two sides; and therefore

$$\begin{aligned} x_1(abc+8l^3)(bx_2^3-cx_3^3)(2l^2x_1^2+bcx_2x_3) \\ = \quad & \alpha U 3l^2bc\,(cx_3^3-bx_2^3) \\ & +\beta\,(bcx_2x_3-l^2x_1^2)(abc+8l^3)\,x_1(bx-cx_3^3) \\ & +\gamma Hbc\,(cx_3^3-bx_2^3). \end{aligned}$$

In order that this may be identically satisfied, we must have

$$\alpha = -1, \quad \beta = -2, \quad \gamma = 3;$$

and therefore the syzygy, of degree 6, and excess of order over class equal to 3, is

$$E\Lambda + UB' + 2\Theta J = 3HB.$$

18. To obtain the other of the specified syzygies, we construct the concomitants determined by the type $z^6u^{12}$; they are

$T$, of class 0, order 0,
$\Theta''$, ,, ,, 2, ,, 2,
$UQ$, ,, ,, 3, ,, 3,
$U\widetilde{K}$, ,, ,, 4, ,, 4,
$\Theta S$, ,, ,, 2, ,, 2,
$\Theta\Theta'$, ,, ,, 4, ,, 4,
$U^2F$, ,, ,, 6, ,, 6,
$\Pi P$, ,, ,, 3, ,, 3,
$E^2$, ,, ,, 6, ,, 6,
$U\Theta P$, ,, ,, 5, ,, 5,
$\Theta^3$, ,, ,, 6, ,, 6;

and therefore the syzygy must have the form

$$\begin{aligned}\alpha\Theta^3+\beta B^2+\gamma U^2F &= \delta U\Theta P\Lambda\\ &\quad+(\epsilon U\bar{K}+Z\Theta\Theta')\,\Lambda^2\\ &\quad+(\eta UQ+\theta HP)\,\Lambda^3\\ &\quad+(\iota\Theta''+\kappa S\Theta)\,\Lambda^4+\lambda T\Lambda^6,\end{aligned}$$

where $\alpha$, $\beta$, ..., $\lambda$ are numerical.

The coefficients of the highest power of $u_1$, viz., of $u_1^6$, must be the same on the two sides. The coefficient on the right-hand side is manifestly divisible by $x_1$; and accordingly

$$\alpha\,(bcx_2x_3-l^2x_1^2)^3+\beta b^2c^2\,(cx_3^3-bx_2^3)^2+\gamma b^2c^2U^2,$$

which is the coefficient on the left-hand side, must be divisible by $x_1$. This requires the relations

$$\alpha=4\beta,\quad \gamma=-\beta;$$

and the coefficient on the left-hand side then is

$$\begin{aligned}&4\beta\,(bcx_2x_3-l^2x_1^2)^3+\beta b^2c^2\left\{(cx_3^3-bx_2^3)^2-U^2\right\}\\ =&-x_1\,12b^2c^2lx_2x_3\,(bx_2^3+cx_3^3)\,\beta\\ &-x_1^2\,48b^2c^2l^2x_2^2x_3^2\beta\\ &-x_1^3\,2ab^2c^2\,(bx_2^3+cx_3^3)\,\beta\\ &+x_1^4\,12bcl\,(l^3-abc)\,x_2x_3\beta\\ &-x_1^6\,(4l^6+a^2b^2c^2)\,\beta.\end{aligned}$$

Taking the coefficient of $u_1^6$ on the right-hand side, arranging it similarly in powers of $x_1$, and equating the terms that involve the respective powers of $x_1$, we have relations as follows.

From the terms involving $x_1^1$,

$$\delta bcx_2x_3\,(-lbc)(bx_2^3+cx_3^3)=-12b^2c^2lx_2x_3\,(bx_2^3+cx_3^3)\,\beta;$$

and therefore

$$\delta=12\beta.$$

From the terms involving $x_1^2$,

$$\delta\,6lx_2x_3bcx_2x_3\,(-lbc)+\zeta\,6b^2c^2l^2x_2^2x_3^2=-48b^2c^2l^2x_2^2x_3^2\beta,$$

so that

$$-6\delta+6\zeta=-48\beta;$$

and therefore

$$\zeta=4\beta.$$

From the terms involving $x_1^3$,

$$\begin{aligned}&\delta l^3 bc\,(bx_2^3+cx_3^3)+\epsilon\,(abc+8l^3)\,bc\,(bx_2^3+cx_3^3)\\&\qquad+\eta\,(abc-10l^3)\,bc\,(bx_2^3+cx_3^3)-\theta lbc\,l^2\,(bx_2^3+cx_3^3)\\&\qquad=-2ab^2c^2\,(bx_2^3+cx_3^3)\,\beta,\end{aligned}$$

so that $$\delta l^3+\epsilon\,(abc+8l^3)+\eta\,(abc-10l^3)-\theta l^3=-2\beta abc,$$

and therefore
$$\epsilon+\eta=-2\beta,$$
$$\delta+8\epsilon-10\eta-\theta=0,$$

that is,
$$\left.\begin{aligned}\epsilon+\eta&=-2\beta\\ \theta-8\epsilon+10\eta&=+12\beta\end{aligned}\right\}.$$

From the terms involving $x_1^4$,

$$\begin{aligned}&\delta\,(-alb^2c^2+l^3bc\,6l)\,x_2x_3+\epsilon\,(abc+8l^3)\,bc\,6lx_2x_3\\&\qquad+\zeta\left\{lbc\,(abc+2l^3)-6bcl^4\right\}x_2x_3+\eta\,6lx_2x_3\,bc\,(abc-10l^3)\\&\qquad+\theta\,lbc\,(abc+2l^3)\,x_2x_3+\iota\,36bcl^4x_2x_3+\kappa Sbcx_2x_3\\&\quad=\ 12bcl\,(l^3-abc)\,x_2x_3\beta,\end{aligned}$$

so that, as $S=l\,(l^3-abc)$, we have

$$\begin{aligned}&\delta\,(6l^3-abc)+6\epsilon\,(abc+8l^3)+\zeta\,(abc-4l^3)\\&\qquad+6\eta\,(abc-10l^3)+\theta\,(abc+2l^3)+36\iota l^3+\kappa\,(l^3-abc)\\&\qquad=12\beta\,(l^3-abc),\end{aligned}$$

and therefore $$12\beta=\ 6\delta+48\epsilon-4\zeta-60\eta+2\theta+36\iota+\kappa,$$
$$-12\beta=-\delta+\ 6\epsilon+\ \zeta+\ 6\eta+\ \theta-\kappa,$$

which, by means of the preceding results, are easily reduced to

$$\theta-\kappa=8\beta,$$
$$36\iota+8\theta+\kappa=28\beta,$$

and, by combining these, also to the simpler form

$$\left.\begin{aligned}\theta+4\iota&=4\beta\\ \theta-\kappa&=8\beta\end{aligned}\right\}.$$

Lastly, from the terms involving $x_1^6$,

$$\begin{aligned}&\delta l^3 abc+\epsilon\,abc\,(abc+8l^3)-\zeta l^3\,(abc+2l^3)+\eta\,abc\,(abc-10l^3)-\theta\,abc\,l^3\\&\qquad-\iota\,(abc+2l^3)^2-\kappa l^2\,(l^4-abc\,l)+\lambda\,(a^2b^2c^2-20abcl^3-8l^6)\\&\qquad\qquad=-\beta\,(4l^6+a^2b^2c^2),\end{aligned}$$

and therefore
$$\left.\begin{aligned}-2\zeta-4\iota-\kappa-\ 8\lambda&=-4\beta\\ \delta+8\epsilon-\zeta-10\eta-\theta-4\iota+\kappa-20\lambda&=0\\ \epsilon+\eta-\iota+\lambda&=-\beta\end{aligned}\right\}.$$

Using the other results obtained, these give

$$\left.\begin{aligned} 4\beta+4\iota+\kappa &= -8\lambda \\ -4\beta-4\iota+\kappa &= \ 20\lambda \\ \beta+1 \quad &= \quad \lambda \end{aligned}\right\},$$

whence $$\lambda = 0, \quad \kappa = 0, \quad \beta+\iota = 0,$$

that is, $$\iota = -\beta.$$

Further, from the other relations, we have

$$\theta = \ 8\beta,$$
$$\epsilon = -\tfrac{4}{3}\beta,$$
$$\eta = -\tfrac{2}{3}\beta;$$

and therefore the syzygy, of degree 6 and of order equal to class, among the concomitants of the ternary cubic, is

$$4\Theta^3+B^2-U^2F = 12U\Theta P\Lambda+(4\Theta\Theta'-\tfrac{4}{3}UK)\Lambda^2+(8HP-\tfrac{2}{3}UQ)\Lambda^3-\Theta''\Lambda^4.$$

The two syzygies thus obtained are of degree 6; all others are of higher degree, and the simplest of them can be obtained in a similar manner. But, before the aggregate of syzygies can be obtained, it seems necessary to possess the expression for the sub-generating function in finite form.

*Note on Bipartite Quantics.*

19. A similar investigation applies to the biternary quantic of order $n$ in the point-variables and class $\nu$ in the line-variables, and the result is to show that the number of hyposyzygetic concomitants of degree $r$, with their order and class connected by the relation

$$m-p = a,$$

is equal to

$$\text{coefficient of } z^r a^{\phi_n} b^{\phi_n} - \text{coefficient of } z^r a^{\phi_n+1} b^{\phi_n-1}$$

in the expansion of $$\frac{1}{\Pi\Pi\Pi\Pi\,(1-za^{k-\kappa}b^{l-\lambda})},$$

the product being taken over the terms corresponding to

$$k, l = 0, 1, \ldots, n, \quad k+l \leqslant n,$$
$$\kappa, \lambda = 0, 1, \ldots, \nu, \quad \kappa+\lambda \leqslant \nu;$$

and the number $\phi_n$ is given by

$$\phi_n = \tfrac{1}{3}\{(n-\nu)\,r-m+p\}.$$

*Thursday, May 12th,* 1898.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Thirteen members present and a visitor.

The provisional action of the officers and members (as noted under April 7th) was ratified.

Miss M. E. Barwell and Mr. W. F. Sheppard were admitted into the Society.

Messrs. William Edward Philip, B.A., Fellow of Clare College, Cambridge; Thomas John I'Anson Bromwich, B.A., Fellow of St. John's College, Cambridge; and John Gaston Leathem, M.A., Fellow of St. John's College, Cambridge, and Isaac Newton Student, were elected members.

Mr. Love communicated a few Notes on some fundamental properties of Manifolds. Mr. Macaulay joined in a discussion of the communication.

The President read abstracts of the following papers:—"On the Numerical Value of $\int_0^h e^{x^2}\,dx$," by Mr. H. G. Dawson; and by Prof. Lamb, "On the Reflection and Transmission of Electric Waves by a Metallic Grating."

Lt.-Col. Cunningham gave an account of "The 77-Squares Puzzle" constructed by Major W. H. Turton, R.E., and exhibited a cardboard specimen of the puzzle. A vote of thanks was unanimously passed to Major Turton for the communication and for the present of the puzzle.* Lt.-Col. Cunningham also communicated the discovery of several high primes of 9, 10, and 11 figures.

Impromptu communications were made by the President—a remark in connexion with the figure of Pascal's theorem—and by Mr. Hargreaves.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LXII., Nos. 388–392.

"Mathematical Questions with their Solutions from the 'Educational Times,'" Vol. LXVIII.

"Journal of the Institute of Actuaries," Vol. XXXIV., Pt. 1, No. 189; April, 1898.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg, Bd. III., Heft 7, 1897, Heft 8, 1898; Leipzig.

"Nyt Tidsskrift for Matematik," A, Aarg. 8, Nr. 7–8; B, Aarg. 9, Nr. 1, 1898; Copenhagen, 1897.

* For a description see Ball's *Mathematical Recreations and Problems* (Third Edition).

"Monatshefte für Mathematik und Physik," Jahrg. IX., Pt. 2; Vienna, 1898.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Série 2, Tome I., Livr. 4–5; La Haye, 1898.

"Bulletin of the American Mathematical Society," Series 2, Vol. IV., No. 7; New York, April, 1898.

"Science Abstracts," Vol. I., Pt. 3; London, March, 1898.

"Bulletin des Sciences Mathématiques," Tome XXII., Mars, Av.; Paris, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. IV., Fasc. 3–4; Napoli, 1898.

"Bulletin de la Société Mathématique de France," Tome XXVI., Nos. 1, 2; Paris, 1898.

"The John Crerar Library, Third Annual Report, for 1897"; Chicago, 1898.

"Annali di Matematica," Serie 3, Tomo I., Fasc. 1; Milano, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 6, 7, 8; Roma, 1898.

"Educational Times," May, 1898.

"Transactions of the Cambridge Philosophical Society," Vol. XVI., Pt. 4; 1898.

"Indian Engineering," Vol. XXIII., Nos. 11–16, March 12–April 16, 1898.

"Transactions of the Royal Irish Academy," Vol. XXXI., Pts. 1–6; Dublin, Nov., 1896–March, 1898.

"International Catalogue of Scientific Literature." Report of the Committee of the Royal Society of London, with Schedules of Classification.

"The 77-Squares Puzzle," presented by the Author, Major W. H. Turton, R.E.

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## *On the Numerical Value of* $\int_0^h e^{x^2} dx$. *By* H. G. Dawson. Received April 30th, 1898. Communicated May 12th, 1898.

Starting from the two known equations

$$e^{-x^2} + e^{-(x-a)^2} + e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} + \&c.$$
$$= \frac{\sqrt{\pi}}{a}\left(1 + 2e^{-(\pi^2/a^2)} \cos\frac{2\pi x}{a} + 2e^{-(4\pi^2/a^2)} \cos\frac{4\pi x}{a} + \&c.\right),$$

$$e^{-x^2} - e^{-(x-a)^2} - e^{-(x+a)^2} + e^{-(x-2a)^2} + e^{-(x+2a)^2} - \&c.$$
$$= \frac{2\sqrt{\pi}}{a}\left(e^{-(\pi^2/4a^2)} \cos\frac{\pi x}{a} + e^{-(9\pi^2/a^2)} \cos\frac{3\pi x}{a} + \&c.\right),$$

we can, by putting $a = 1$, and making a few obvious transformations, deduce the two equations

$$e^{-x^2}\phi(x) = \sqrt{\pi}\left(1 + \frac{2\cos 2\pi x}{e^{\pi^2}} + \frac{2\cos 4\pi x}{e^{4\pi^2}} + \frac{2\cos 6\pi x}{e^{9\pi^2}} + \&c.\right),$$

$$e^{-x^2}\psi(x) = \sqrt{\pi}\, e^{\frac{1}{4}}\left(1 - \frac{2\cos 2\pi x}{e^{\pi^2}} + \frac{2\cos 4\pi x}{e^{4\pi^2}} - \frac{2\cos 6\pi x}{e^{9\pi^2}} + \&c.\right),$$

where $\phi(x) \equiv 1 + \frac{e^{2x}+e^{-2x}}{e} + \frac{e^{4x}+e^{-4x}}{e^4} + \ldots + \frac{e^{2mx}+e^{-2mx}}{e^{m^2}} + \ldots,$

and $\psi(x) = e^x + e^{-x} + \frac{e^{3x}+e^{-3x}}{e^2} + \ldots + \frac{e^{(2m+1)x}+e^{-(2m+1)x}}{e^{m(m+1)}} + \ldots .$

Approximately we have

$$\sqrt{\pi}\, e^{x^2}\left(1 + \frac{2\cos 2\pi x}{e^{\pi^2}}\right) = \phi_1(x),$$

$$e^{\frac{1}{4}}\sqrt{\pi}\, e^{x^2}\left(1 - \frac{2\cos 2\pi x}{e^{\pi^2}}\right) = \psi_1(x);$$

whence $2e^{\frac{1}{4}}\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \left\{e^{\frac{1}{4}}\phi_1(x) + \psi_1(x)\right\} dx \quad (h>2);$

therefore $2\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \left\{\phi_1(x) + e^{-\frac{1}{4}}\psi_1(x)\right\} dx,$

$$2\sqrt{\pi}\int_0^h e^{x^2}dx = \int_0^h \phi_1(x)\,dx + e^{-\frac{1}{4}}\int_0^h \psi_1(x)\,dx.$$

Now $\int_0^h \phi_1(x)\,dx = h + \frac{e^{2h}-e^{-2h}}{2e} + \frac{e^{4h}-e^{-4h}}{4e^4} + \frac{e^{6h}-e^{-6h}}{6e^9} + \ldots \equiv \phi_2(h),$

$$\int_0^h \psi_1(x)\,dx = \frac{e^h - e^{-h}}{1} + \frac{e^{3h}-e^{-3h}}{3e^2} + \frac{e^{5h}-e^{-5h}}{5e^6} + \ldots \equiv \psi_2(h).$$

Now, by aid of the tables given by Professor Newman in *Cambridge Philosophical Transactions*, Vol. XIII., supplemented by those of Dr. Glaisher, which are printed immediately after Professor Newman's, it was found, in general, possible to write down the values of $\phi_2(h)$, $\chi_2(h)$.

When it became necessary to calculate positive powers of $e$ whose index was greater than 2, Dr. Glaisher's Table II. was no longer available, and then use was made of Guderman's *Theorie der Potenzial Functionen*, Tafel II., in conjunction with Vega's *Thesaurus Logarithmorum* (Leipzig, 1794).

The original terms of the series $\phi_2(h)$, $\chi_2(h)$ were taken to eight places of decimals.

By this means the values of $2\sqrt{\pi}\int_0^h e^{x^2}dx$ were found to eight places for values (of $h$) which increased from 0 to 2 by intervals of magnitude ·01.

The values of $\int_0^h e^{x^2}dx$ were calculated by aid of an ordinary seven-place set of logarithm tables.

In this calculation we have neglected $\int_0^h e^{x^2}\frac{\cos 4\pi x}{e^{4\pi^2}}dx$ and smaller quantities; the value of

$$\int_0^h e^{x^2}\frac{\cos 4\pi x}{e^{4\pi^2}}dx < \frac{1}{4\pi}e^{-4(\pi^2-1)} < \frac{1}{12}e^{-34} < 2\times 10^{-16}.$$

I append tables of the values of $\int_0^h e^{x^2}dx$.

| $h$ | $\int_0^h e^{x^2}dx$ | $h$ | $\int_0^h e^{x^2}dx$ | $h$ | $\int_0^h e^{x^2}dx$ |
|---|---|---|---|---|---|
| ·01 | ·010000 | ·37 | ·387601 | ·73 | ·883332 |
| ·02 | ·020003 | ·38 | ·399111 | ·74 | ·900496 |
| ·03 | ·030008 | ·39 | ·410709 | ·75 | ·917916 |
| ·04 | ·040021 | ·40 | ·422397 | ·76 | ·935600 |
| ·05 | ·050042 | ·41 | ·434180 | ·77 | ·953553 |
| ·06 | ·060073 | ·42 | ·446060 | ·78 | ·971785 |
| ·07 | ·070114 | ·43 | ·458040 | ·79 | ·990305 |
| ·08 | ·080171 | ·44 | ·470122 | ·80 | 1·009120 |
| ·09 | ·090244 | ·45 | ·482313 | ·81 | 1·028239 |
| ·10 | ·100334 | ·46 | ·494614 | ·82 | 1·047669 |
| ·11 | ·110445 | ·47 | ·507027 | ·83 | 1·067419 |
| ·12 | ·120578 | ·48 | ·519558 | ·84 | 1·087500 |
| ·13 | ·130736 | ·49 | ·532209 | ·85 | 1·107922 |
| ·14 | ·140920 | ·50 | ·544987 | ·86 | 1·128696 |
| ·15 | ·151132 | ·51 | ·557892 | ·87 | 1·149828 |
| ·16 | ·161376 | ·52 | ·570930 | ·88 | 1·171331 |
| ·17 | ·171652 | ·53 | ·584104 | ·89 | 1·193216 |
| ·18 | ·181963 | ·54 | ·597418 | ·90 | 1·215497 |
| ·19 | ·192311 | ·55 | ·610876 | ·91 | 1·238180 |
| ·20 | ·202699 | ·56 | ·624484 | ·92 | 1·261280 |
| ·21 | ·213128 | ·57 | ·638244 | ·93 | 1·284809 |
| ·22 | ·223602 | ·58 | ·652163 | ·94 | 1·308782 |
| ·23 | ·234121 | ·59 | ·666243 | ·95 | 1·333207 |
| ·24 | ·244689 | ·60 | ·680492 | ·96 | 1·358100 |
| ·25 | ·255307 | ·61 | ·694912 | ·97 | 1·383476 |
| ·26 | ·265979 | ·62 | ·709509 | ·98 | 1·409349 |
| ·27 | ·276707 | ·63 | ·724289 | ·99 | 1·435735 |
| ·28 | ·287493 | ·64 | ·739255 | 1·00 | 1·462652 |
| ·29 | ·298339 | ·65 | ·754415 | 1·01 | 1·490109 |
| ·30 | ·309248 | ·66 | ·769772 | 1·02 | 1 518127 |
| ·31 | ·320223 | ·67 | ·785335 | 1·03 | 1·546721 |
| ·32 | ·331266 | ·68 | ·801106 | 1·04 | 1·575912 |
| ·33 | ·342381 | ·69 | ·817094 | 1·05 | 1·605716 |
| ·34 | ·353568 | ·70 | ·833303 | 1·06 | 1·636153 |
| ·35 | ·364833 | ·71 | ·849742 | 1·07 | 1·667240 |
| ·36 | ·376176 | ·72 | ·866415 | 1·08 | 1·699002 |

| $h$ | $\int_0^h e^{x^2}dx$ | $h$ | $\int_0^h e^{x^2}dx$ | $h$ | $\int_0^h e^{x^2}dx$ |
|---|---|---|---|---|---|
| 1·09 | 1·731457 | 1·40 | 3·240894 | 1·71 | 6·886604 |
| 1·10 | 1·764627 | 1·41 | 3·312892 | 1·72 | 7·076003 |
| 1·11 | 1·798534 | 1·42 | 3·386950 | 1·73 | 7·272034 |
| 1·12 | 1·833201 | 1·43 | 3·463141 | 1·74 | 7·474966 |
| 1·13 | 1·868655 | 1·44 | 3·541542 | 1·75 | 7·685085 |
| 1·14 | 1·904921 | 1·45 | 3·622235 | 1·76 | 7·902688 |
| 1·15 | 1·942021 | 1·46 | 3·705302 | 1·77 | 8·128087 |
| 1·16 | 1·979985 | 1·47 | 3·790831 | 1·78 | 8·361608 |
| 1·17 | 2·018840 | 1·48 | 3·878912 | 1·79 | 8·603593 |
| 1·18 | 2·058614 | 1·49 | 3·969639 | 1·80 | 8·854398 |
| 1·19 | 2·099341 | 1·50 | 4·063110 | 1·81 | 9·114396 |
| 1·20 | 2·141046 | 1·51 | 4·159428 | 1·82 | 9·383977 |
| 1·21 | 2·183766 | 1·52 | 4·258700 | 1·83 | 9·663552 |
| 1·22 | 2·227532 | 1·53 | 4·361035 | 1·84 | 9·953550 |
| 1·23 | 2·272378 | 1·54 | 4·466551 | 1·85 | 10·254406 |
| 1·24 | 2·318340 | 1·55 | 4·575367 | 1·86 | 10·566612 |
| 1·25 | 2·365457 | 1·56 | 4·687609 | 1·87 | 10·890680 |
| 1·26 | 2·413770 | 1·57 | 4·803408 | 1·88 | 11·227050 |
| 1·27 | 2·463311 | 1·58 | 4·922901 | 1·89 | 11·576332 |
| 1·28 | 2·514128 | 1·59 | 5·046230 | 1·90 | 11·939070 |
| 1·29 | 2·566264 | 1·60 | 5·173545 | 1·91 | 12·315857 |
| 1·30 | 2·619762 | 1·61 | 5·304999 | 1·92 | 12·703175 |
| 1·31 | 2·674669 | 1·62 | 5·440755 | 1·93 | 13·114102 |
| 1·32 | 2·731034 | 1·63 | 5·580982 | 1·94 | 13·536895 |
| 1·33 | 2·788907 | 1·64 | 5·725857 | 1·95 | 13·976416 |
| 1·34 | 2·848341 | 1·65 | 5·875565 | 1·96 | 14·433417 |
| 1·35 | 2·909391 | 1·66 | 6·030289 | 1·97 | 14·908699 |
| 1·36 | 2·972109 | 1·67 | 6·190240 | 1·98 | 15·403060 |
| 1·37 | 3·036555 | 1·68 | 6·355621 | 1·99 | 15·917402 |
| 1·38 | 3·102793 | 1·69 | 6·526656 | 2·00 | 16·45263 |
| 1·39 | 3·170884 | 1·70 | 6·703571 | | |

The foregoing table is true to six figures; two tests can be used to try its accuracy; the first is as follows: if $u_h$ denote $\int_0^h e^{x^2}dx$, then

$$\Delta^2 u_h = 2h\delta\,(1+h\delta)\,\Delta' u_h.$$

The second is $\quad \Delta u_h = e^{h^2}\left\{1+h\delta+\tfrac{1}{3}(1+2h^2)\,\delta^2\right\}\delta,$

where $\delta = \cdot 01$, $\quad \Delta u_h = u_{(h+\delta)} - u_h, \quad \Delta' u_h = u_h - u_{(h-\delta)}.$

## *On the Reflection and Transmission of Electric Waves by a Metallic Grating.*

*By* Prof. Horace Lamb, F.R.S. Received May 11th, 1898. Read May 12th, 1898.

The main problem of this paper consists in the calculation of the disturbance produced in a train of electric waves by a plane grating composed of parallel, equal, and equidistant metallic strips. The treatment is approximate, and involves the assumption that the wave-length is large compared with the distance between the centres of consecutive strips; the application is, therefore, rather to Hertzian waves than to phenomena of ordinary Optics. The previous mathematical literature bearing directly on the subject consists of an investigation by Prof. J. J. Thomson,* and of two papers by Lord Rayleigh.† It will be seen that the author is under great obligations to each of these sources.

The work of §§ 1, 2, 3 is preliminary, and consists in the solution of several electrostatical problems. With modern methods these present no difficulty; but the results appeared to possess sufficient interest of their own to warrant the expenditure of a little trouble in the way of graphical representation. The diagrams obtained serve, moreover, to illustrate very clearly some of the conclusions arrived at afterwards in the discussion of the wave problem.

The problem of § 4 is introduced on account of its close connexion with what precedes, although it has no bearing on the main subject of the paper.

1. Let it be required to find the disturbance produced in a uniform electric field by a grating (of the kind above described) placed with its plane perpendicular to the lines of force. We will suppose that the axis of $x$ is normal to the plane of the grating, and the axis of $y$ in this plane, at right angles to the lengths of the strips.

In the case of a field symmetrical on the two sides of the grating, it will be found that (with a special choice of units) the potential- and stream-functions are given by the formula

$$\cosh w = \mu \cosh z, \qquad (1)‡$$

* *Recent Researches on Electricity and Magnetism*, § 359. The problem attacked is that of reflection by a grating of cylindrical wires. Some notes on this question are included in the present paper, § 7.

† *Phil. Mag.*, April and July, 1897. See also *Theory of Sound*, § 292.

‡ [*July*, 1898.—This is not new, as the author had supposed. It was set in the Mathematical Tripos, Part II., 1895, and is due to Mr. Larmor.]

where $$w = \phi + i\psi, \quad z = x + iy, \tag{2}$$

and the constant $\mu$ is supposed greater than unity. This result was obtained originally by a modification of Schwarz's method, but it will be sufficient here to verify it *à posteriori*. The formula (1) gives

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh x\cos y \\ \sinh\phi\sin\psi &= \mu\sinh x\sin y \end{aligned}\right\}. \tag{3}$$

The locus $\phi = 0$ consists of those portions of the axis of $y$ for which

$$1 > \mu\cos y > -1;$$

these represent the breadths of the metallic strips, so that (on the scale of our formulæ) the half-breadth of a strip is $\sin^{-1}(1/\mu)$. For other portions of the plane, $\phi$ is, so far, indeterminate as to sign; we will take it to be everywhere positive. Then $\psi$ is indeterminate to the extent of a term $2s\pi$, but we may suppose that the lines $\psi = 0$, $\psi = \pm\pi$, $\psi = \pm 2\pi$, ... consist partly of the lines $y = 0$, $y = \pm\pi$, $y = \pm 2\pi$, ..., respectively, and partly of those portions of the axis of $y$ for which

$$|\,\mu\cos y\,| > 1;$$

these correspond to the widths of the apertures between the successive strips. The half-breadth of an aperture, on the present scale, is $\cos^{-1}(1/\mu)$. At the centres of the apertures—for example, at the origin—we have $\phi = \cosh^{-1}\mu$; these are points of zero force, and the equipotential- and force-lines there meet at angles of 45°.

I have thought it worth while to trace the curves

$$\phi = \text{const.}, \quad \psi = \text{const.},$$

using the formula

$$\left.\begin{matrix} \cosh 2x \\ \cos 2y \end{matrix}\right\} = p \pm \sqrt{(p^2 - 2q + 1)}, \tag{4}$$

where

$$p = \frac{1}{2\mu^2}(\cosh 2\phi + \cos 2\psi), \quad q = \frac{1}{2\mu^2}(1 + \cosh 2\phi\cos 2\psi); \tag{5}$$

this is easily deduced from (3). The value of $\mu$ adopted for convenience of calculation was

$$\mu = \cosh\tfrac{1}{5}\pi = 1{\cdot}2040,$$

whence $$\sin^{-1}(1/\mu) = {\cdot}312\pi, \quad \cos^{-1}(1/\mu) = {\cdot}188\pi;$$

this gives the relative breadths of the strips and apertures. The curves are shewn in Fig. 1.

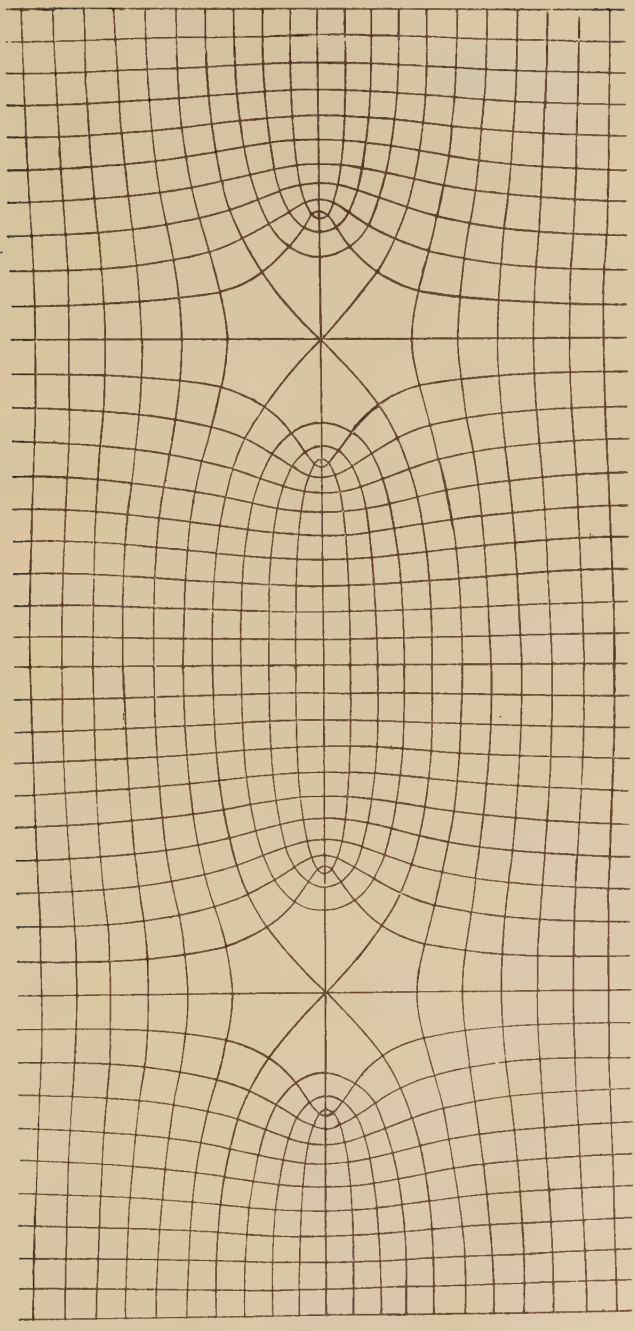

FIG. 1.

The formulæ (3), and the diagram, admit of the usual variety of interpretations in Electrostatics, Electric Conduction, Hydrodynamics, &c. If we introduce a more general linear unit, and denote the breadth of each aperture by $a$, that of each strip by $b$, we may write

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh\frac{\pi x}{a+b}\cos\frac{\pi y}{a+b} \\ \sinh\phi\sin\psi &= \mu\sinh\frac{\pi x}{a+b}\sin\frac{\pi y}{a+b} \end{aligned}\right\}, \tag{6}$$

where

$$\mu = \sec\frac{\pi a}{2(a+b)} = \operatorname{cosec}\frac{\pi b}{2(a+b)}. \tag{7}$$

A first application of (6) is to find the capacity of a condenser formed by two metal plates at the same potential with an insulated grating half way between them, the distance between the two external plates being large compared with the interval $a+b$. For large positive values of $x$, the formulæ (6) give

$$\phi = \frac{\pi x}{a+b} + \log\mu, \quad \psi = \frac{\pi y}{a+b}. \tag{8}$$

The charge per unit area of one of the plates is $1/4(a+b)$ in the present units, and the capacity of the system per unit area is, therefore

$$\frac{1}{2\pi\left\{x + \frac{a+b}{\pi}\log\sec\frac{\pi a}{2(a+b)}\right\}}, \tag{9}$$

where $x$ is the half-distance between the external plates. If the central plate had been entire, the capacity would have been $1/2\pi x$; the effect of the apertures is therefore equivalent to increasing the distance between the central plate and each external plate by an amount $c$, given by

$$c = \frac{a+b}{\pi}\log\sec\frac{\pi a}{2(a+b)}. \tag{10}$$

The numerical values subjoined shew what a very large proportion of the central plate can be cut away without sensibly altering the capacity, so long as the period $(a+b)$ of the residual structure is large compared with the distance between the plates:

$$\frac{a}{a+b} = 0,\ \cdot1,\ \cdot2,\ \cdot3,\ \cdot4,\ \cdot5,\ \cdot6,\ \cdot7,\ \cdot8,\ \cdot9,\ 1{\cdot}0$$

$$\frac{c}{a+b} = 0,\ \cdot004,\ \cdot016,\ \cdot037,\ \cdot067,\ \cdot110,\ \cdot169,\ \cdot251,\ \cdot374,\ \cdot590,\ \infty.$$

For another interpretation of (6) we may imagine that a current is flowing along a metal strip, of breadth $a+b$, which is interrupted by a central transverse non-conducting partition (*e.g.*, a narrow cut in the metal), of breadth $a$, or by two lateral transverse cuts, opposite to one another, each of breadth $\frac{1}{2}a$. It appears that the additional resistance thus introduced is equal to that of a length $2c$ of the strip.

2. The function $\phi$ in (3) is a periodic even function of $y$, the period being $\pi$. It can therefore be expanded by Fourier's Theorem in a series of cosines of multiples of $2y$, the coefficients being functions of $x$ whose general form is to be determined by substitution in the equation

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0. \tag{11}$$

Thus, for $x$ positive we find, having regard to the conditions to be satisfied for large values of $x$,

$$\phi=\log\mu+x+\Sigma_1^\infty C_m e^{-2mx}\cos 2my. \tag{12}$$

The precise values of the coefficients $C_m$ hardly concern us, but it may be worth while to record them. The expansion of the function

$$w=\cosh^{-1}(\mu\cosh z) \tag{13}$$

can be effected directly, with the assistance of Lagrange's Theorem, and in this way we obtain the form (12), with

$$C_m=(-)^{m-1}\frac{1}{m}\left\{1-\frac{m^2}{1^2.\mu^2}+\frac{m^2(m^2-1^2)}{1^2.2^2.\mu^4}-\frac{m^2(m^2-1^2)(m^2-2^2)}{1^2.2^2.3^2.\mu^6}+\dots\right\}. \tag{14}$$

It may be noted that this contains $1-1/\mu^2$ as a factor.*

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* [*July*, 1898.—We have

$$C_m=(-)^{m-1}\frac{1}{m}F\left(m,-m,1,\frac{1}{\mu^2}\right)$$

$$=(-)^{m-1}\frac{1}{m}\left(1-\frac{1}{\mu^2}\right)F\left(1+m,1-m,1,\frac{1}{\mu^2}\right),$$

in the hypergeometric notation.

At the suggestion of one of the referees the proof of (14) is indicated. If we put $y=0$ in (13), we have

$$\phi=\cosh^{-1}(\mu\cosh x)=\log\left\{\mu\cosh x+\sqrt{(\mu^2\cosh^2 x-1)}\right\}$$

$$=\log\mu+x+\log\left[\frac{1+t}{2}+\sqrt{\left\{\left(\frac{1+t}{2}\right)^2-\frac{1}{\mu^2}\right\}}\right],$$

where $t=e^{-2x}$. If we write

$$\zeta=\frac{1+t}{2}+\sqrt{\left\{\left(\frac{1+t}{2}\right)^2-\frac{1}{\mu^2}\right\}},$$

The comparison of (12) with (3) leads to a number of results in definite integrals. For example, using square brackets to denote that $x$ has been put $= 0$ in the values of the enclosed functions, we have

$$\int_0^{\frac{1}{2}\pi} [\phi]\, dy = \tfrac{1}{2}\pi \log \mu, \tag{15}$$

$$\int_0^{\frac{1}{2}\pi} \left[\frac{d\phi}{dx}\right] dy = \tfrac{1}{2}\pi. \tag{16}$$

Now it appears from (3) that from $y = 0$ to $y = \cos^{-1}(1/\mu)$ we have

$$[\psi] = 0, \quad [\phi] = \cosh^{-1}(\mu \cos y),$$

whilst, from $y = \cos^{-1}(1/\mu)$ to $y = \frac{1}{2}\pi$,

$$[\phi] = 0, \quad [\psi] = \cos^{-1}(\mu \cos y).$$

Hence (15) gives the theorem

$$\int_0^{\beta} \cosh^{-1}\left(\frac{\cos y}{\cos \beta}\right) dy = \tfrac{1}{2}\pi \log \sec \beta. \tag{17}$$

This is easily verified. If we denote the value of the definite integral on the left hand by $f(\beta)$, we find

$$f'(\beta) = \tan \beta \int_0^{\beta} \frac{\cos y\, dy}{\sqrt{(\cos^2 y - \cos^2 \beta)}} = \tfrac{1}{2}\pi \tan \beta, \tag{18}$$

whence (17) follows by integration, since $f(0) = 0$.* The verification of (16) is still simpler; we have only to put $d\phi/dx = d\psi/dy$.

---

we find $$\zeta = 1 + t\left(1 - \frac{1}{\mu^2 \zeta}\right) = 1 + t f(\zeta),$$

say. If $x$ be positive, $t$ will be less than unity, and $\log \zeta$ can be expanded in powers of $t$ by Lagrange's Theorem, thus

$$\log \zeta = \log u + \Sigma_1^{\infty} \frac{t^m}{m!} \left(\frac{d}{du}\right)^{m-1} \frac{\{f(u)\}^m}{u},$$

where $u$ is to be put $= 1$, finally. Working this out, we find

$$\log \zeta = \Sigma_1^{\infty} C_m t^m,$$

where $C_m$ has the value given above. Hence the value of $\phi$ for $y = 0$ and $x > 0$ is

$$\phi = \log \mu + x + \Sigma_1^{\infty} C_m e^{-2mx}.$$

It easily follows that (12) gives the general value of $\phi$ for $x > 0$.]

* The theorem (17) is, substantially, one of a number given by Lobatschewsky, *Kasan Memoirs*, 1836 (Part I., p. 149), but the proof is quite different. The formula

$$\int_0^{\beta} \frac{y \sin y\, dy}{\sqrt{(\sin^2 \beta - \sin^2 y)}} = \tfrac{1}{2}\pi \log \sec \beta,$$

which is immediately deduced from (17) by a partial integration, was obtained by Legendre, *Exercices de Calcul Intégral*, t. I., Supplément, Case ix. (1811), as a particular case of a general theorem involving elliptic integrals.

3. When the field of force is not symmetrical with respect to the plane of the grating, the requisite formulæ are obtained by the addition of a term $Az$ to $w$ in (1), so that

$$\cosh(w+Az) = \mu \cosh z. \tag{19}$$

The most interesting special case is where the constant $A$ is so chosen as to annul the force at a distance on one side, say that for which $x$ is negative. We then have $A = -1$, and the equation (19) reduces to the form

$$\tanh z = \frac{\cosh w - \mu}{\sinh w}, \tag{20}$$

whence

$$z = \tfrac{1}{2}\log\frac{e^w - \mu}{\mu - e^{-w}}. \tag{21}$$

If we write $\mu = e^a$, this can be put into the shape

$$z = \tfrac{1}{2}w + \tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(w-a)}{\sinh\frac{1}{2}(w+a)}, \tag{22}*$$

or

$$x+iy = \tfrac{1}{2}(\phi+i\psi) + \tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(\phi-a)\cos\frac{1}{2}\psi + i\cosh\frac{1}{2}(\phi-a)\sin\frac{1}{2}\psi}{\sinh\frac{1}{2}(\phi+a)\cos\frac{1}{2}\psi + i\cosh\frac{1}{2}(\phi+a)\sin\frac{1}{2}\psi}. \tag{23}$$

This gives, after a little manipulation,

$$\left.\begin{aligned} x &= \tfrac{1}{2}\phi + \tfrac{1}{4}\log\frac{\cosh(\phi-a)-\cos\psi}{\cosh(\phi+a)-\cos\psi}, \\ y &= \tfrac{1}{2}\psi + \tfrac{1}{2}\tan^{-1}\frac{\sinh a\,\sin\psi}{\cosh\phi - \cosh a\,\cos\psi} \end{aligned}\right\}. \tag{24}$$

By means of these equations the curves

$$\phi = \text{const.}, \quad \psi = \text{const.}$$

can be traced without much difficulty. Their forms are shown in Fig. 2 (p. 530) for the case of $a = \frac{1}{5}\pi$. The diagram affords the best verification of the formulæ, but the following points may be noticed.

In the first place, the line $\psi = 0$ consists of distinct portions, according as $\phi \gtrless a$. For $\phi > a$ we may take

$$y = 0, \quad x = \tfrac{1}{2}\phi + \tfrac{1}{2}\log\frac{\sinh\frac{1}{2}(\phi-a)}{\sinh\frac{1}{2}(\phi+a)}, \tag{25}$$

the value of $x$ ranging from $-\infty$ to $+\infty$ as $\phi$ increases from $a$ to $\infty$. This gives the line of force through the origin, which is the centre of

* The solution for the present case was originally obtained in this form by Schwarz's method.

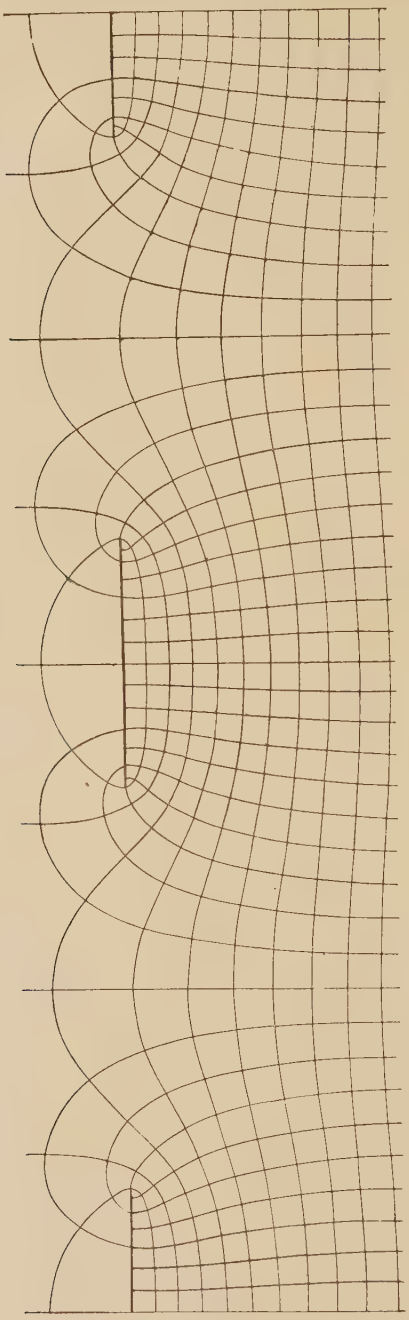

Fig. 2.

an aperture, and the lines of force through the centres of consecutive apertures are given in like manner by

$$y = \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \ldots,$$

corresponding to

$$\psi = \pm 2\pi, \quad \pm 4\pi, \quad \pm 6\pi, \quad \ldots.$$

If, on the other hand, $\psi = 0$ and $\phi < \alpha$, we have

$$y = \tfrac{1}{2}\tan^{-1} 0, \quad x = \tfrac{1}{2}\phi + \tfrac{1}{2}\log\frac{\sinh\tfrac{1}{2}(\alpha-\phi)}{\sinh\tfrac{1}{2}(\alpha+\phi)}. \tag{26}$$

The value of $x$ ranges from 0 to $-\infty$ as $\phi$ increases from 0 to $\alpha$. The value to be attributed to $\tan^{-1} 0$ is determined by considerations of continuity. Thus, if we assign a very small fixed positive value to $\psi$, and then imagine $\phi$ to decrease from $\infty$ through $\alpha$ to 0, the fraction

$$\frac{\sinh\alpha\,\sin\psi}{\cosh\phi - \cosh\alpha\,\cos\psi}$$

will pass from a small positive value through infinity to a small negative value. Hence, if $\psi$ approach zero from the positive side, the value of $y$ in (26) will be $\frac{1}{2}\pi$. If $\psi$ were to approach zero from the negative side, we should have $y = -\frac{1}{2}\pi$. The portions of the lines

$$y = \pm\tfrac{1}{2}\pi, \quad \pm\tfrac{3}{2}\pi, \quad \pm\tfrac{5}{2}\pi, \quad \ldots$$

for which $x$ is negative are lines of force abutting on the backs of the metal strips.

Again, for $\psi = \pi$, we have

$$y = \tfrac{1}{2}\pi, \quad x = \tfrac{1}{2}\phi + \tfrac{1}{2}\log\frac{\cosh\tfrac{1}{2}(\phi-\alpha)}{\cosh\tfrac{1}{2}(\phi+\alpha)}; \tag{27}$$

this value of $x$ ranges from 0 to $\infty$ as $\phi$ increases from 0 to $\infty$. And, generally, the portions of the lines

$$y = \pm\tfrac{1}{2}\pi, \quad \pm\tfrac{3}{2}\pi, \quad \pm\tfrac{5}{2}\pi, \quad \ldots$$

for which $x$ is positive are lines of force abutting on the fronts of the metal strips.

If in (24) we make $\phi = 0$, we get

$$x = 0, \quad y = \tfrac{1}{2}\psi + \tfrac{1}{2}\tan^{-1}\frac{\sinh\alpha\,\sin\psi}{1-\cosh\alpha\,\cos\psi}. \tag{28}$$

This gives a series of isolated portions of the axis of $y$, representing the breadths of the metal strips. These are bounded by the points for which

$$\cos\psi = e^{-\alpha} = 1/\mu,$$

and therefore

$$\frac{\sinh\alpha\,\sin\psi}{1-\cosh\alpha\,\cos\psi} = \tan\psi. \tag{29}$$

For example, if we make $\psi$ increase from 0 to $\pi$, $y$ will diminish from $\frac{1}{2}\pi$ to the minimum value $\cos^{-1}e^{-a}$, and then increase again up to the value $\frac{1}{2}\pi$.

It is further to be noticed that the lines $\phi = \text{const.}$ fall into two systems separated by the curves $\phi = a$, which have the lines

$$y = 0, \ \pm\pi, \ \pm 2\pi, \ \ldots$$

as asymptotes. For $\phi < a$, the equipotential lines are closed. This is indicated clearly in the figure, which may be compared with the corresponding diagram for a grating of cylindrical wires given by Maxwell.*

The formulæ (24), when generalized as to the linear unit involved, lead to a determination of the capacity of a condenser formed of two parallel plates, one of which is entire, whilst the other is a grating. If the distance $x$ between these plates be large compared with the interval $a+b$, where $a$ and $b$ have the same meanings as in § 1, we find for the capacity in question

$$\frac{1}{4\pi\left\{x + \frac{a+b}{2\pi}\log\sec\frac{\pi a}{2(a+b)}\right\}}. \tag{30}$$

It is also easy to shew that the charge on the grating is divided between its front and back surfaces in the ratio $a+2b : a$.

4. Although the matter is not very closely connected with the main object of this paper, it may be worth while to notice the interpretation of the formulæ (3) when the constant $\mu$ is *less* than unity. They give the distribution of electric force in a condenser one of whose plates has a series of parallel equidistant ribs projecting at right angles to its surface. Thus, if, altering the linear unit, we write

$$\cosh w = \mu \cosh\frac{\pi z}{a}, \tag{31}$$

or

$$\left.\begin{aligned} \cosh\phi\cos\psi &= \mu\cosh\frac{\pi x}{a}\cos\frac{\pi y}{a} \\ \sinh\phi\sin\psi &= \mu\sinh\frac{\pi x}{a}\sin\frac{\pi y}{a} \end{aligned}\right\}, \tag{32}$$

we find that the line $\phi = 0$ now consists of the whole of the axis of $y$, together with those portions of the lines

$$y = 0, \ \pm a, \ \pm 2a, \ \pm 3a, \ \ldots$$

which extend from the axis of $y$ to the points determined by

$$\cosh\frac{\pi x}{a} = \frac{1}{\mu}. \tag{33}$$

* *Electricity and Magnetism*, Vol. I., Fig. xiii.

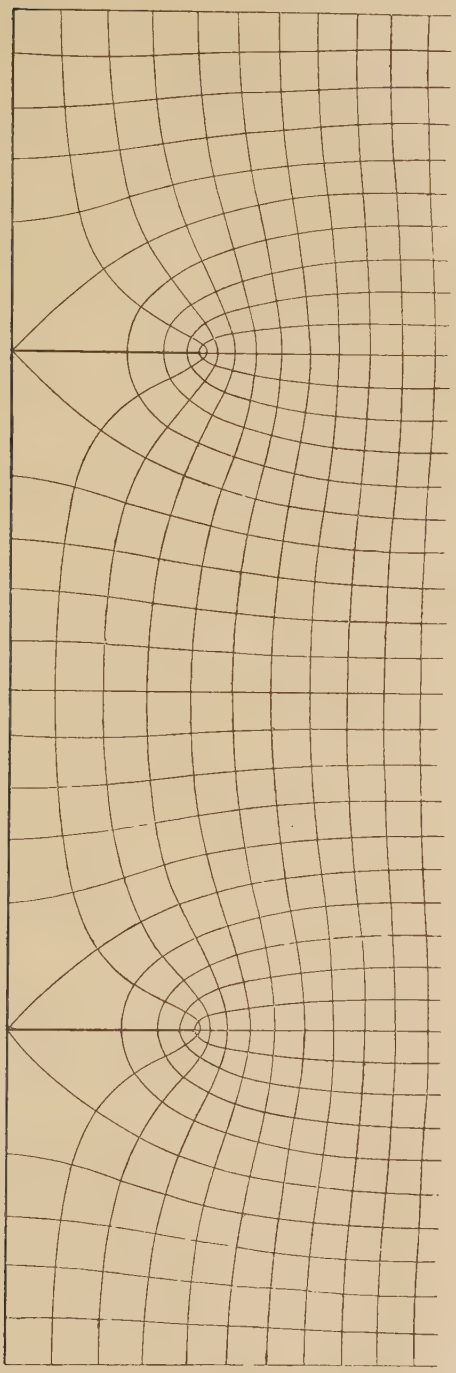

Fig. 3.

Hence, if $b$ be the depth of a rib, we have

$$\mu = \operatorname{sech} \frac{\pi b}{a}. \tag{34}$$

The curves $\phi = \text{const.}$, $\psi = \text{const.}$ are shown in Fig. 3, for the case of $\mu = 1/\sqrt{2}$.

Proceeding as in § 1, we find that the capacity (per unit area) of a condenser of the above type is

$$\frac{1}{4\pi\left(x - \frac{a}{\pi} \log \cosh \frac{\pi b}{a}\right)}, \tag{35}$$

$x$ denoting the distance between the plates, which is assumed to be large compared with $b$. The effect of the ribs in increasing the capacity is therefore equivalent to diminishing the distance between the plates by the amount

$$\frac{a}{\pi} \log \cosh \frac{\pi b}{a}. \tag{36}$$

Again, if we take the curves $\phi = \text{const.}$ to be lines of electric flow, we get the case of a current flowing along a relatively broad strip which is interrupted along one edge, or along the medial line, by a series of parallel, equal, and equidistant transverse cuts, one behind the other. If $b$ be the length of the cuts, their effect (when at the edge) is equivalent to diminishing the breadth of the strips by the amount (36). As numerical examples, we find, for

$$b/a = \cdot 1, \quad \cdot 2, \quad \cdot 3, \quad \cdot 4, \quad \cdot 5, \quad \cdot 6, \quad \cdot 7, \quad \cdot 8, \quad \cdot 9, \quad 1 \cdot 0,$$

$$\frac{\log \cosh \pi b/a}{\pi b/a} = \cdot 156, \ \cdot 296, \ \cdot 415, \ \cdot 511, \ \cdot 586, \ \cdot 645, \ \cdot 691, \ \cdot 729, \ \cdot 756, \ \cdot 780.$$

The latter numbers give the ratio which the aforesaid diminution bears to the breadth $b$ of the cuts. For increasing values of $b/a$ this ratio tends obviously to the value unity.

In the case of cuts across the medial line the quantity (36) must be multiplied by 2, $b$ now denoting the half-length of a cut.

5. Proceeding to the problem of electric waves, and assuming that all our functions involve the two dimensions $x$ and $y$ only, we have to satisfy an equation of the form

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + k^2 u = 0, \tag{37}$$

subject to the proper condition at the surfaces of the metal. If the

electrical vibrations be sufficiently rapid, the condition is that the component of electric force tangential to the surface, just outside the metal, must be zero.

We take separately the cases where the direction of the electric force is perpendicular and parallel, respectively, to the plane $xy$. In the former case, the equation (37) is satisfied by the electric force ($R$), and the condition which is to hold at the metal surfaces is $R = 0$, simply.

Let us now suppose that a train of waves of length $2\pi/k$ and amplitude unity is incident directly on the grating from the side on which $x$ is positive. Omitting the time-factor $e^{ikVt}$, where $V$ is the wave-velocity, this train may be represented symbolically by

$$R = e^{ikx}. \tag{38}$$

If in place of the grating we had an uninterrupted sheet of metal, the reflected wave would be represented by $-e^{ikx}$, and the total disturbance on the side of $x$ positive would be given by

$$R = e^{ikx} - e^{-ikx}, \tag{39}$$

for this satisfies (37) and makes $R = 0$ for $x = 0$.

In the actual circumstances we assume that, on the positive side of the grating,

$$R = e^{ikx} - e^{-ikx} + \chi, \tag{40}$$

and, on the negative side,

$$R = \chi', \tag{41}$$

where $\chi$ and $\chi'$ must, of course, satisfy (37). Over the portions of the plane $x = 0$ which are occupied by the metal strips, we must have

$$\chi = 0, \quad \chi' = 0; \tag{42}$$

whilst the continuity of $R$ and $dR/dx$ demands that

$$\chi = \chi', \quad \frac{d\chi}{dx} + 2ik = \frac{d\chi'}{dx} \tag{43}$$

over the apertures. The latter condition will be satisfied provided we make

$$\frac{d\chi}{dx} = -ik, \quad \frac{d\chi'}{dx} = ik. \tag{44}$$

If the differential equation to be satisfied by $\chi$ had been

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \tag{45}$$

in place of (37), the conditions relating to this function would (save as to the factor $ik$) have been the same as in the hydrodynamical problem where a system of equal, parallel, and equidistant

blades, all in the same plane, is moved perpendicular to itself in a frictionless liquid, which may have an independent general motion in the same direction. The function $\chi/ik$ would, in fact, give the value of the velocity-potential on the positive side of the plane $x=0$. It must be borne in mind in this analogy that the blades correspond to the *apertures* of our grating, and the intervals between them to the metallic strips.* Now the *relative* motion of the fluid in such a case is shown in Fig. 1, to which the formulæ (6) apply; and a value of $\chi$ satisfying the conditions (42) and (44) would therefore be

$$\chi = -ikx + C\phi, \tag{46}$$

where $\phi$ is the positive function determined by (6).

This value of $\chi$ is necessarily periodic with respect to $y$, the period being $a+b$; and the line $y=0$ is a line of symmetry. Hence, expanding by Fourier's Theorem in a series of cosines of multiples of $2\pi y/(a+b)$, we find

$$\chi = C\log\mu + \left(\frac{\pi C}{a+b} - ik\right) x + C\Sigma_1^\infty C_m e^{-\kappa_m x}\cos\frac{2m\pi y}{a+b}, \tag{47}$$

where

$$\kappa_m = \frac{2m\pi}{a+b}, \tag{48}$$

and the first two terms have been determined from a comparison of (46) and (8). The terms involving $e^{\kappa_m x}$ are excluded by the condition of finiteness for $x=\infty$.

The true value of $\chi$ can be similarly expanded in a Fourier's series, the forms of the coefficients as functions of $x$ being determined by means of the differential equation (37). The resulting form is

$$\chi = B_0 e^{-ikx} + \Sigma_1^\infty B_m e^{-\lambda_m x}\cos\frac{2m\pi y}{a+b}, \tag{49}$$

provided

$$\lambda_m^2 = \frac{4m^2\pi^2}{(a+b)^2} - k^2. \tag{50}$$

Since, by hypothesis, $a+b$ is small compared with the wave-length $2\pi/k$, the right-hand side of (50) is positive. Hence the quantities $\lambda_m$ are real, and, moreover, differ respectively very little from $\kappa_m$. Terms involving $e^{\lambda_m x}$ are excluded as before by the condition of finiteness for $x=\infty$, so that the waves represented by $\chi$ are ultimately plane. The fact that they must travel outwards from the grating justifies the omission of the term in $e^{ikx}$.

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* The foregoing method of analysing the conditions to be satisfied is taken from Lord Rayleigh's paper, *Phil. Mag.*, April, 1897.

For small values of $x$, the expansion (49) reduces to the form

$$\chi = B_0(1-kx) + \Sigma_1^\infty B_m e^{-\kappa_m x}\cos\frac{2m\pi y}{a+b}, \qquad (51)^*$$

approximately; and it appears, on comparison with (47), that the conditions incumbent on $\chi$ will all be satisfied, subject to an error of the order $k^2(a+b)^2/4\pi^2$, if

$$B_m = CC_m, \quad B_0 = C\log\mu = 1+\frac{i\pi C}{k(a+b)}. \qquad (52)$$

Substituting the value of $\mu$ from (7), we find

$$B_0 = \frac{ikc}{1+ikc}, \qquad (53)$$

where $c$ is the linear magnitude defined by (10).

The conditions for $\chi'$ are all fulfilled if we suppose that its value at any point $P'$ on the negative side of the grating is equal, both in magnitude and in sign, to the value of $\chi$ at the *image* $P$ of $P'$ on the positive side. Hence the Fourier expansion of $\chi'$ is

$$\chi' = B_0 e^{ikx} + \Sigma_1 B_m e^{\lambda_m x}\cos\frac{2m\pi y}{a+b}. \qquad (54)$$

In the above investigation, the coefficient of the primary wave has been taken to be unity. On the same scale, the coefficients of the reflected and transmitted waves are $-1+B_0$ and $B_0$, or

$$-\frac{1}{1+ikc} \quad \text{and} \quad \frac{ikc}{1+ikc},$$

respectively. Hence the *intensities* $I$, $I'$ (say) of these waves, in terms of that of the primary wave as unit, are given by

$$I = \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \qquad (55)$$

It appears from the numerical values given near the end of § 1 that the value of $c/(a+b)$ may be very moderate even when the breadth ($b$) of the metal strips is relatively small. Hence even a very open grating (*i.e.*, one in which a very small proportion of the area is occupied by metal) will produce almost total reflection of waves in which the electric displacement is parallel to the bars, provided the wave-length ($2\pi/k$) be moderately large compared with the period $a+b$ of the structure.

* *Cf.* J. J. Thomson, *loc. cit.*

The circumstances when the reflection is practically total are clearly illustrated in Fig. 2, where the lines $\phi = \text{const.}$ represent the lines of (alternating) magnetic force in the immediate neighbourhood of the grating. A similar interpretation applies also to Maxwell's diagram already referred to.

6. When the electric force is everywhere parallel to the plane $xy$, the magnetic force ($\gamma$) is everywhere perpendicular to this plane, and satisfies an equation of the form (37). The components of displacement-current in the dielectric are then

$$\frac{1}{4\pi}\frac{d\nu}{dy}, \quad -\frac{1}{4\pi}\frac{d\gamma}{dx}, \quad 0,$$

and the condition to be satisfied at the surfaces of the metal strips is $d\gamma/dn = 0$, where $dn$ denotes an element of the normal.

Considering a train of waves represented by

$$\gamma = e^{ikx}, \tag{56}$$

and therefore incident directly on the grating from the side of $x$ positive, we should have, if the apertures were filled up,

$$\gamma = e^{ikx} + e^{-ikx} \tag{57}$$

on the positive side, and $\gamma = 0$ on the negative side. In the actual case we assume, for the positive side,

$$\gamma = e^{ikx} + e^{-ikx} + \chi, \tag{58}$$

and, for the negative side, $\qquad \gamma = \chi', \tag{59}$

where $\chi$ and $\chi'$ must satisfy (37); and the conditions to be fulfilled over those portions of the line $x = 0$ which correspond to the metal strips are

$$\frac{d\chi}{dx} = 0, \quad \frac{d\chi'}{dx} = 0, \tag{60}$$

whilst over the apertures we must have

$$2 + \chi = \chi', \quad \frac{d\chi}{dx} = \frac{d\chi'}{dx}, \tag{61}$$

on account of the continuity of $\gamma$ and $d\gamma/dx$. The former of the conditions (61) will be satisfied provided

$$\chi = -1, \quad \chi' = 1. \tag{62}$$

If the differential equation (37) were replaced by (45), the conditions applicable to $\chi$ would be similar to those which govern the

electric potential in the case of a grating placed transverse to the lines of force in a uniform field; but it must be remembered in this analogy that the bars in one problem correspond to the apertures in the other, and *vice versâ*.* We should therefore have

$$\chi = -1 + C\phi, \tag{63}$$

where $\phi$ is determined by (6); but, if we still continue to denote by $a$ and $b$ the breadths of the apertures and of the strips, respectively, in our grating, the formula (7), giving the value of $\mu$, must be replaced by

$$\mu = \sec \frac{\pi b}{2(a+b)} = \operatorname{cosec} \frac{\pi a}{2(a+b)}. \tag{64}$$

The Fourier expansion of (63) is

$$\chi = -1 + C \log \mu + \frac{\pi C}{a+b} x + C \Sigma_1^\infty C_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}, \tag{65}$$

whilst that of the true function $\chi$ has the form

$$\chi = C_0 e^{-ikx} + \Sigma_1^\infty B_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a+b}, \tag{66}$$

where $\kappa_m$ and $\lambda_m$ are defined by (48) and (50). For small values of $x$ the latter expansion takes the form

$$\chi = C_0 (1 - ikx) + \Sigma_1^\infty B_m e^{-\kappa_m x} \cos \frac{2m\pi y}{a+b}. \tag{67}$$

By comparison we find that (66) will satisfy approximately all the conditions of the problem, provided

$$B_m = CC_m, \quad C_0 = -1 + C \log \mu = \frac{i\pi C}{k(a+b)}. \tag{68}$$

Hence

$$C_0 = -\frac{1}{1 + ikc_1}, \tag{69}$$

provided

$$c_1 = \frac{a+b}{\pi} \log \sec \frac{\pi b}{2(a+b)}. \tag{70}$$

As regards $\chi'$, all the conditions are satisfied if we suppose that its value at any point $P'$ on the negative side of the grating is equal in absolute magnitude, but opposite in sign, to that of $\chi$ at the image $P$ of $P'$ on the positive side. Hence the Fourier expansion of $\chi'$ is

$$\chi' = -C_0 e^{ikx} - \Sigma_1^\infty B_m e^{\lambda_m x} \cos \frac{2m\pi y}{a+b}. \tag{71}$$

* See the footnote on p. 536.

The coefficients of the reflected and transmitted waves are $1+C_0$ and $-C_0$, or

$$\frac{ikc_1}{1+ikc_1} \quad \text{and} \quad \frac{1}{1+ikc_1},$$

respectively, that of the primary being taken as unit. Hence the intensities $J$, $J'$ of these waves are given by

$$J=\frac{k^2c_1^2}{1+k^2c_1^2}, \qquad J'=\frac{1}{1+k^2c_1^2}. \tag{72}$$

For sufficiently great wave-lengths there is very little reflection, even when the intervals $a$ between the strips are very narrow compared with the breadths of the strips themselves.

* [These results, combined with those of § 5, serve to explain the polarizing efficacy of a grating, as employed originally by Hertz. As a numerical example, let us suppose that the wave-length is ten times the interval $a+b$. Then, even if one-tenth only of the area of the grating be occupied by the metal strips, the coefficients of reflection and transmission are, for electrical vibrations parallel to the lengths,

$$I=\cdot 879, \quad I'=\cdot 121.$$

The same numbers in reverse order give the coefficients of reflection and transmission for electrical vibrations transverse to the length, when the *apertures* occupy only one-tenth of the area; viz.,

$$J=\cdot 121, \quad J'=\cdot 879.$$

The coefficient of reflection in the one case, and of transmission in the other, increases rapidly with the wave-length.]

In the immediate neighbourhood of the grating the lines $\gamma=$ const. coincide with the curves $\phi=$ const. of Fig. 1. These give the directions of the electric oscillation; but it must be remembered that the portions of the vertical line of symmetry which represented the metal strips in the electrostatic problem of § 1 now correspond to apertures, and *vice versâ*.

The mathematical conditions to which the magnetic force $\gamma$ is subject in the problem of this section are exactly those which apply to the velocity-potential in the case of sound-waves incident directly on a grating of the kind considered.† It appears that a series of narrow slits, with relatively broad opaque intervals, may transmit a *very* large proportion of the original sound, provided the wave-length be sufficiently great.

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* Paragraph inserted July, 1898.

† *Cf.* Rayleigh, *Phil. Mag.*, April, 1897, p. 272.

7. As mentioned in the introduction, the problem of reflection by a grating composed of parallel cylindrical bars has been discussed by Prof. J. J. Thomson in the case where the electric displacement is everywhere in the direction of the length. The result arrived at is that the reflection is total, without any limitation as to the thinness of the bars. It may be worth while to examine this point, and to complete the investigation by considering also the case where the electric displacement is transverse.

If, as in § 1, we write

$$w = \phi + i\psi, \quad z = x + iy, \tag{73}$$

the potential- and stream-functions corresponding to a system of equal and equidistant line-sources cutting the plane $xy$ perpendicularly at the points $(0, 0)$, $(0, \pm a)$, $(0, \pm 2a)$, ..., are given by the formula

$$w \propto \log z + \log(z - ia) + \log(z + ia) + \log(z - 2ia) + \log(z + 2ia) + \ldots, \tag{74}$$

or, say,

$$w = \log \sinh \frac{\pi z}{a}. \tag{75}$$

This makes

$$\left.\begin{aligned} \phi &= \frac{1}{2} \log \frac{1}{2} \left( \cosh \frac{2\pi x}{c} - \cos \frac{2\pi y}{c} \right) \\ \psi &= \tan^{-1} \left\{ \frac{\tan \frac{\pi y}{c}}{\tanh \frac{\pi x}{a}} \right\} \end{aligned}\right\}, \tag{76}$$

in agreement with Maxwell, *Electricity and Magnetism*, § 203.

The case of a row of *double* sources having their axes parallel to $x$ is obtained by differentiating (75) with respect to $z$; thus we find

$$w = \frac{\pi}{a} \log \coth \frac{\pi z}{a}, \tag{77}$$

$$\left.\begin{aligned} \phi &= \frac{\pi}{a} \frac{\sinh \frac{2\pi x}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \\ \psi &= -\frac{\pi}{a} \frac{\sin \frac{2\pi y}{a}}{\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a}} \end{aligned}\right\}. \tag{78}$$

If the double sources have their axes parallel to $y$, we have merely to interchange the values of $\phi$ and $\psi$ in these formulæ.

These results enable us to solve very simply a number of problems in Hydrodynamics, Magnetic Induction, and so on. For example, the stream-function for a liquid flowing through a grating of parallel cylindrical bars is

$$\psi = V\left\{y - \frac{\pi b^2}{a}\frac{\sin\frac{2\pi y}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}\right\}, \qquad (79)^*$$

where $V$ is the velocity at infinity. This formula is easily verified, since, for small values of $x$ and $y$, it reduces to the form

$$\psi = Vy\left(1 - \frac{b^2}{x^2+y^2}\right). \qquad (80)$$

It is here assumed that the radius $b$ of a bar is small compared with the distance $a$ between the axes of consecutive bars.

Again, in the case of a liquid flowing *parallel* to a row of cylindrical obstacles, we have

$$\psi = V\left\{x - \frac{\pi b^2}{a}\frac{\sinh\frac{2\pi x}{a}}{\cosh\frac{2\pi x}{a} - \cos\frac{2\pi y}{a}}\right\}. \qquad (81)$$

Returning to the formula (75), we find, if the real part of $z$ be positive,

$$w = \log\tfrac{1}{2}\left(e^{\pi z/a} - e^{-\pi z/a}\right)$$

$$= \frac{\pi z}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{-2m\pi z/a}, \qquad (82)$$

whence

$$\phi = \frac{\pi x}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{-2m\pi x/a}\cos\frac{2m\pi y}{a}. \qquad (83)$$

Since $\phi$ is an even function of $x$, we have, for $x$ negative,

$$\phi = -\frac{\pi x}{a} - \log 2 - \Sigma_1^\infty \frac{1}{m} e^{2m\pi x/a}\cos\frac{2m\pi y}{a}. \qquad (84)$$

For small values of $x$, we have, from (76),

$$\phi = \log\frac{\pi r}{a}, \qquad (85)$$

where

$$r = \sqrt{(x^2+y^2)}.$$

When the electric force ($R$) is everywhere parallel to the bars of the grating, we assume, for $x$ positive,

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* The curves $\psi$ = const. have been traced from this formula by Prof. Hele Shaw, *Trans. Inst. Nav. Arch.*, 1898.

$$R = e^{ikx} + Ae^{-ikx} + \Sigma_1^\infty D_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{86}$$

and, for $x$ negative,

$$R = Be^{ikx} + \Sigma_1^\infty D_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{87}$$

where $\lambda_m$ is determined by

$$\lambda_m^2 = \frac{4m^2\pi^2}{a^2} - k^2. \tag{88}$$

For small values of $x$, we may ignore the difference between $\lambda_m$ and $2m\pi/a$; and, if we further put $D_m = -C/m$, the formulæ (86) and (87) may be put into the shapes

$$R = 1 + A + ik(1-A)\,x + C\left(\phi - \frac{\pi x}{a} + \log 2\right), \tag{89}$$

and

$$R = B + ikBx + C\left(\phi + \frac{\pi x}{a} + \log 2\right), \tag{90}$$

respectively, where $\phi$ is defined by (83) and (84). The continuity of $R$ and $dR/dx$ requires that

$$1 + A = B, \quad ik(1-A) - \frac{\pi}{a}C = ikB + \frac{\pi}{a}C. \tag{91}$$

At the surface of the metal wires, where $r = b$ (say), we must have $R = 0$. This cannot be satisfied exactly; but, if the ratio $b/a$ be sufficiently small, it is satisfied approximately if we make

$$B + C \log \frac{2\pi b}{a} = 0. \tag{92}*$$

If we now write

$$c = \frac{a}{\pi} \log \frac{a}{2\pi b}, \tag{93}$$

we find, from (91) and (92), that

$$A = -\frac{1}{1+ikc}, \quad B = \frac{ikc}{1+ikc}. \tag{94}$$

The intensities of the reflected and transmitted waves, in terms of that of the primary wave, are therefore

$$= \frac{1}{1+k^2c^2}, \quad I' = \frac{k^2c^2}{1+k^2c^2}. \tag{95}$$

If the wave-length is at all large compared with $c$, $kc$ is small, and the reflection is almost total. But, for any given wave-length (large compared with $a$), $kc$ may be made as great as we please by sufficiently diminishing the radius $b$ of the wires. In this way we can pass to the case of free transmission.

* [*July*, 1898.—Viz., this makes $R = 0$ for $x = 0$, $r = b$.]

When the electric force is everywhere parallel to the plane $xy$, it is convenient (as in § 6) to adopt as dependent variable the magnetic force ($\gamma$), which is everywhere perpendicular to this plane. We assume, for $x$ positive,

$$\gamma = e^{ikx} + Ae^{-ikx} + \Sigma E_m e^{-\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{96}$$

and, for $x$ negative,

$$\gamma = Be^{ikx} - \Sigma E_m e^{\lambda_m x} \cos \frac{2m\pi y}{a}, \tag{97}$$

where $\lambda_m$ is defined by (88). If we put $E_m = 2\pi C/a$, and ignore the difference between $\lambda_m x$ and $2m\pi x/a$, this gives, for small values of $x$,

$$\gamma = 1 + A + ik(1-A)x + C\left(\frac{d\phi}{dx} - \frac{\pi}{a}\right), \tag{98}$$

and

$$\gamma = \quad B + ikBx + C\left(\frac{d\phi}{dx} + \frac{\pi}{a}\right), \tag{99}$$

respectively. The continuity of $\gamma$ and $d\gamma/dx$ on the two sides of the plane $x = 0$ requires that

$$1 + A - \frac{\pi}{a}C = B + \frac{\pi}{a}C, \quad 1 - A = B. \tag{100}$$

The surface-condition is that $d\gamma/dr = 0$ for $r = b$. If we put $x = r\cos\theta$, $d\phi/dx = \cos\theta/r$, this gives

$$ikB - \frac{C}{b^2} = 0. \tag{101}$$

Hence, if we write

$$c_1 = \frac{\pi b^2}{a}, \tag{102}$$

we get

$$A = \frac{ikc_1}{1+ikc_1}, \qquad B = \frac{1}{1+ikc_1}. \tag{103}$$

The intensities of the reflected and transmitted waves are now

$$J = \frac{k^2c_1^2}{1+k^2c_1^2}, \qquad J' = \frac{1}{1+k^2c_1^2}. \tag{104}$$

If the half wave-length be large compared with $b^2/a$, we have free transmission, with hardly any reflection.

As in the problem of § 6, this last calculation has also an acoustical interpretation, and serves to further illustrate the "extreme smallness of the obstruction offered by fine wires or fibres to the passage of sound." *

* Rayleigh, *Sound*, Vol. II., § 343.

*Thursday, June 9th,* 1898.

Prof. E. B. ELLIOTT, F.R.S., President, in the Chair.

Eleven members present and a visitor (Prof. F. Morley).

Prof. H. A. Lorentz, of Leyden, and M. Emile Picard, of Paris, were elected (Honorary) Foreign Members; and Mr. John Henry Michell, M.A., late Fellow of Trinity College, Cambridge, Assistant Professor of Mathematics in the University of Melbourne, was elected an ordinary member.

The President briefly announced the recent deaths of Mr. Henry Perigal (elected January 23rd, 1868), and of the Rev. Dr. Percival Frost, F.R.S. (elected December 9th, 1869).

The following papers were read by their authors:—

Point Groups in a Plane, and their effect in determining Algebraic Curves: Mr. F. S. Macaulay.

On a Regular Rectangular Configuration of Ten Lines: Prof. F. Morley.

On the Conformal Representation of a Pentagon on a Half Plane: Miss M. E. Barwell.

Messrs. Berry, Cunningham, Burnside, Love, the Chairman, and Miss Hardcastle took part in discussions upon one or more of the papers.

The following communications were made in abstract:—

On the General Theory of Anharmonics: Prof. E. O. Lovett.

On the Calculus of Equivalent Statements (Eighth Paper): Mr. H. MacColl.

On the Continuous Group that is defined by any given Group of Finite Order (Second Paper): Prof. W. Burnside.

On certain Regular Polygons in Modular Network: Prof. L. J. Rogers.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 4, 5.

"Proceedings of the Royal Society," Vol. LXIII., Nos. 393–95.

"Proceedings of the Royal Society of Edinburgh," Vol. XXI., 1895–6, 1896–7.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLII., Pt. 2; Manchester, 1897–98.

"Bulletin des Sciences Mathématiques," Tome XXI., "Tables des Matières": Tome XXII., Mai, 1898; Paris.

"Bulletin of the American Mathematical Society," Vol. IV., No. 8, May, 1898; New York.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 1–23, 1898.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 9, 10, Sem. 1; Roma, 1898.

"Educational Times," June, 1898.

"Journal für die reine und angewandte Mathematik," Bd. CXIX., Heft 2; Berlin, 1898.

"Indian Engineering," Vol. XXIII., Nos. 17–20; April 23–May 14, 1898.

Whitehead, A. N.—"Treatise on Universal Algebra," Vol. I., 4to; Cambridge, 1898.

Frolov, M.—La Théorie des Parallèles," pamphlet, 8vo; Paris, 1898.

Rayet, G.—"Observations Pluviométriques et Thermométriques" de Juin 1896 à Mai 1897; Bordeaux, 1897.

"Procès-Verbaux des Séances de la Société des Sciences Physiques et Naturelles de Bordeaux," 1896–7; Paris.

"Journal de l'Ecole Polytechnique de Paris," Série II., Cahier 2; Paris, 1897.

## *On the Continuous Group that is defined by any given Group of Finite Order. (Second Paper.)* By W. Burnside. Received June 8th, 1898. Read June 9th, 1898.

In the present paper the analysis of the continuous group $G$ defined by the equations

$$x'_s = \Sigma y_{s^{-1}i} x_i \quad (s,\ i = 1,\ 2,\ \ldots,\ n)$$

(where $y_{pq} = y_t$,

if $S_p S_q = S_t$;

$S_1 (= 1)$, $S_2$, ..., $S_n$ being the operations of a discontinuous group $g$ of finite order $n$) is brought to a definite conclusion. In a former paper with the same title,* it is shown that that group is the direct product of an Abelian group of order $r$ and a semi-simple group of order $n-r$, $r$ being the number of conjugate sets of operations in $g$.

It is here shown that $r$ integers

$$\mu_1 (= 1),\ \mu_2,\ \ldots,\ \mu_r$$

exist such that

$$\mu_1^2 + \mu_2^2 + \ \ldots\ + \mu_r^2 = n,$$

* *Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 207–224.

and that $G$ is the direct product of $r$ groups

$$G_1,\ G_2,\ \ldots,\ G_r,$$

which are simply isomorphic with the general linear homogeneous groups in

$$\mu_1,\ \mu_2,\ \ldots,\ \mu_r$$

variables respectively. Incidentally it is also shown that the determinant formed from the multiplication table of $g$, when in it the symbols of the operations are regarded as independent variables, has $r$ distinct irreducible factors; and that the power to which any one of these factors enters into the determinant is equal to the degree of the factor. These are two of the main results proved by Herr Frobenius in his memoir "Über die Primfactoren der Gruppendeterminante" (*Berliner Sitzungsberichte*, 1896, pp. 1343–1382).

It is further shown that, if $g$ can be represented as a group of linear substitutions performed on $m$ symbols, so that to the operation $S_k$ of $g$ there corresponds the substitution

$$x'_s = \sum_{i=1}^{i=m} a_{sik} x_i \quad (s = 1,\ 2,\ \ldots,\ m),$$

then the equations

$$x'_s = \sum_{i,k} a_{sik} x_i y_k \quad (s, i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$$

define a finite continuous group which is the direct product of a number of groups each of which is simply isomorphic with a general linear homogeneous group.

The simultaneous consideration of the discontinuous group $g$, and of the continuous group $G$ that is allied to it, appears to me likely to facilitate the discussion of certain of the properties of $g$. In particular, it obviously has a bearing on the question of the smallest number of variables in which $g$ can be expressed as a group of linear substitutions, *i.e.*, on what Prof. Klein calls the degree of the normal-problem connected with $g$. Again, the properties of the group-determinant above mentioned, when investigated in this connexion, appear in a particularly clear and simple light.

In a first section are given some properties of the characteristic determinant of a simply transitive linear homogeneous group which are required for a discussion of $G$. These are not intended to be exhaustive; and some of them, including the contents of § 6, are equivalent to results obtained by Herr Molien in his memoir "Über Systeme höherer complexer Zahlen" (*Math. Ann.*, XLI., pp. 83–156, 1892). The form in which the properties are here presented and the methods of proof are quite distinct from those of Herr Molien.

[*Sept.* 19*th*, 1898.—Since communicating this paper to the Society I have seen a further memoir by Herr Frobenius, "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen," *Berliner Sitzungsberichte*, 1897, pp. 994–1015, which deals with the questions discussed here in §§ 11–13. In it reference is made to a memoir by Herr Molien, "Eine Bemerkung zur Theorie der homogenen Substitutionsgruppen," *Sitzungsberichte der Naturforscher-Gesellschaft zu Dorpat*, 1897, discussing the same question. This memoir I have not been able to gain access to.]

## 1.

1. The equations of a simply transitive linear homogeneous group $G$ in $n$ variables can always be written in the form

$$x'_s = \sum_{i,k} a_{sik} x_i y_k \quad (s,\ i,\ k = 1,\ 2,\ \ldots,\ n).$$

In my paper "On Linear Homogeneous Groups whose Operations are Permutable" I have shown that, by making, if necessary, a linear transformation of variables and parameters, the coefficients may be taken such that

$$a_{s1k} = 0,\ a_{sk1} = 0,\ s \neq k;\ a_{s1s} = 1;\ a_{ss1} = 1.$$

When these conditions are satisfied, $G$ is said to be in "standard form"; and the equations of $G'$, the reciprocal group, are

$$x'_s = \sum_{i,k} a_{ski} x_i y_k \quad (s,\ i = 1,\ 2,\ \ldots,\ n).$$

2. The characteristic determinant of the general finite operation of $G$ is

$$\begin{vmatrix} \Sigma a_{11k} y_k + \lambda, & \Sigma a_{12k} y_k, & \ldots, & \Sigma a_{1nk} y_k \\ \Sigma a_{21k} y_k, & \Sigma a_{22k} y_k + \lambda, & \ldots, & \Sigma a_{2nk} y_k \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma a_{n1k} y_k, & \Sigma a_{n2k} y_k, & \ldots, & \Sigma a_{nnk} y_k + \lambda \end{vmatrix}.$$

If $G$ is, as supposed, in standard form, the only elements in which $y_1$ occurs are those of the leading diagonal, and in each of these it occurs with the coefficient unity. The characteristic determinant of the general finite operation of the group is therefore the same as the determinant of the finite equations of the group with respect to the $x$'s if in the latter $y_1$ be replaced by $y_1 + \lambda$. Now the determinant of the finite equations of the group

$$x'_s = \Sigma a_{sik} x_i y_k,$$

with respect to the $x$'s, is the same as that of the finite equations

$$x'_s = \Sigma \alpha_{ski} x_i y_k$$

of the reciprocal group, with respect to the $y$'s. If, then, we consider simultaneously $G$ and its reciprocal group $G'$, the characteristic determinant of the general operation of either is obtained from the determinant of the finite equations of the other with respect to the parameters on replacing $x_1$ by $x_1 + \lambda$.

Let $D_x$ represent the determinant of the finite equations of $G$ with respect to the $y$'s, so that

$$D_x = \left| \sum_i \alpha_{sik} x_i \right| ,$$

the term written being the $k^{\text{th}}$ element of the $s^{\text{th}}$ row. If in $D_x$ the variables be transformed by any operation of the group, then

$$D_{x'} = \left| \sum_i \alpha_{sik} \sum_{j,l} \alpha_{ijl} x_j y_l \right| .$$

Using in this the relation

$$\sum_i \alpha_{sik} \alpha_{ijl} = \sum_i \alpha_{ilk} \alpha_{sji},$$

$$D_{x'} = \left| \sum_i \sum_l \alpha_{ilk} y_l \sum_j \alpha_{sji} x_j \right|$$

$$= D_y D_x,$$

by the ordinary rule for the multiplication of determinants, $D_y$ representing the result of writing $y$'s for $x$'s in $D_x$.

The determinant $D_x$ is therefore a homogeneous function of the $x$'s which is changed into a multiple of itself by every operation of $G$.

3. Let now $P_x$ be any homogeneous irreducible function of the $x$'s which has this property, so that

$$P_{x'} = f(y_1, y_2, \ldots, y_n) P_x.$$

In this equation put

$$x_1 = 1, \quad x_2 = x_3 = \ldots = x_n = 0,$$

so that, in virtue of the finite equations of the group,

$$x'_1 = y_1, \quad x'_2 = y_2, \quad \ldots, \quad x'_n = y_n.$$

Then $$P_y = f(y_1, y_2, \ldots, y_n) P_{1,0,\ldots}.$$

If the term of the highest degree in $x_1$ in $P_x$ is wanting, this shows that $P_x$ is zero. We may therefore assume that the coefficient of the term of the highest degree in $x_1$ is unity, and then

$$P_{x'} = P_y P_x.$$

Take $$x_1' = D_x, \quad x_2' = \ldots = x_n' = 0,$$

so that the $y$'s are rational integral functions of the $x$'s of degree $n-1$. If $m$ is the degree of $P_x$, the equation becomes

$$D_x^m = P_y P_x,$$

and therefore $P_x$, being by supposition irreducible, must be a factor of $D_x$.

The relation $$P_{x'} = P_y P_x$$

may be regarded from the point of view of the reciprocal group; for, if in the equations

$$x_s' = \Sigma a_{sik} x_i y_k$$

we regard the $y$'s as the original variables, the $x$'s as the parameters, and the $x'$'s as the transformed variables, the equations are in fact the finite equations of the reciprocal group; and $P_{x'}$ is the result of performing on the $y$'s in $P_y$ any operation of the reciprocal group. A homogeneous function of the variables which is transformed into a multiple of itself by every operation of $G$ has therefore the same property in respect of $G'$.

Lastly, every factor of $D_x$ has the property in question. For let $P_x$ be any factor of $D_x$. Since

$$D_{x'} = D_y D_x,$$

it follows that $$P_{x'} = Q_y R_x,$$

where $Q_y$ and $R_x$ are factors of $D_y$ and $D_x$ respectively. In this equation put

$$x_1 = 1, \quad x_2 = \ldots = x_n = 0,$$

so that $$x_r' = y_r \quad (r = 1, 2, \ldots, n).$$

Then $$P_y = Q_y R_{1,0,\ldots},$$

and therefore $$P_{x'} = P_y R_x \div R_{1,0,\ldots}.$$

Hence $P_y$ is changed into a multiple of itself by every operation of the reciprocal group, and therefore

$$R_x \div R_{1,0,\ldots} = P_x \div P_{1,0,\ldots}.$$

If $D_x'$ is the determinant of the finite equations of the reciprocal group with respect to the parameters, it follows that every irreducible function of the $x$'s which has the property

$$P_{x'} = P_y P_x$$

is a factor of $D_x'$, and that every factor of $D_x'$ has this property. The two determinants $D_x$ and $D_x'$ therefore have the same irreducible factors; and they can only differ, if at all, in the powers in which these factors enter into them.

4. Every function $P_x$ which has the property denoted by the equation

$$P_{x'} = P_y P_x,$$

in respect of the substitutions of $G$, is an invariant of the derived group of $G$. In fact, if $X$ is one of the differential operations of $G$, the result of carrying out the corresponding infinitesimal operation on $P_x$ is $(1+\delta t X)P_x$. For this infinitesimal operation, $P_y$ differs from unity by a quantity proportional to $\delta t$, and therefore

$$XP_x = \lambda P_x,$$

where $\lambda$ is a constant. Similarly, if $Y$ is any other differential operator of $G$,

$$YP_x = \mu P_x,$$

where $\mu$ is a constant; and therefore

$$(XY)P_x = 0.$$

In particular, every factor of $D_x$ is an invariant of the derived groups of both $G$ and $G'$.

5. To avoid interrupting the argument later, it will be convenient here to deal with the case in which $D_x$ has only one distinct invariant factor. In this case $D_x$ and $D_x'$ are identical, and the determinants of the finite equations of $G$ or $G'$ with respect to the variables are also identical, being given by writing $y$'s for $x$'s in $D_x$.

Suppose, first, that this irreducible factor is linear, so that

$$D_x = (x_1 + a_2x_2 + \ldots + a_nx_n)^n.$$

The characteristic equation of the general finite operation of $G$ is then

$$(\lambda + y_1 + a_2y_2 + \ldots + a_ny_n)^n = 0.$$

Take, now, new variables, $\xi$'s, and new parameters, $\eta$'s, linear functions of the old ones, such that

$$\xi_1 = x_1 + a_2x_2 + \ldots + a_nx_n,$$

$$\eta_1 = y_1 + a_2y_2 + \ldots + a_ny_n.$$

The characteristic equation of the general operation of the group is then

$$(\lambda + \eta_1)^n = 0,$$

and the first of its finite equations is

$$\xi_1' = \eta_1\xi_1.$$

Moreover, $\xi_1$ is the only linear function of the variables which is transformed into a multiple of itself by every operation of the group. In the equations thus written put $\xi_1$ and $\xi_1'$ zero. The remaining $n-1$ equations (the first only disappearing) have the group property, and are the finite equations of a linear homogeneous group (not necessarily

transitive) in the $n-1$ variables in which the parameters enter linearly and homogeneously. Moreover, the characteristic equation of the general operation of this group is

$$(\lambda+\eta_1)^{n-1}=0.$$

If, in the finite equations of this group, $\eta_1\xi_r$ be written for $\xi_r'$ for each suffix $r$ from 2 to $n$, one (or more, if the group is intransitive) set of values of the ratios $\xi_2 : \xi_3 : \ldots : \xi_n$ is determined such that the set of values or point $(\xi_2, \xi_3, \ldots, \xi_n)$ is changed into $(\eta_1\xi_2, \eta_1\xi_3, \ldots, \eta_1\xi_n)$ by every one of its operations. Hence, by a suitable change of variables, one or more new variables can be chosen each of which is changed into $\eta_1$ times itself by every operation of this group. If $\xi_2$ be such a symbol, then for every operation of $G$ the symbol replacing $\xi_2$ is of the form $\eta\xi_1+\eta_1\xi_2$, where $\eta$ is a linear function of the parameters.

Repeating this process, it follows that the finite equations of $G$ can be thrown into the form

$$\begin{aligned}
\xi_1' &= \eta_1\xi_1,\\
\xi_2' &= \eta_2\xi_1+\eta_1\xi_2,\\
\xi_3' &= \eta_3\xi_1+a\xi_2+\eta_1\xi_3,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,\\
\xi_n' &= \eta_n\xi_1+b\xi_2+c\xi_3+\ldots+\eta_1\xi_n.
\end{aligned}$$

Since the group is transitive, $\eta_1, \eta_2, \ldots, \eta_n$ must be linearly independent; and, since $\eta_1$ corresponds to the operation which changes every symbol in the same ratio, $a, b, c, \ldots$ are linear functions of $\eta_2, \eta_3, \ldots$. The group thus expressed is in standard form, and the finite equations of $G'$ are therefore obtained by interchanging $\xi$'s and $\eta$'s on the right-hand side (*i.e.*, writing $\xi_p\eta_q$ for $\eta_p\xi_q$). Now, the first $t$ equations of $G$ form a transitive group in $t$ variables, and the first $t$ equations of $G'$ form the reciprocal group. Hence each of these groups is simply transitive, and therefore no parameter with a greater suffix than $t$ can occur in the equation for $\xi_t'$.

Moreover, in the expression for $\xi_t'$ the only term in which $\eta_t$ occurs is $\eta_t\xi_1$, since in the corresponding equation of $G'$ the only term in which $\xi_t$ occurs is $\eta_1\xi_t$. Hence in the group given by the first $t$ equations of $G$ the infinitesimal operation corresponding to $\eta_t$ is self-conjugate.

Suppose, now, that the first $t-1$ equations of $G$ taken alone constitute an Abelian group. Then, since $G$ is its own parameter group,

the sub-group of $G$ generated by the infinitesimal operations corresponding to $\eta_1, \eta_2, \ldots, \eta_{t-1}$ is Abelian. But, in the group constituted by the first $t$ equations of $G$, the operation corresponding to $\eta_t$ is self-conjugate. Hence this group is Abelian. The first two equations of $G$, however, clearly constitute an Abelian group; and therefore, by induction, $G$ itself must be Abelian.

A simply transitive group, whose finite equations are linear and homogeneous in both the variables and the parameters, while its characteristic equation has only one distinct root, is therefore necessarily Abelian.

6. Suppose, secondly, that the irreducible factor of which $D_x$ is a power is not linear, and let

$$P(x_1, x_2, \ldots, x_n)$$

represent it, where $P$ is of the $\mu^{\text{th}}$ degree in the variables, and

$$D_x = P^\nu,$$

so that

$$n = \mu\nu.$$

The characteristic equation of the general operation of $G$ or $G'$ is

$$\{P(y_1+\lambda, y_2, \ldots, y_n)\}^\nu = 1.$$

If

$$P(y_1+\lambda, y_2, \ldots, y_n) \equiv (\lambda+\lambda_1)(\lambda+\lambda_2)\ldots(\lambda+\lambda_\mu),$$

so that $-\lambda_1, -\lambda_2, \ldots, -\lambda_\mu$ are the $\mu$ distinct roots of the characteristic equation, a general substitution of $G'$ can, by properly choosing the variables, be thrown into the form

$$z'_{i1} = \lambda_i z_{i1}, \quad z'_{i2} = \lambda_{i2} z_{i2} + \lambda_i z_{i2}, \quad \ldots, \quad z'_{i\nu} = \lambda_{i,\nu-1} z_{i,\nu-1} + \lambda_i z_{i\nu}$$
$$(i = 1, 2, \ldots, \mu).$$

Every substitution on the $\mu\nu$ variables which is permutable with this, and therefore every substitution of $G$, must transform among themselves each set of $z$'s with the same first suffix. Hence the characteristic determinant of $G$ in its new form consists of the product of $\mu$ distinct factors $f_1(\lambda), f_2(\lambda), \ldots, f_\mu(\lambda)$, rational in $\lambda$ and the parameters, each of which is the characteristic determinant of the group so far as it affects a single set of $z$'s. The characteristic determinant of $G$ being invariant, we have

$$f_1(\lambda) f_2(\lambda) \ldots f_\mu(\lambda) = \left[P(y_1+\lambda, y_2, \ldots, y_n)\right]^\nu.$$

Hence, since $P$ is irreducible, $\mu = \nu$,

and

$$f_1(\lambda) = f_2(\lambda) = \ldots = P(y_1+\lambda, y_2, \ldots, y_n).$$

This second case can therefore only occur when $n$ is a square.

Suppose, next, that any operation of $G$ leaves all the $z$'s of one set unchanged. For such an operation $P(y_1+\lambda, y_2, \ldots, y_n)$ must be $(\lambda+1)^\mu$, and its first invariant factor must be $\lambda+1$; and then each of the $z$'s is unchanged. Hence $G$ is simply isomorphic with the group which arises when we consider the effect of the substitutions of $G$ on one set of the $z$'s only. This is a group of order $\mu^2$, linear and homogeneous in $\mu$ variables. There is but one such group, namely the general linear homogeneous group in $\mu$ variables.

Hence, when the determinant of the finite equations of a simply transitive group in $n$ variables is the power of a single irreducible non-linear factor, $n$ must be a square, and the group is simply isomorphic with the general linear homogeneous group in $\sqrt{n}$ variables.

Further, it is clear from this discussion that, in the case under consideration, the finite equations of the group $G$ may be written in the form

$$\left.\begin{aligned} z'_{i1} &= u_{11}z_{i1}+u_{12}z_{i2}+\ldots+u_{1\mu}z_{i\mu} \\ z'_{i2} &= u_{21}z_{i1}+u_{22}z_{i2}+\ldots+u_{2\mu}z_{i\mu} \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ z'_{i\mu} &= u_{\mu 1}z_{i1}+u_{\mu 2}z_{i2}+\ldots+u_{\mu\mu}z_{i\mu} \end{aligned}\right\} \quad (i=1, 2, \ldots, \mu),$$

the $u$'s being the parameters and $n$ being equal to $\mu^2$. The finite equations of $G'$ in terms of the same variables are

$$\left.\begin{aligned} z'_{1i} &= u_{11}z_{1i}+u_{12}z_{2i}+\ldots+u_{1\mu}z_{\mu i} \\ z'_{2i} &= u_{21}z_{1i}+u_{22}z_{2i}+\ldots+u_{2\mu}z_{\mu i} \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ z'_{\mu i} &= u_{\mu 1}z_{1i}+u_{\mu 2}z_{2i}+\ldots+u_{\mu\mu}z_{\mu i} \end{aligned}\right\} \quad (i=1, 2, \ldots, \mu).$$

The determinant of the finite equations of $G$ with respect to the parameters is the $\mu^{\text{th}}$ power of

$$\begin{vmatrix} z_{11}, & z_{12}, & \ldots, & z_{1\mu} \\ z_{21}, & z_{22}, & \ldots, & z_{2\mu} \\ \ldots & \ldots & \ldots & \\ z_{\mu 1}, & z_{\mu 2}, & \ldots, & z_{\mu\mu} \end{vmatrix},$$

and this is changed into a multiple of itself, as it should be, whether either the variables in every line or the variables in every column are transformed among themselves by any the same linear homogeneous substitution.

## II.

7. It will be convenient to modify slightly the notation used in my former paper, so that the continuous group $G$, which is connected with a given group $g$ of finite order $n$, shall be in what I have called "standard form."

With this end in view, the finite equations of $G$ are taken in the form

$$x'_s = \sum_{i=1}^{i=n} y_{si^{-1}} x_i \quad (s = 1, 2, \ldots, n),$$

where $$y_{si^{-1}} = y_t,$$

if $$S_s S_i^{-1} = S_t;$$

$S_s$, $S_i$, $S_t$ being three of the operations of $g$.

If $G'$ is the reciprocal group, its finite equations are

$$x'_s = \sum_{i=1}^{i=n} y_{i^{-1}s} x_i \quad (s = 1, 2, \ldots, n);$$

and every operation of $G'$ is permutable with every operation of $G$. Moreover, every linear homogeneous substitution on the $n$ symbols which is permutable with every operation of $G$ is contained in $G'$, and conversely. The sub-group $H$ of $G$ constituted by the totality of its self-conjugate operations is the greatest common sub-group of $G$ and $G'$. It is shown in my former paper (pp. 213, 221) that the order of $H$ is $r$, where $r$ is the number of distinct conjugate sets of operations in $g$; and that the finite equations of $H$ are

$$x'_s = \sum_{i=1}^{i=n} e_{si^{-1}} x_i \quad (s = 1, 2, \ldots, n),$$

where the double suffixes of the $e$'s are reduced by the same law as those of the $y$'s, and

$$e_i = e_j,$$

if $S_i$ and $S_j$ are conjugate operations of $g$.

Of the $e$'s only $r$ are independent. The $r$ independent symbols will be taken to be

$$\epsilon_1, \epsilon_2, \ldots, \epsilon_r;$$

and, if $S_i$ belongs to the $k^{\text{th}}$ conjugate set of operations in $g$, when these sets are arranged in some definite sequence to which the sequence of the $\epsilon$'s corresponds, then

$$e_i = \epsilon_k.$$

The first conjugate set will be taken to be that consisting of the identical operation alone, so that

$$e_1 = \epsilon_1.$$

8. The characteristic equation of the general finite operation of $H$ is

$$\begin{vmatrix} e_1+\lambda, & e_{2^{-1}}, & e_{3^{-1}}, & \ldots, & e_{n^{-1}} \\ e_2, & e_1+\lambda, & e_{23^{-1}}, & \ldots, & e_{2n^{-1}} \\ e_3, & e_{32^{-1}}, & e_1+\lambda, & \ldots, & e_{3n^{-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_n, & e_{n2^{-1}}, & e_{n3^{-1}}, & \ldots, & e_1+\lambda \end{vmatrix} = 0.$$

The determinant on the left-hand side of this equation will be denoted by $D_{e,\lambda}$, and the corresponding determinant when $\lambda$ is zero by $D_e$; so that $D_{e_i\lambda}$ is the result of writing $e_1+\lambda$ for $e_1$ in $D_e$.

It is shown in my paper "On Linear Continuous Homogeneous Groups whose Operations are Permutable"* that the characteristic determinant of the general finite operation of an Abelian group which is linear and homogeneous both in the variables and the parameters is the product of factors which are linear and homogeneous in $\lambda$ and the parameters. We may therefore write

$$D_{e,\lambda} = \prod_{i=1}^{i=s} (\lambda+\epsilon_1+a_{i2}\epsilon_2+\ldots+a_{ir}\epsilon_r)^{m_i},$$

where

$$\sum_{i=1}^{i=s} m_i = n.$$

If $n_k$ is the number of operations in the $k^{\text{th}}$ conjugate set of $g$, the symbol $\epsilon_k$ will occur $n_k$ times in each line of $D_e$, and, therefore, $nn_k$ times in all. Now it may be shown, and the result is most important for what follows, that the minor corresponding to $\epsilon_k$ is in each place the same. To prove this, let $T$ be the operation of $H$ whose parameters

are

$$e_1, e_2, \ldots, e_n;$$

and let

$$\eta_1, \eta_2, \ldots, \eta_n$$

be the parameters of $T^{-1}$.

Since $T^{-1}T$ is the identical operation, the $e$'s and $\eta$'s are connected by the relations

$$\sum_{s=1}^{s=n} \eta_{si^{-1}} e_{is^{-1}} = 1,$$

$$\sum_{s=1}^{s=n} \eta_{si^{-1}} e_{js^{-1}} = 0 \quad (i \neq j),$$

for all values of $i$ and $j$.

* *Proc. Lond. Math. Soc.*, Vol. xxix., p. 332; where other references are given.

Hence $\eta_{si^{-1}}$ is the minor of $e_{is^{-1}}$ in the determinant

$$\begin{vmatrix} e_{11^{-1}}, & e_{12^{-1}}, & e_{13^{-1}}, & \ldots, & e_{1n^{-1}} \\ e_{21^{-1}}, & e_{22^{-1}}, & e_{23^{-1}}, & \ldots, & e_{2n^{-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_{n1^{-1}}, & e_{n2^{-1}}, & e_{n3^{-1}}, & \ldots, & e_{nn^{-1}} \end{vmatrix},$$

where the element $e_{is^{-1}}$ is to be picked out before the double suffixes are simplified; and, after simplifying, the $e$'s may be replaced by the equivalent $\epsilon$'s. But, $T^{-1}$ being an operation of $H$, $\eta_{si^{-1}}$ depends only on the conjugate set in $g$ to which $S_s S_i^{-1}$ belongs. Hence, in the above determinant, which is $D_e$, the minors of $e_{is^{-1}}$ and $e_{jt^{-1}}$ are the same if $S_i S_s^{-1}$ and $S_j S_t^{-1}$ belong to the same conjugate set in $g$. In other words, when in $D_e$ the $e$'s are replaced by $\epsilon$'s, the minor of $\epsilon_k$ is the same in every place in which it occurs. Let $M_k$ denote the minor of $\epsilon_k$. Then, by differentiating the determinant,

$$M_k = \frac{1}{nm_k} \frac{\partial D_e}{\partial \epsilon_k}.$$

Hence, $m_i$ being the order to which any linear factor

$$\lambda + \epsilon_1 + a_{i2}\epsilon_2 + \ldots + a_{ir}\epsilon_r$$

enters in $D_{e,\lambda}$, the same factor enters to the order $m_i - 1$ in every first minor of $D_{e,\lambda}$. The first invariant-factor of $D_{e,\lambda}$ therefore contains every linear factor of the determinant, and each one in the first degree. I have shown (*loc. cit.*, p. 347) that, where this is the case, $n$ independent linear functions of the variables can be chosen so that each of them is changed into a multiple of itself by every operation of the group; the multipliers being the roots of the characteristic equation taken negatively, each occurring as often as the corresponding root. The order of such a group is obviously equal to the number of distinct multipliers. Now the order of $H$ is $r$. Hence $s = r$, and the equation

$$D_{e,\lambda} = 0$$

has just $r$ distinct roots. Let these roots be

$$-\eta_1, \ -\eta_2, \ \ldots, \ -\eta_r,$$

so that $$D_{e,\lambda} = (\lambda + \eta_1)^{m_1} (\lambda + \eta_2)^{m_2} \ldots (\lambda + \eta_r)^{m_r},$$

where $$\sum_{i=1}^{i=r} m_i = n.$$

Then the result of this discussion is that $n$ independent linear functions of the $x$'s

$$\xi_{ij} \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m_i),$$

may be chosen so that the finite equations of $H$ take the form

$$\xi'_{ij} = \eta_i \xi_{ij} \quad (j = 1, 2, \ldots, m_i;\ i = 1, 2, \ldots, r).$$

9. Suppose, next, that the finite equations of $G$ and $G'$ are expressed in terms of the $\xi$'s. Since every operation of either $G$ or $G'$ is permutable with every operation of $H$, every operation of $G$ and of $G'$ must transform among themselves each set of $\xi$'s with the same first suffix. Let $G_i$ and $G'_i$ be the groups to which $G$ and $G'$ reduce when we consider only their effect on the set of $m_i$ symbols

$$\xi_{ij} \quad (j = 1, 2, \ldots, m_i).$$

Since the finite equations of $G$ and $G'$ contain the parameters linearly and homogeneously, the same must be true of the finite equations of $G_i$ and $G'_i$; and, since $G$ and $G'$ are transitive, $G_i$ and $G'_i$ must be transitive in the $m_i$ symbols on which they operate. Also, since every operation of $G$ is permutable with every operation of $G'$, the same is true of $G_i$ and $G'_i$. Hence $G_i$ and $G'_i$ must be simply transitive in the $m_i$ symbols on which they operate; in other words, they are each of order $m_i$. Since $G_i$ is a simply transitive group in $m_i$ variables whose finite equations are linear and homogeneous both in the variables and parameters, these finite equations can be written in the form

$$\xi'_{is} = \sum_{j,k}^{1, \ldots, m_i} a_{sjk} \xi_{ij} \eta_{ik} \qquad (s = 1, 2, \ldots, m_i),$$

where $\eta_{ik}$ $(k = 1, 2, \ldots, m_i)$, the $m_i$ independent parameters, are independent linear functions of the original $y$'s. Moreover, by carrying out, if necessary, a linear substitution on the $\xi$'s, which leaves the form of $H$ unaltered, these equations may be brought to standard form; and the finite equations of $G'_i$ are then

$$\xi'_{is} = \sum_{j,k}^{1, \ldots, m_i} a_{skj} \xi_{ij} \eta_{ik} \qquad (s = 1, 2, \ldots, m_i).$$

Consider, now, the $n$ linear functions

$$\eta_i \quad (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, m_i)$$

of the original parameters thus introduced. If they are not independent, the finite equations of $G$, when expressed in terms of the $\xi$'s,

would contain fewer than $n$ independent parameters. This is impossible, since $G$ is a group of order $n$. Hence the $n$ new parameters are independent, and therefore the group $G_i$, to which $G$ reduces when we consider only its effect on the symbols $\xi_{ik}$ $(k = 1, 2, \ldots, m_i)$, is a sub-group of $G$. Moreover, this sub-group and the sub-group $G_j$, which affects only the symbols $\xi_{jk}$ $(k = 1, 2, \ldots, m_j)$, have no operation in common except identity; and, since the symbols affected by the two sub-groups are distinct, every operation of one is permutable with every operation of the other. It follows, then, that $G$ is the direct product of

$$G_1,\ G_2,\ \ldots,\ G_r.$$

For each suffix $i$, the self-conjugate operations of $G_i$ form a group of order 1. In fact, the self-conjugate operations of $G_i$ are also self-conjugate operations of $G$; hence, if the self-conjugate operations of $G_i$ formed a sub-group whose order were greater than 1, the self-conjugate operations of $G$ would form a sub-group of order greater than $r$.

10. Consider, next, the determinant of the finite equations of $G_i$ with respect to the parameters. It, and each of its irreducible factors, is an invariant of the derived group of $G_i$, and, therefore, also of the derived group of $G$. The $r$ determinants of the $r$ groups $G_i$ $(i = 1, 2, \ldots, r)$ therefore give $r$ invariants of the derived group of $G$; and, since the variables that enter into any one of these are distinct from those which enter into any other, these $r$ invariants are independent. Now the derived group of $G$, being a sub-group of order $n-r$ of a transitive group of order $n$ in $n$ variables, cannot have more than $r$ independent invariants. Hence the determinant of the finite equations of $G_i$, with respect to the parameters, must be the power of a single irreducible factor for every value of $i$. For, if it contained two or more irreducible factors, each of them would be invariants of the derived group of $G$, and at least two of them would be independent; this would, however, make the number of independent invariants of the derived group of $G$ greater than $r$. Now the self-conjugate operations of $G_i$ form a group of order 1. Hence, if $m_i$ is greater than 1, the irreducible factor in question cannot be linear (§ 5). It follows therefore (§ 6) that, if $m_i$ is greater than unity, it must be a square, and $G_i$ is simply isomorphic with the general linear homogeneous group in $\sqrt{m_i}$ variables.

As regards the continuous group $G$ of order $n$ in $n$ variables, which is defined by the group $g$ of finite order $n$ containing $r$ distinct sets of

conjugate operations, the result of the present section may be summarized thus:—

*The continuous group $G$ is the direct product of $r$ groups simply isomorphic with the general linear homogeneous groups in $\mu_1 (=1)$,* $\mu_2, \ldots, \mu_r$ variables, where the integers $\mu_1, \mu_2, \ldots$ are such that $\sum_1^r \mu_s^2 = n$.*

As regards the group $g$ of finite order $n$, the results may also be stated as follows:—

*The determinant formed from the multiplication table of $g$, when the symbols of the operations are regarded as $n$ independent algebraical quantities, contains $r$ distinct irreducible factors. Each of these factors enters into the determinant to a power equal to its degree†; and any irreducible factor of degree $m$ is expressible in the form*

$$\begin{vmatrix} z_{11}, & z_{12}, & \ldots, & z_{1m} \\ z_{21}, & z_{22}, & \ldots, & z_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ z_{m1}, & z_{m2}, & \ldots, & z_{mm} \end{vmatrix},$$

*where the $z$'s are independent linear functions of the $n$ algebraical quantities.*

11. Let $P_x$ be an irreducible factor of $D_x$ whose degree is $\mu_i$. Then it has been seen (§ 6) that $\mu_i^2$ independent linear functions of the $x$'s can be formed such that

$$P_x = \begin{vmatrix} \xi^i_{11}, & \xi^i_{12}, & \ldots, & \xi^i_{1\mu_i} \\ \xi^i_{21}, & \xi^i_{22}, & \ldots, & \xi^i_{2\mu_i} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \xi^i_{\mu_i 1}, & \xi^i_{\mu_i 2}, & \ldots, & \xi^i_{\mu_i \mu_i} \end{vmatrix},$$

while the functions in each line (or column) of the determinant are transformed among themselves by every operation of $G$ (or $G'$).

Now, $$P_{x'} = P_y P_x;$$

hence, if $\eta^i_{pq}$ $(p, q = 1, 2, \ldots, \mu_i)$ denote the same $\mu_i^2$ linear functions

* That one at least of the $\mu$'s must be unity follows from noticing that the determinant $D_y$ obviously contains the linear factor $y_1 + y_2 + \ldots + y_n$.

† Frobenius: *Berliner Sitzungsberichte*, 1896, pp. 1363, 1372.

of the $y$'s that the $\xi^i$'s are of the $x$'s, the $\eta^i$'s may be taken as the independent parameters of $G_i$, and the finite equations of $G_i$ take the form

$$\left.\begin{aligned}
\xi'^i_{s1} &= \eta^i_{11}\,\xi^i_{s1} + \eta^i_{12}\,\xi^i_{s2} + \ldots + \eta^i_{1\mu_i}\,\xi^i_{s\mu_i} \\
\xi'^i_{s2} &= \eta^i_{21}\,\xi^i_{s1} + \eta^i_{22}\,\xi^i_{s2} + \ldots + \eta^i_{2\mu_i}\,\xi^i_{s\mu_i} \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\xi'^i_{s\mu_i} &= \eta^i_{\mu_i 1}\xi^i_{s1} + \eta^i_{\mu_i 2}\xi^i_{s2} + \ldots + \eta^i_{\mu_i\mu_i}\xi^i_{s\mu_i}
\end{aligned}\right\} (s = 1, 2, \ldots, \mu_i).$$

When the $y$'s are replaced by $e$'s, these equations reduce to

$$\xi'^i_{pq} = \eta_i \xi^i_{pq} \quad (p, q = 1, 2, \ldots \mu_i);$$

and therefore the $\eta^i$'s are such that, on replacing $y$'s by $e$'s,

$$\eta^i_{pq} = 0 \quad (p \neq q),$$

$$\eta^i_{pp} = \eta_i.$$

The $n$ operations of $G$ given by

$$y_1 = \ldots = y_{k-1} = y_{k+1} = \ldots = y_n = 0,\ y_k = 1 \quad (k = 1, 2, \ldots, n)$$

constitute a discontinuous sub-group of finite order $n$ simply isomorphic with $g$. Hence the distinct operations of $G_i$ which correspond to these values of the parameters constitute a discontinuous sub-group $g_i$ of $G_i$ with which $g$ is isomorphic. If the isomorphism is not simple, let

$$S_1,\ S_2,\ \ldots,\ S_m$$

constitute the self-conjugate sub-group of $g$ which corresponds to the identical operation of $g_i$. Then

$$S_t S_1,\ S_t S_2,\ \ldots,\ S_t S_m$$

correspond to the same operation of $g_i$, and therefore in the $\eta^i$'s the symbols

$$y_{t1},\ y_{t2},\ \ldots,\ y_{tm}$$

occur only in the linear combination

$$y_{t1} + y_{t2} + \ldots + y_{tm}$$

for each value of $t$. The $\eta^i$'s are therefore linear functions of $\frac{n}{m}$ linear functions of the $y$'s; and, since they are independent and $\mu_i^2$ in number,

$$\frac{n}{m} \geqslant \mu_i^2,$$

*i.e.*,

$$m \leqslant \frac{n}{\mu_i^2}.$$

Hence, unless $g$ has a self-conjugate sub-group whose index is equal to or greater than $\mu_i^2$, $g_i$ must be simply isomorphic with $g$. In particular, if $g$ is simple, $g_i$ is simply isomorphic with $g$, except when $\mu_i$ is unity; in this case $g_i$ clearly consists of the identical operation only. Again, when $g$ contains a simple self-conjugate sub-group of index 2, $g_i$ and $g$ are simply isomorphic unless $\mu_i$ is unity.

Now $g$, and the groups of smaller order with which $g$ is isomorphic, can, in general, be represented as groups of linear substitutions in a variety of ways, and the question arises as to which of these different modes of representation occur among the groups $g_i$. This question is best considered by approaching it from a rather different point of view.

12. Let $$S_1 (=1),\ S_2,\ \ldots,\ S_\nu$$ be the operations of a group $g'$ of finite order $\nu$, which can be represented as a group of linear substitutions performed on the $m$ symbols

$$x_1,\ x_2,\ \ldots,\ x_m;$$

and let $$x_s' = \sum_{i=1}^{i=m} a_{sik} x_i \quad (s = 1, 2, \ldots, m)$$

be the substitution corresponding to the operation $S_k$ of $g'$.

Then, if $$S_k S_l = S_p,$$

$$\sum_{j=1}^{j=m} a_{sjl} a_{jik} = a_{sip}$$

for all values of $s$ and $i$.

Let $Y$ and $Z$ represent the two linear substitutions

$$\left.\begin{aligned} x_s' &= \sum_{i,k} a_{sik} x_i y_k \\ \text{and} \quad x_s' &= \sum_{i,k} a_{sik} x_i z_k \end{aligned}\right\} \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, \nu \end{pmatrix};$$

where $$y_1,\ y_2,\ \ldots,\ y_\nu,$$

$$z_1,\ z_2,\ \ldots,\ z_\nu,$$

are any two sets of constants.

The linear substitution $YZ$ is given by the equations

$$\begin{aligned} x_s' &= \sum_{j,l} a_{sjl} y_l \sum_{i,k} a_{jik} x_i z_k \\ &= \sum_{i,l,k} x_i y_l z_k \sum_{j} a_{sjl} a_{jik} \\ &= \sum_{i,l,k} a_{sip} x_i y_l z_k, \end{aligned}$$

where the suffix $p$ is determined by the relation

$$S_k S_l = S_p.$$

If, then, we write $$w_p = \Sigma y_l z_k,$$

where the summation is extended to all pairs of suffixes such that

$$S_k S_l = S_p,$$

the linear substitution $YZ$ is given by the equations

$$x'_s = \Sigma a_{sip} x_i w_p.$$

The relation of the $w$'s to the $y$'s and $z$'s may be more conveniently represented by means of the notation with multiple suffixes already used, such that

$$y_{ab\ldots d} = y_e$$

and $$z_{ab\ldots d} = z_e,$$

if $$S_a S_b \ldots S_d = S_e.$$

With this notation the $w$'s are given in terms of the $y$'s and $z$'s by the equations

$$w_p = \sum_{l=1}^{l=\nu} z_{pl^{-1}} y_l = \sum_{k=1}^{k=\nu} z_k y_{k^{-1}p} \quad (p = 1, 2, \ldots, \nu).$$

These are the finite equations of the continuous group $G'$ of order $\nu$ in $\nu$ variables, defined by the multiplication table of the group $g'$. The totality of the linear substitutions of the form $Y$, when the $y$'s are regarded as arbitrary parameters, form therefore a continuous group $\Gamma'$, with which $G'$ is isomorphic.

The isomorphism between $G'$ and $\Gamma'$ is clearly not necessarily, nor in general, simple. In fact, the order of $\Gamma'$, a linear homogeneous group in $m$ variables, cannot exceed $m^2$, and there is no such limitation on $\nu$.

13. Suppose, next, that $g$ is a continuous group of order $n\,(=\mu\nu)$, and that $g$ contains a self-conjugate sub-group $h$ of order $\mu$, in respect of which it is isomorphic with $g'$. If

$$S_{\alpha_1}\,(=1),\ S_{\alpha_2},\ \ldots,\ S_{\alpha_\mu}$$

are the operations of $h$, a set of $\nu$ operations

$$S_1\,(=1),\ S_2,\ \ldots,\ S_\nu$$

may be chosen so that the totality of the operations of $g$ are represented by the $\nu$ sets

$$S_t S_{\alpha_1},\ S_t S_{\alpha_2},\ \ldots,\ S_t S_{\alpha_\mu} \quad (t = 1, 2, \ldots, \nu),$$

while these sets combine among themselves in the same way as the operations of $g'$.

Using the same suffix notation as before, consider the linear substitution $Y$, or

$$x'_s = \Sigma \beta_{si(ta_r)} x_i y_{ta_r} \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ t = 1, 2, \ldots, \nu \\ r = 1, 2, \ldots, \mu \end{pmatrix},$$

where $$\beta_{si(ta_r)} = \alpha_{sit}.$$

It contains, formally, $n$ arbitrary parameters corresponding to the $n$ operations of $g$. If the double suffixes $ta_r$ are replaced by single suffixes extending from 1 to $n$, $Y$ may be written in the form

$$x'_s = \Sigma \beta_{sik} x_i y_k \quad \begin{pmatrix} s, i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, n \end{pmatrix},$$

and, exactly as in the former case, it may be verified that $YZ$ is

$$x'_s = \Sigma \beta_{sik} x_i w_k,$$

where $$w_k = \sum_{l=1}^{i=n} z_{kl^{-1}} y_l \quad (k = 1, 2, \ldots, n).$$

The set of equations $$x'_s = \Sigma \beta_{sik} x_i y_k$$

therefore define a continuous group $\Gamma$ with which the continuous group $G$, of order $n$, formed from the multiplication table of $g$, is isomorphic.

Let $$\Delta_y = \left| \sum_k \beta_{sik} y_k \right|$$

denote the determinant of the general finite operation of $\Gamma$. The only inadmissible systems of values of the parameters are those given by equating $\Delta_y$ to zero. But, since $G$ is isomorphic with $\Gamma$, these systems of values must all occur among those obtained by equating $D_y$ to zero. The irreducible factors of $\Delta_y$ are therefore factors of $D_y$, and we may write

$$\Delta_y = P_{a_1}^{\nu_1} P_{a_2}^{\nu_2} \ldots P_{a_t}^{\nu_t},$$

where $$P_{a_1}, P_{a_2}, \ldots, P$$

are irreducible factors of $D_y$ of degrees

$$\mu_{a_1}, \mu_{a_2}, \ldots, \mu_a,$$

which correspond to the self-conjugate sub-groups of $G$,

$$G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_t},$$

whose orders are $$\mu^2_{\alpha_1}, \mu^2_{\alpha_2}, \ldots, \mu^2_{\alpha_t}.$$

If $P_i$ is an irreducible factor of $D_y$ which does not occur in $\Delta_y$, and $G_i$ the corresponding self-conjugate sub-group of $G$ of order $\mu_i^2$, then to every operation of $G_i$ must correspond the identical operation of $\Gamma$. On the other hand, no operation of $G_{\alpha_s}$, except the identical operation, can correspond to the identical operation of $\Gamma$; for the characteristic determinant of the general finite operation of $\Gamma$ contains that of $G_{\alpha_s}$ as a factor. Hence $\Gamma$ is simply isomorphic with the direct product of $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_t}$.

The characteristic equation of the most general self-conjugate operation of $\Gamma$ will then have $t$ distinct roots, and therefore the $m$ variables of $\Gamma$, or suitably chosen linear functions of them which may be taken as new variables, are capable of being divided into $t$ sets, such that the variables of each set are transformed among themselves by all the operations of $\Gamma$.

The numbers of variables in the $t$ sets are clearly

$$\nu_1\mu_{\alpha_1}, \nu_2\mu_{\alpha_2}, \ldots, \nu_t\mu_{\alpha_t}.$$

Further, since the finite equations of $\Gamma$ are linear in the parameters, the $\nu_s\mu_{\alpha_s}$ symbols of the $s^{\text{th}}$ set may be replaced by $\nu_s$ new sets, of $\mu_{\alpha_s}$ symbols each, such that the symbols of each set are transformed among themselves by all the operations of $\Gamma$.

This completes the analysis of the group $\Gamma$, and affords a partial answer to the question suggested at the end of § 11, which may be stated thus :—If a discontinuous group $g'$, with which $g$ is simply or multiply isomorphic, can be represented as a group of linear substitutions performed on $m$ symbols, and if it is impossible, by choosing new variables, to divide the variables into sets such that those of each set are transformed among themselves by every operation of $g'$, then $m$ must be one of the integers $\mu_s$ defined in § 10.

*On the General Theory of Anharmonics.* *By* Prof. Edgar O. Lovett.

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It is the object of this note to show how Lie's theory of the projective group may be made to yield Clifford's statement* of the general theory of anharmonics.

A geometrical configuration depending on $m \,.\, n$ coordinates, say a system of $m$ points in a space of $n$ dimensions, has at least $mn-r$ absolute invariants under an $r$-parameter Lie group of point transformations; these $mn-r$ invariant functions, and all others, are solutions of the complete system of simultaneous partial differential equations

$$\sum_1^m i(X_1 f)_i = 0, \quad \sum_1^m i(X_2 f)_i = 0, \quad \ldots, \quad \sum_1^m i(X_r f)_i = 0, \tag{1}$$

where the

$$X_k f = \sum_1^n i \xi_{ik}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} \quad (k = 1, \ldots, r) \tag{2}$$

are the $r$ independent infinitesimal point transformations which generate the $r$-parameter continuous group, and $(X_k f)_j$ represents the result of replacing $x_1, \ldots, x_n$ by $x_1^{(j)}, \ldots, x_n^{(j)}$, respectively, in $(X_k f)$,

$$x_1^{(j)}, \; x_2^{(j)}, \; \ldots, \; x_n^{(j)} \quad (j = 1, 2, \ldots, m) \tag{3}$$

being the $mn$ independent coordinates which determine the system of points.

1. Consider the general projective group of the $xy$-plane, an 8-parameter Lie group generated by the following independent infinitesimal point transformations—

$$\frac{\partial f}{\partial x}, \; \frac{\partial f}{\partial y}, \quad x\frac{\partial f}{\partial y}, \; y\frac{\partial f}{\partial x}, \quad x\frac{\partial f}{\partial x}, \; y\frac{\partial f}{\partial y}, \quad x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y}, \quad xy\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}. \tag{4}$$

* Clifford: "On the General Theory of Anharmonics," *Proc. Lond. Math. Soc.*, Vol. II.

A plane hexagon regarded as a system of six points depending upon the twelve coordinates, $x_1, y_1, \ldots, x_6, y_6$, possesses at least four independent invariants under the above group. These and all others are found by integrating the complete system of partial differential equations

$$\left.\begin{aligned} &\sum_1^6 i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i x_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i y_i\frac{\partial\phi}{\partial x_i} = 0, \\ &\sum_1^6 i x_i\frac{\partial\phi}{\partial x_i} = 0, \quad \sum_1^6 i y_i\frac{\partial\phi}{\partial y_i} = 0, \quad \sum_1^6 i\left(x_i^2\frac{\partial\phi}{\partial x_i} + x_i y_i\frac{\partial\phi}{\partial y_i}\right) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sum_1^6 i\left(x_i y_i\frac{\partial\phi}{\partial x_i} + y_i^2\frac{\partial\phi}{\partial y_i}\right) = 0 \end{aligned}\right\}. \quad (5)$$

That this system of partial differential equations possesses no more than four independent solutions is readily seen by constructing the matrix

$$\left\|\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & y_1^2 & y_2^2 & y_3^2 & y_4^2 \end{array}\right|, \quad (6)$$

formed by the coefficients of the $\frac{\partial\phi}{\partial x_i}$ and $\frac{\partial\phi}{\partial y_i}$, and observing that all eighth order determinants of this matrix do not vanish; for example, the determinants composed of the first eight columns.

The first and second equations of the system (5) show that the

invariant function, $\phi$, is a function of the variables

$$u_{i-1} = x_i - x_1, \quad v_i = y_i - y_1 \quad (i = 2, 3, \ldots, 6). \quad (7)$$

The introduction of these new variables in the third, fourth, fifth, and sixth equations of the system reduces them to the forms, respectively,

$$\sum_1^5 i u_i \frac{\partial \phi}{\partial v_i} = 0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial u_i} = 0, \quad \sum_1^5 i u_i \frac{\partial \phi}{\partial u_i} = 0, \quad \sum_1^5 i v_i \frac{\partial \phi}{\partial v_i} = 0. \quad (8)$$

The first and second of these equations demand that $\phi$ shall be a function of the determinants

$$w_{ij} = | u_i, v_j | , \quad (9)$$

where $ij$ takes successively the values of the ten combinations of the five integers 1, 2, 3, 4, 5, taken two at a time. On introducing the variables (9) in the third and fourth equations of (8), we find the form

$$\Sigma w_{ij} \frac{\partial \phi}{\partial w_{ij}} = 0, \quad (10)$$

which requires that the function $\phi$ be a function of the ratios of the determinants $w_{ij}$. Finally, by direct substitution in the transformed forms of the seventh and eighth equations of the system (5), we have the result that the following nine functions are solutions of the system, and hence invariant under the group (4),

$$\frac{w_{\alpha\beta}(w_{\lambda\mu} + w_{\mu\nu} + w_{\nu\lambda})}{w_{ab}(w_{lm} + w_{mn} + w_{nl})}, \quad (11)$$

where $(a, b, l, m, n) = (1, 2, 3, 4, 5)$,

and $(\alpha\beta\lambda\mu\nu)$ is any other of the permutations of the five integers 1, 2, 3, 4, 5.

These invariants are readily interpreted geometrically; thus, for example, the expression

$$\frac{w_{13}(w_{24} + w_{45} + w_{52})}{w_{ab}(w_{lm} + w_{mn} + w_{nl})}, \quad (12)$$

when simplified, becomes successively

$$\frac{|\,u_1, v_3\,|\,\{\,|\,u_2, v_4\,|\,+\,|\,u_4, v_5\,|\,+\,|\,u_5, v_2\,|\,\}}{|\,u_1, v_2\,|\,\{\,|\,u_3, v_4\,|\,+\,|\,u_4, v_5\,|\,+\,|\,u_5, v_3\,|\,\}}$$

$$= \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_4-x_1 & y_4-y_1 \end{vmatrix} \cdot \begin{vmatrix} x_3-x_1 & y_3-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix} \cdot \begin{vmatrix} x_4-x_1 & y_4-y_1 & 1 \\ x_5-x_1 & y_5-y_1 & 1 \\ x_6-x_1 & y_6-y_1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \end{vmatrix}$$

$$= \frac{124.356}{123.456}, \tag{13}$$

where 123 means the area of the triangle formed by the points 1, 2, 3. The other invariants are composed of corresponding terms formed from the twenty triangles made by the original six points. These nine invariants are not independent, since they may be shown to satisfy five independent identities of the form

$$123.456 - 124.563 + 125.634 - 126.345 \equiv 0. \tag{14}$$

To prove this statement, we first establish the identity (14) by observing that, since $ijk = -kji$, the left-hand member is identical with the expression

$$123.456 + 124.365 + 125.634 + 126.543, \tag{15}$$

which, in terms of the quantities $w_{ij}$, in virtue of (12) and (13), may be written

$$\begin{aligned} &w_{12}(w_{34}+w_{45}+w_{53}) + w_{13}(w_{25}+w_{54}+w_{42}) \\ &+ w_{14}(w_{23}+w_{35}+w_{52}) + w_{15}(w_{43}+w_{32}+w_{24}), \end{aligned} \tag{16}$$

or

$$\begin{aligned} &(w_{12}w_{34}+w_{23}w_{14}+w_{31}w_{24}) + (w_{12}w_{45}+w_{24}w_{15}+w_{41}w_{25}) \\ &+ (w_{12}w_{53}+w_{25}w_{13}+w_{51}w_{23}) + (w_{13}w_{54}+w_{35}w_{14}+w_{51}w_{34}), \end{aligned} \tag{17}$$

which, by noting the definition (9) of the $w_{ij}$'s, is zero, by virtue of the identity

$$| a_1, b_2 | . | a_3, b_4 | + | a_2, b_3 | . | a_1, b_4 | + | a_3, b_1 | . | a_2, b_4 | \equiv 0. \tag{18}$$

In the second place, to determine the number of independent identities of the form (14), we remark that the demonstration of (14) just given demonstrates simultaneously the following system of identities derived by permutation:

$$\left.\begin{aligned}
&\left.\begin{aligned}
&123.456-124.563+125.634-126.345=0,\\
&124.563-125.634+126.345-123.456=0,\\
&125.634-126.345+123.456-124.563=0,\\
&126.345-123.456+124.563-125.634=0;
\end{aligned}\right\}\\
&\left.\begin{aligned}
&134.562-135.624+136.245-132.456=0,\\
&135.624-136.245+132.456-134.562=0,\\
&136.245-132.456+134.562-135.624=0;
\end{aligned}\right\}\\
&\left.\begin{aligned}
&145.623-146.235+142.356-143.562=0,\\
&146.235-142.356+143.562-145.623=0;
\end{aligned}\right\}\\
&156.234-152.346+153.462-154.623=0;\\
&\left.\begin{aligned}
&234.561-235.614+236.145-231.456=0,\\
&235.614-236.145+231.456-234.561=0,\\
&236.145-231.456+234.561-235.614=0;
\end{aligned}\right\}\\
&\left.\begin{aligned}
&245.613-246.135+241.356-243.561=0,\\
&246.135-241.356+243.561-245.613=0;
\end{aligned}\right\}\\
&256.134-251.346+253.461-254.613=0;\\
&\left.\begin{aligned}
&345.612-346.125+341.256-342.561=0,\\
&346.125-341.256+342.561-345.612=0;
\end{aligned}\right\}\\
&356.124-351.246+352.461-354.612=0;\\
&456.123-451.236+452.361-453.612=0
\end{aligned}\right\}. \tag{19}$$

By designating the first ten terms in the left-hand column from the top down or the bottom up by $a$, $b$, $c$, ..., $j$, respectively, and observing

that the first four equations are identical, the next three, and so on, the system (19) reduces to the following system of ten identities:

$$a-b+c-d=0, \tag{20}$$
$$a-e+f-g=0, \tag{21}$$
$$b-e+h-i=0, \tag{22}$$
$$c-f+h-j=0, \tag{23}$$
$$a+h-i+j=0, \tag{24}$$
$$b+f-g+j=0, \tag{25}$$
$$c+e-g+i=0, \tag{26}$$
$$c-d+e+j=0, \tag{27}$$
$$b-d+f+i=0, \tag{28}$$
$$a-d+g+h=0. \tag{29}$$

Of these, the first five are clearly independent; (21), (22), and (24) give (25); (22), (23), and (25) give (26); (20), (21), and (25) give (27); (22), (23), and (27) give (28); (21), (22), and (28) give (29); or, in other words, there exist the following five independent relations connecting the left-hand members of the above identities, viz.,

$$\left.\begin{aligned}(21)+(22)=(24)+(25)=(28)+(29),\\ (22)+(23)=(25)+(26)=(27)+(28),\\ (20)+(25)=(21)+(27)\end{aligned}\right\} \tag{30}$$

Hence, there are five, and but five, independent identities of the form (14) connecting the areas of triangles formed by the six points 1, 2, 3, 4, 5, 6 in a plane. Accordingly, Clifford's statement,* "There

* *Vide loc. cit.*, § 3.

It is interesting to remark here that, by taking the corrected form (14) of the identity (*a*) and permuting cyclically, we obtain the following six identities:

$$\left.\begin{aligned}123.456+124.365+125.634+126.543=0,\\ 234.561+235.416+236.145+231.654=0,\\ 345.612+346.521+341.256+342.165=0,\\ 456.123+451.632+452.361+453.216=0,\\ 561.234+562.143+563.412+564.321=0,\\ 612.345+613.254+614.523+615.432=0\end{aligned}\right\}. \tag{*b*}$$

The sum of the left-hand members of the first, third, and fifth of these identities is

are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123.456+124.563+125.634+126.345\equiv 0, \qquad (a)$$

with five others obtained from this by permutation," requires modification with regard both to the form of the identities and their number.

To return to the original point, the independent absolute invariants of the system of six points in the plane under the general projective group of the plane are thus determined both in form and number.

2. A system of eight points in ordinary space is determined by twenty-four coordinates. The eight points form seventy different tetrahedra. The volumes of these tetrahedra arrange themselves in pairs in thirty-five different products to satisfy twenty-five independent identical relations of the form

$$1234.5678+1235.6784+1236.7845+1237.8456+1238.4567=0. \quad (31)$$

This identity is established in a manner wholly analogous to the method of proving the identity (14) in the plane: the identity is a direct consequence of the determinantal relation

$$\begin{aligned}&|\,a_1, b_2, c_3\,|\,.\,|\,a_4, b_5, c_6\,| - |\,a_2, b_3, c_4\,|\,.\,|\,a_1, b_5, c_6\,|\\ &\quad + |\,a_3, b_4, c_1\,|\,.\,|\,a_2, b_5, c_6\,| - |\,a_4, b_1, c_2\,|\,.\,|\,a_3, b_5, c_6\,| = 0, \qquad (32)\end{aligned}$$

which itself results immediately from the identity already used

$$|\,a_1, b_2,\,|\,.\,|\,a_3, b_4\,| + |\,a_2, b_3\,|\,.\,|\,a_1, b_4\,| + |\,a_3, b_1\,|\,.\,|\,a_2, b_4\,| = 0.$$

The general projective group of ordinary space is a 15-parameter Lie group generated by the following fifteen independent infinitesimal transformations:

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zero, which is also the sum of the left-hand members of the second, fourth, and sixth; hence but four are independent. They contain all possible products except 135.624; if we write down the identity involving this term, and permute cyclically, the system formed of these six identities and of the six (*b*) is found to reduce to eight different identities among which three relations appear, leaving but five independent as before.

$$\left.\begin{array}{c} p \equiv \dfrac{\partial f}{\partial x}, \quad q \equiv \dfrac{\partial f}{\partial y}, \quad r \equiv \dfrac{\partial f}{\partial z}, \\ xp,\ yp,\ zp,\ \ xq,\ yq,\ zq,\ \ xr,\ yr,\ zr, \\ x(xp+yq+zr),\ \ y(xp+yq+zr),\ \ z(xp+yq+zr). \end{array}\right\} \tag{33}$$

The nine independent ratios of the terms of the identity (31), and of those identities derived from it by permutation, are solutions of the simultaneous system

$$\sum_1^8 i \frac{\partial \phi}{\partial x_i} = \sum_1^8 i \frac{\partial \phi}{\partial y_i} = \sum_1^8 i \frac{\partial \phi}{\partial z_i} = \sum_1^8 i x_i \frac{\partial \phi}{\partial x_i}$$

$$= \ldots = \sum_1^8 i \left\{ z_i \left( x_i \frac{\partial \phi}{\partial x_i} + y_i \frac{\partial \phi}{\partial y_i} + z_i \frac{\partial \phi}{\partial z_i} \right) \right\} = 0, \tag{34}$$

and hence are the invariants of the original system of eight points in space of three dimensions by the projective group of that space.

3. The generalization of the identities (14) and (31), together with a study of the Lie group which represents the general projective group of $n$-dimensional space, furnishes a basis for the complete generalization of Clifford's statement of the theory of anharmonics for two dimensions.

Consider the expression formed by the sum of the following binary products of determinants:

$$\left.\begin{array}{rl} & |\ 1, 2, 3, 4, 5, \ldots, n, \quad |\ .\ |\ n+1, n+2, n+3, \ldots, 2n\ |\ , \\ (-1)^n & |\ 2, 3, 4, 5, 6, \ldots, n, n+1\ |\ .\ |\ \ 1, n+2, n+3, \ldots, 2n\ |\ , \\ & |\ 3, 4, 5, 6, \ldots, n, n+1, 1\ |\ .\ |\ \ 2, n+2, n+3, \ldots, 2n\ |\ , \\ (-1)^n & |\ 4, 5, 6, \ldots, n, n+1, 1, 2\ |\ .\ |\ \ 3, n+2, n+3, \ldots, 2n\ |\ , \\ & \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots, \\ (-1)^n & |\ n+1, 1, 2, 3, \ldots, n-1\ |\ .\ |\ \ n, n+2, n+3, \ldots, 2n\ | \end{array}\right\}, \tag{35}$$

where, for brevity,

$$|\ 1, 2, 3, 4, 5, \ldots, n\ | \equiv |\ a_1^{(1)}, a_2^{(2)}, a_3^{(3)}, \ldots, a_n^{(n)}\ |\ . \tag{36}$$

By expanding all the determinants with regard to the elements of their left-hand columns, we see that the coefficient of the general term $a_j^{(r)}$, $a_k^{(s)}$ of the above sum is equal to the sum of determinant products, which sum is of the *form*

$$\begin{aligned}
&| 1, 2, 3, 4, \ldots, n-1 | \,.\, | n, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+(-1)^{n-1} &| 2, 3, 4, \ldots, n-1, n | \,.\, | 1, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+ &| 3, 4, \ldots, n-1, n, 1 | \,.\, | 2, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+(-1)^{n-1} &| 4, \ldots, n-1, n, 1, 2 | \,.\, | 3, n+1, n+2, n+3, \ldots, 2(n-1) | \\
+ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
+(-1)^{n-1} &| n, 1, 2, 3, \ldots, n-2 | \,.\, | n-1, n+1, n+2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n+3, \ldots, 2(n-1) | \,. \quad (37)
\end{aligned}$$

By treating this sum (37) in the same manner, its general term is found to have a coefficient of the same form in which $n$ is again decreased by unity; carrying this through $n-2$ steps, we arrive finally at a coefficient of the form

$$| 1, 2 | \,.\, | 3, 4 | + | 2, 3 | \,.\, | 1, 4 | + | 3, 1 | \,.\, | 2, 4 | ,$$

which is obviously zero; hence the original sum of the determinant products (35) is zero, that is,

$$\Sigma | \alpha, \beta, \gamma, \delta, \epsilon, \ldots, \mu | \,.\, | \nu, n+2, n+3, \ldots, 2n | \equiv 0, \qquad (38)$$

where $\alpha\beta\gamma\delta\epsilon \ldots \mu\nu$

takes successively the values of the sequence

$$1, 2, 3, 4, 5, \ldots, n, n+1$$

and its cyclical permutations, and is always positive, or alternately positive and negative, according as $n$ is even or odd.

A system of $\left[2(n+1)\right]$ points in a space of $n$ dimensions depends upon $2n(n+1)$ coordinates and forms $\left[2(n+1)\right]! / \left[(n+1)!\right]^2$ solids

each having $n+1$ vertices. The volumes of these solids satisfy identities of the form

$$\begin{aligned}
&\big[1, 2, 3, 4, \ldots, n, \quad n+1\big].\big[n+2, n+3, n+4, \ldots, 2(n+1)\big] \\
\pm&\big[1, 2, 3, 4, \ldots, n, \quad n+2\big].\big[n+3, n+4, \ldots, 2(n+1), n+1\big] \\
+&\big[1, 2, 3, 4, \ldots, n, \quad n+3\big].\big[n+4, \ldots, 2(n+1), n+1, n+2\big] \\
\pm&\big[1, 2, 3, 4, \ldots, n, \quad n+4\big].\big[n+5, \ldots, 2(n+1), n+1, n+2, n+3\big] \\
+&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\pm&\big[1, 2, 3, 4, \ldots, n, 2(n+1)\big].\big[n+1, n+2, n+3, \ldots, 2n+1\big] \equiv 0, \quad (39)
\end{aligned}$$

the upper or lower signs being taken according as $n$ is odd or even. This identity follows directly from (38). The other identities, $\big[2(n+1)\big]!/2\big[(n+1)!\big]^2-(n^2+2)$ in number, are derived from this one by permutation.

The general projective group of $n$-dimensional space is an $\big[n(n+2)\big]$ parameter group generated by the following $\big[n(n+2)\big]$ independent infinitesimal transformations

$$\frac{\partial f}{\partial x_i}, \quad x_i\frac{\partial f}{\partial x_k}, \quad x_i\sum_1^n j x_j\frac{\partial f}{\partial x_j} \quad (i, k = 1, \ldots, n).$$

The $n^2$ independent ratios of products of the form of those entering (39) are the $\big[2n(n+1)-n(n+2)\big]$ absolute invariants of the system of $\big[2(n+1)\big]$ points by the general projective group of $n$-dimensional space.

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*Zeroes of the Bessel Functions.* *By* H. M. Macdonald. Communicated (provisionally) April 7th, 1898. Received July 16th, 1898.

The zeroes of any holomorphic function can be approximated to by finding the zeroes of the polynomials obtained by stopping at any term of the power series representing the function. This method indicates the nature of the zeroes, but is unsuitable for the calculation of their values. In the neighbourhood of an essential singularity the function has an infinite number of zeroes, and all these zeroes can be quickly calculated from the expression for the function in that neighbourhood. This is the method applied by Stokes to calculate

the zeroes of $J_0(z)$.* When all the zeroes associated with the essential singularities of the function have been found, there will remain a finite number of zeroes or none. In this paper it is shown, § 1, that the number of zeroes of a holomorphic function $f(z)$ inside the space bounded by a contour for which $|f(z)|$ is constant exceeds the number of the zeroes of the function $f'(z)$ in the same space by unity. When, by successive applications of the theorem, a derived function is found which has no zeroes which are not associated with an essential singularity, the number of zeroes of $f(z)$ not associated with an essential singularity is known. In § 3 the theorem of § 1 is applied to prove the separation of the real zeroes of a function with real coefficients by the real zeroes of its derived function. In § 4 a method is given of calculating the zeroes of $f(z)$, when those of $f'(z)$ are known. It is shown, §§ 5–7, that the zeroes of $J_n(z)/z^n$ are all real and associated with the essential singularity at infinity when $n$ is real and greater than $-1$, and in §§ 8, 9 it is proved that all the real zeroes of $J_n(z)/z^n$, when $n$ is real and less than $-1$, and not an integer, are associated with the essential singularity at infinity. When $n$ is a negative integer $-m$, $J_n(z)/z^n$ has, in addition, $2m$ real zeroes coincident at the origin. It is proved in § 10 that, when $n = -m-\nu$, $m$ being a positive integer, and $1>\nu>0$, $J_n(z)/z^n$ has a finite number $2m$ of zeroes which are not associated with the essential singularity at infinity. Formulæ for calculating these $2m$ zeroes are given in § 11. It appears that, $n$ being real, starting with any positive value of $n$, the zeroes nearest the origin approach it as $n$ diminishes, two of them reaching it when $n$ has diminished to $-1$, and two more reaching it whenever $n$ passes through a negative integral value; these zeroes then become complex for values of $n$ not integral.

1. The number of zeroes of the function $f(z)$ in the space bounded by a contour at each point of which $|f(z)| = m_1$ exceeds the number of zeroes of the derived function $f'(z)$ in the same space by unity, the function $f(z)$ being holomorphic throughout the space and having no singularity in it.

The number $N$ of zeroes of $f(z)$ inside the contour $|f(z)| = m_1$ is finite, since $f(z)$ has no essential singularity in the space. When $m$ is very small, the portion of the curve $|f(z)| = m$ inside the contour $|f(z)| = m_1$ consists of $N$ closed curves surrounding the

* *Camb. Phil. Trans.*, Vol. IX.

$N$ zeroes of $f(z)$ in the space; as $m$ increases, these ovals increase until two of them coalesce, the point at which they coalesce being a double point on the curve corresponding to that particular value of $m$. When $m$ has increased to its final value $m_1$, the $N$ closed curves have coalesced into one closed curve, and therefore $N-1$ double points have been passed through. Each of these double points is a zero of $f'(z)$, for, if $f(z) = \phi + \iota\psi$, where $\phi$ and $\psi$ are functions of $x$ and $y$ with real coefficients, at a double point on the curve $\phi^2+\psi^2 = \text{const.}$ $\dfrac{\partial\phi}{\partial x}$ and $\dfrac{\partial\psi}{\partial x}$ vanish, that is, $f'(z)$ vanishes. Hence the number of zeroes $f'(z)$ inside the contour is $N-1$. To complete the proof, it should be observed that two ovals cannot coalesce at more than one point, $f(z)$ being single valued. The proof assumes the zeroes of $f(z)$ inside the contour to be all different; the case where $f(z)$ has equal zeroes is immediately reducible to the above by dividing out the factor common to $f(z)$ and $f'(z)$. If $f'(z)$ has two zeroes equal, two of the double points coalesce, that is, three ovals coalesce at the same point.

Similarly, it can be shown that the number of zeroes of $f'(z)$ in the space between the contours $|f(z)| = m_1$ and $|f(z)| = m_2$ is equal to the number of zeroes of $f(z)$ in the same space, $f(z)$ being holomorphic throughout the space and having no singularity* in it.

2. The converses of the above theorems are obviously true, that, if inside a contour for each point of which $|f(z)| = m_1$, $f'(z)$ has $N-1$ zeroes, $f(z)$ has $N$ zeroes; and that, if $f'(z)$ has $N$ zeroes in the space between two contours $|f(z)| = m_1$ and $|f(z)| = m_2$, $f(z)$ has $N$ zeroes in the same space, $f(z)$ being holomorphic throughout the space, and having no singularity in it.

A simple proof of the theorem that a polynomial of the $n^{\text{th}}$ degree has $n$ zeroes can be obtained from the above. For let the curve $|f(z)| = m_1$ consist of $N'$ ovals; then, if $N$ is the total number of zeroes of $f'(z)$, there are $N-N'+1$ zeroes of $f'(z)$ inside the $N'$ ovals (there being $N'-1$ double points to be passed through to coalesce the $N'$ ovals). Therefore the total number of zeroes of $f(z)$ is $N-N'+1+N'$, being one more for each oval, that is, $N+1$. Now a polynomial of degree unity has one zero; therefore the polynomial of degree $n$ has $n$ zeroes.

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* When $f(z)$ has an essential singularity at infinity, the contours which enclose it are of parabolic form.

3. If a holomorphic function $f(z)$ has all its coefficients real, and all its zeroes real and different, then between two consecutive zeroes of $f(z)$ there is one zero and one only of $f'(z)$.

In this case the curve $|f(z)| = m$ is symmetrical with respect to the real axis, and the double points therefore lie on the real axis, there being one and one only between two consecutive zeroes of $f(z)$, for, if there were more than one, there would be ovals between these two zeroes of $f(z)$ when $m$ is sufficiently diminished; that is, they would not be consecutive zeroes. The theorem is clearly true in the case where all the zeroes of the function inside a contour $|f(z)| = \text{const.}$ are real, the zeroes of $f'(z)$ inside this contour separating them.

4. The zeroes of $f(z)$ can be obtained in the form of a series when those of $f'(z)$ are known.

Let $z_0$ be a zero of $f'(z)$ and let $f(z_0) = a$; then

$$f(z)-a = \frac{(z-z_0)^2}{2!}f''(z_0)+\frac{(z-z_0)^3}{3!}f'''(z_0)+\&c.;$$

that is, $$f(z)-a = (z-z_0)^2\chi(z).$$

Then the function $f(z)$ has two zeroes continuous with the two zeroes $z_0$ of $f(z)-a$; they lie inside the contour determined by the condition $|f(z)-a| > |a|$, and are given by

$$z = z_0+\sum_1^\infty \frac{e^{\pm\frac{1}{2}n\pi i}}{n!}\frac{d^{n-1}}{dz^{n-1}}\left\{\frac{a}{\chi(z)}\right\}^{\frac{1}{2}n}.$$

5. The zeroes of the function $J_n(z)/z^n$ are all real when $n$ is real and greater than $-1$, $J_n(z)$ being Bessel's function. This is usually established by considering $J_n(z)/z^n$ as the limit of a polynomial when its degree is indefinitely increased. It may be proved as follows: $2^n\pi(n)\,J_n(z)/z^n \equiv u_n$ is a holomorphic function both of $z$ and $n$, when $|n|<1$,* and therefore its zeroes are continuous with those of $u_{\frac{1}{2}}$ or $u_{-\frac{1}{2}}$ (Picard, *Traité d'Analyse*, t. II., chap. ix.), each zero of $u_n$ being given by a Lagrange series, each term of which is real when $n$ is. Therefore all the zeroes of $u_n$ are real when $n$ is real and $1>n>-1$.

Again, $$\frac{d}{dz}\frac{J_n(z)}{z^n} = -\frac{J_{n+1}(z)}{z^n};$$

therefore, by § 3 above, $J_{n+1}(z)/z^n$ has all its zeroes real, there being one and one only between two consecutive zeroes of $J_n(z)/z^n$; that

* $u^n$ is a holomorphic function of $n$ when the real part of $n$ is greater than $-1$.

is, all the zeroes of $J_{n+1}(z)/z^{n+1}$ are real if the zeroes of $J_n(z)/z^n$ are, and they have been shown to be real when $n$ is real and $1 > n > -1$; hence the zeroes of $J_n(z)/z^n$ are all real when $n$ is real and greater than $-1$.

6. The zeroes of $J_n(z)/z^n$ are separated by the zeroes of $J_{n+1}(z)/z^n$, one zero of $J_{n+1}(z)/z^n$, and one only, lying between two consecutive zeroes of $J_n(z)/z^n$.

This is a particular case of § 3 above; other proofs of it have been given by Hobson, *Proc. Lond. Math. Soc.*, Vol. XXVII.; Van Vleck, *Amer. Jour. of Math.*, Vol. XIX. It can also be established as follows: If $\alpha$ and $\beta$ are two consecutive real zeroes of $J_n(z)/z^n$, the recurrence formula at once gives

$$\int_\alpha^\beta \frac{J_{n+1}(z)}{z^n}\,dz = 0;$$

and therefore $J_{n+1}(z)/z^n$ vanishes once at least when $z$ is between $\alpha$ and $\beta$. Again, if $\alpha'$ and $\beta'$ are two consecutive real zeroes of $z^{n+1}J_{n+1}(z)$,

$$\int_{\alpha'}^{\beta'} z^{n+1}J_n(z)\,dz = 0;$$

and therefore $z^{n+1}J_n(z)$ vanishes once at least when $z$ is between $\alpha'$ and $\beta'$. The two statements taken together show that between any two consecutive real zeroes of $J_n(z)/z^n$ there is one, and only one, zero of $J_{n+1}(z)/z^n$, the zero $z = 0$ of $J_{n+1}(z)/z^n$ being the one which separates the least positive and negative zeroes of $J_n(z)/z^n$. The theorem holds for the real zeroes of $J_n(z)/z^n$ whatever real value $n$ has.

7. When $n$ is real and greater than $-1$, all the zeroes of $J_n(z)/z^n$ are given by Stokes' formula.

The formula for the $m^{\text{th}}$ positive zero in order of magnitude is

$$\zeta_m = \alpha - \frac{4n^2-1}{8\alpha} - \frac{4(4n^2-1)(28n^2-31)}{3(8\alpha)^3} - \&c.,$$

where

$$\alpha = \frac{\pi}{4}(2n+4m-1),$$

and $n$ is real and greater than $-1$.

It follows from the form of this expression that, if all the zeroes of $J_n(z)/z^n$ are given by it, then all those of $J_{n+1}(z)/z^{n+1}$ are given by the corresponding formula when $n+1$ is written for $n$ (§ 6). It is therefore sufficient to show that all the zeroes are given by this

formula when $\frac{1}{2} > n > -1$. All the zeroes are given by it when $n$ is $\frac{1}{2}$ or $-\frac{1}{2}$, they being then the zeroes of $\frac{\sin z}{z}$ and of $\cos z$. Now $J_n(z)/z^n$ is holomorphic in both $z$ and $n$, when $|n| < 1$; therefore all its zeroes are continuous with those of $\cos z$. Hence all the zeroes of $J_n(z)/z^n$ are given by Stokes' formula when $n$ is real and greater than $-1$. This may be expressed by saying that all the zeroes of $J_n(z)/z^n$ are associated with the essential singularity at infinity when $n$ is real and greater than $-1$.

8. The least positive real zero of $z^n J_{-n}(z)$ is less than the least positive real zero of $J_n(z)/z^n$, and less than the least positive real zero of $z^{n+1}J_{-n-1}(z)$, $n$ being real and positive.

The first result follows at once from the relation between $J_n(z)$ and $J_{-n}(z)$, which are independent solutions of Bessel's differential equation, there being a zero of $z^n J_{-n}(z)$ between the origin and the least positive zero of $J_n(z)/z^n$. Writing

$$u_n = \left(\frac{2}{z}\right)^n J_n(z),$$

the relation between $u_n$ and $u_{-n}$ is

$$\frac{z^2}{4} u_{n+1} u_{-n} + u_n u_{-n-1} = -\frac{\sin n\pi}{\pi}.$$

Now, when $z = 0$,

$$u_n = \frac{1}{\Pi(n)} \quad \text{and} \quad u_{-n-1} = -\frac{\sin n\pi}{\pi}\Pi(n),$$

and, when $z$ is equal to $z_0$, the least positive zero of $u_{-n}$, $u_n$ is positive, and therefore $u_{-n-1}$ has the same sign as it has when $z = 0$; hence the least positive real zero of $u_{-n}$ is less than the least positive real zero of $u_{-n-1}$. Similar results hold for the negative real zero of $u_{-n}$ nearest to the origin.

9. All the real zeroes of $u_{-n}$, when $n$ is real and positive, are associated with the essential singularity at infinity.

The formula for the zeroes, which corresponds to Stokes' formula for $n$ positive, and which is derived from the semi-convergent form of $J_{-n}(z)$, is easily shown to be the same as that quoted, § 7, where $\alpha$ is now $\frac{\pi}{4}(-2n+4k+4m-1)$, $k$ being the greatest positive integer less than $n$. It is obvious that the zeroes of $u_{-n}$ given by this formula separate those given by the corresponding formula for the

zeroes of $u_n$, the least positive zero of $u_{-n}$ being less than the least positive zero of $u_n$, whence it follows (as there can only be one zero of $u_{-n}$ between two consecutive real zeroes of $u_n$) that all the real zeroes of $u_{-n}$ are given by the formula; that is, they are all associated with the essential singularity at infinity.

10. When $m+1>n>m$, $m$ being an integer, $z^n J_{-n}(z)$ has $2m$ complex zeroes, which, when $m$ is even, are arranged in $m/2$ groups of four, each zero of a group having the same modulus, and, when $m$ is odd, in $(m-1)/2$ such groups of four together with two pure imaginary zeroes.

Writing, as before, $$u_n = \left(\frac{2}{z}\right)^n J_n(z),$$

the relation between $u_n$ and $u_{-n}$ is

$$\frac{z^2}{4} u_{n+1} u_{-n} + u_n u_{-n-1} = -\frac{\sin n\pi}{\pi}.$$

When $1>n>0$, putting $z = \iota y$ ($y$ being real), $u_{-n}$ is positive for all values of $y$, and increases with $y$. Now

$$\frac{\partial u_{-n-1}}{\partial y} = 2y u_{-n};$$

therefore $u_{-n-1}$ increases with $y$. Again $u_{-n-1}$ is negative when $y=0$, and positive when $y=\infty$; therefore $u_{-n-1}$ changes sign once as $z$ goes from 0 to $\iota\infty$; that is, $u_{-n-1}$ has one pure imaginary zero between $z=0$ and $z=\iota\infty$. Hence, when $2>n>1$, $u_{-n}$ has a pair of pure imaginary zeroes which are equal and of opposite sign. When $2>n>1$, $u_{-n-1}$ is positive when $z=0$, and decreasing for $u_{-n}$ is then negative; therefore $u_{-n-1}$ decreases to a minimum when $u_{-n}$ vanishes, and then increases indefinitely with $y$. Now, when $u_{-n}$ vanishes,

$$u_{-n-1} = -\frac{\sin n\pi}{\pi u_n},$$

a positive quantity, for $u_n$ is positive and $\sin n\pi$ is negative; therefore $u_{-n-1}$ is positive for all pure imaginary values of $z$. Hence, when $3>n>2$, $u_{-n}$ has no pure imaginary zeroes. Similarly it follows that, when $4>n>3$, $u_{-n}$ has a pair of pure imaginary zeroes, and, when $5>n>4$, $u_{-n}$ has no pure imaginary zeroes. Therefore, when $2m+1>n>2m$, $u_{-n}$ has no pure imaginary zeroes, and, when $2m+2>n>2m+1$, $u_{-n}$ has a pair of equal imaginary zeroes of opposite sign.

Since the least positive real zero of $u_{-n+1}$ is less than that of $u_{-n}$,

$$| u_{-n} |_{z=x_0} > | u_{-n} |_{z=0},$$

where $x_0$ is the least positive real zero of $u_{-n+1}$. When $u_{-n}$ has no pure imaginary zero, $u_{-n}$ is positive when $z=0$, and increases as $z$ goes from 0 to $\infty$; therefore $u_{-n}$ takes the value $| u_{-n} |_{z=x_0}$ at some point on the positive imaginary axis, and also at an equidistant point on the negative imaginary axis. When $u_{-n}$ has pure imaginary zeroes, it is negative at the origin, and its modulus decreases as $z$ goes from 0 to an imaginary zero of $u_{-n}$ along the axis of imaginaries, and then increases indefinitely; therefore $u_{-n}$ will take the value $| u_{-n} |_{z=x_0}$ at two points on the axis of imaginaries equidistant from the origin, and further from it than the pure imaginary zeroes. Let the contour be drawn for each point of which

$$| u_{-n} | = | u_{-n} |_{z=\pm x_0};$$

the portion of it between $z=-x_0$ and $z=x_0$ will, by the above, surround the origin; hence there is a closed curve, enclosing the origin, which is part of a contour for which $| u_{-n} |$ is slightly less than $| u_{-n} |_{z=\pm x_0}$. The number of zeroes of $u_{-n}$ inside this closed curve will, by § 1, exceed the number of zeroes of $\frac{du_{-n}}{dz}$ inside it by unity. Now

$$\frac{du_{-n}}{dz} = -2zu_{-n+1},$$

and $z=0$ is a zero of $zu_{-n+1}$; therefore the number of zeroes inside the closed curve exceeds the number of zeroes of $u_{-n+1}$ inside it by 2. It has been shown that the two real zeroes of $u_{-n+1}$ nearest the origin are nearer the origin than the two nearest real zeroes of $u_{-n}$; therefore the zeroes of $u_{-n}$ and $u_{-n+1}$ inside this contour are complex. Now, when $1>n>0$, $u_{-n}$ has no zeroes which are not real, therefore, when $2>n>1$, $u_{-n}$ has two zeroes which are not real; hence, when $m+1>n>m$, $m$ a positive integer, $u_{-n}$ has $2m$ zeroes which are not real. Since $u_{-n}$ involves only even powers of $z$, and all its coefficients are real, if it has a pure imaginary zero, it must have an equal pure imaginary zero with opposite sign; and, if it has a complex zero, it must have an equal complex zero with opposite sign, and two other zeroes conjugate to these two, forming a group of four with the same modulus. Hence the theorem is completely established. The number of the complex zeroes of $u_{-n}$ has been obtained by Hurwitz, *Math. Ann.*, Vol. XXXIII., p. 246, by a different method.

11. *Investigation of a Formula for the Complex Zeroes of* $J_n(z)/z^n$. It has been shown above that, when $m = n+\mu$, where $m$ is a positive integer, and $1>\mu>0$, $u_{-n}$ has $2m$ complex zeroes inside a certain closed curve. When $\mu = 0$, these $2m$ zeroes become the $2m$ zeroes of $x^{2m}u_m$ inside the closed curve, and these are the $2m$ zeroes of $x^{2m} = 0$; therefore, when $n = m+\mu$, the $2m$ complex zeroes of $u_{-n}$ are continuous with the zeroes of $x^{2m} = 0$. To obtain an expression for these zeroes it is necessary to express $u_{-n}$ in a suitable form. The expression for $u_{-n}$ in a series is given by

$$u_{-n} = \sum_{s=0}^{s=\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-n+s)};$$

that is,
$$u_{-n} = \sum_{s=0}^{s=\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-m-\mu+s)};$$

whence
$$u_{-n} = \sum_{0}^{m-1} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(-m-\mu+s)} + \sum_{m}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s)\,\Pi(s-m-\mu)}.$$

Now, $\Pi(-m-\mu+s)\,\Pi(m+\mu-s-1) = \pi \operatorname{cosec}(m+\mu-s)\,\pi$;

therefore

$$u_{-n} = (-)^m \frac{\sin \mu\pi}{\pi} \sum_{0}^{m-1} \left(\frac{x^2}{4}\right)^s \frac{\Pi(m+\mu-s-1)}{\Pi(s)} + \sum_{0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^{m+s}}{\Pi(s-\mu)\,\Pi(m+s)}.$$

Hence the zeroes of $u_{-n}$ are those of the expression

$$\frac{\sin \mu\pi}{\pi} X + x^{2m} F,$$

where
$$X = \sum_{0}^{m-1} \left(\frac{x^2}{4}\right)^s \frac{\Pi(m+\mu-s-1)}{\Pi(s)}$$

and
$$F = \frac{1}{2^{2m}} \sum_{0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\Pi(s-\quad\Pi(m+s)},$$

the zeroes associated with the essential singularity at infinity being continuous with those of $F = 0$, and the complex zeroes (those associated with the essential singularity at the origin) being given by

$$x = \gamma \left\{ \frac{\sin \mu\pi}{\pi} \frac{X}{F} \right\}^{\frac{1}{2m}},$$

where $$\gamma = e^{\pm[(2r+1)/2m]\pi i},$$

$r$ having values 0 to $m-1$ inclusive.

Expanding by Lagrange's theorem,

$$x = \sum_1^\infty \frac{\gamma^k}{k!}\left(\frac{\sin\mu\pi}{\pi}\right)^{k/2m}\left\{\left(\frac{d}{dx}\right)^{k-1}\left(\frac{X}{F}\right)^{k/2m}\right\}_{x=0},$$

or, since $X$ and $F$ only involve even powers of $x$,

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2k+1!}\left(\frac{\sin\mu\pi}{\pi}\right)^{(2k+1)/2m}\left\{\left(\frac{d}{dx}\right)^{2k}\left(\frac{X}{F}\right)^{(2k+1)/2m}\right\}_{x=0}.$$

Writing $$X = F\sum_0^\infty \frac{a_s}{s!}\left(\frac{x^2}{4}\right)^s,$$

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2^{2k}(2k+1)\,k!}\left(\frac{\sin\mu\pi}{\pi}\right)^{(2k+1)/2m}\Delta^k a_0^{(2k+1)/2m},$$

where $$\Delta a_s^p = p a_s^{p-1} a_{s+1}.$$

The above series for the roots converges quickly when $\mu$ is small, but the convergence becomes very slow when $\mu$ is nearly unity. For values of $\mu$ near to unity it is better to use another series obtained as follows: The expression $u_{-n}$ can be written in the form

$$X_1 - x^{2m+2}F_1,$$

where $$F_1 = \frac{1}{2^{2m+2}}\sum_0^\infty\left(-\frac{x^2}{4}\right)^s \frac{1}{\Pi(s-\mu+1)\,\Pi(s+m+1)},$$

and $X_1$ is a polynomial of degree $2m$ in $x$. The complex zeroes of $u_{-n}$ are now given by

$$x = \gamma\left(\frac{X_1}{F_1}\right)^{1/(2m+2)},$$

where $$\gamma = \pm e^{r\pi i/(m+1)},$$

and $r$ has values 1 to $m$ inclusive; that is,

$$x = \sum_0^\infty \frac{\gamma^{2k+1}}{2^{2k}(2k+1)\,k!}\Delta^k b_0^{(2k+1)/(2m+2)},$$

where $$X_1 = F_1\sum_0^\infty \frac{b_s}{s!}\left(\frac{x^2}{4}\right)^s.$$

This series is in ascending powers of $(1-\mu)^{1/(2m+2)}$, and is rapidly convergent when $\mu$ is nearly unity. The zeroes which correspond to $\gamma = \pm 1$ are the two real zeroes of $u_{-n}$ nearest the origin.

*The Catenary, and Associated Trajectory, on the Paraboloid, the Cone, &c.* *By* A. G. Greenhill. Read March 12th, 1896. Received August 10th, 1896. Received, in revised form, February 13th, 1898.

In a preceding article (*Proc. Lond. Math. Soc.*, Vol. xxvii) the properties of the catenary curve assumed by a chain wrapped on a sphere have been considered, concluding with an investigation of the shape of the curve when the sphere is spinning about a vertical axis with such rapidity that the influence of gravity may be left out of account.

But, as the analytical results of this last problem are practically the same as those required for the catenary curve of a chain wrapped on a vertical paraboloid of revolution, we resume the investigations and extend them to the allied problems of the catenary on the cone, &c., and, at the same time, consider the associated problems of the motion of a particle on a surface of revolution.

The theory is illustrated at length by working out the simplest *pseudo-elliptic* cases, by means of which the construction of a catenary or trajectory is made to depend upon tabular matter in mathematical tables, in conjunction with the tables for $F\phi$ and $E\phi$, given in Legendre's *Fonctions Elliptiques*, t. II.

The analysis required in these applications has been developed in papers in the *Proceedings of the London Mathematical Society* :—

"Pseudo-Elliptic Integrals and their Dynamical Applications," Vol. xxv;

"The Dynamics of a Top," Vols. xxvi and xxvii;

"The Spherical Catenary," Vol. xxvii;

"The Transformation and Division of Elliptic Functions," Vol. xxvii;

and, to save repetition, the results are quoted in the sequel, with a reference to the volume and the page, as (*L.M.S.*, xxv, p. 195), &c.

The following Dissertations discuss the same subject:—

Bertram, *Diss.*, Marburg, 1876.
Osswald, J., *Diss.*, Freiburg, 1876.
Neumann, L., *Diss.*, Freiburg, 1878.
Schönlicht, L., *Diss.*, Freiburg, 1884.
Sonntag, *Diss.*, Marburg, 1886.

It will be noticed that in all these dynamical applications which require the elliptic integral of the third kind the Jacobian parameter is a fraction of the *imaginary* period, so that the integrals are of the *circular* form, in Legendre's terminology; and tables of the Jacobian $\Theta$ functions, even if accessible to us, would be practically useless for our purposes.

But, by choosing as parameters the rational fractions of the imaginary period, beginning with the simplest fractions, we are able to utilize the pseudo-elliptic cases worked out in *L.M.S.*, xxv, to explore the analytical field with a number of well determined particular cases.

A dynamical desideratum appears, then, to be the tabulation of the function

$$\left\{\frac{\Theta\left(u-\frac{rK'i}{n}\right)}{\Theta\left(u+\frac{rK'i}{n}\right)}\right\}^n,$$

where $n = 2, 3, 4, 5, 6, \ldots,$

in the form $$\frac{A+iB}{A-iB},$$

where $A$ and $B$ are single-valued elliptic functions of $u$, these being the functions analogous to $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, mentioned by Halphen, *Fonctions Elliptiques*, I, p. 222.

It is easy to translate the pseudo-elliptic results of *L.M.S.*, xxv, into this new notation: thus, for instance, from p. 212,

$$\left\{\frac{\Theta(u-\tfrac{1}{2}K'i)}{\Theta(u+\tfrac{1}{2}K'i)}\right\}^2 = \frac{\operatorname{cn} u \operatorname{dn} u + i(1+u)\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u - i(1+u)\operatorname{sn} u};$$

and, from p. 218,

$$e^{3iI(\frac{1}{3}\omega_3)} = \left\{\frac{\Theta(u-\frac{2}{3}K'i)}{\Theta(u+\frac{2}{3}K'i)}\right\}^3$$

$$= \frac{(1-c+c^2)\,s-(c-c^2)^2+\frac{1}{2}i\sqrt{S}}{(1-c+c^2)\,s-(c-c^2)^2-\frac{1}{2}i\sqrt{S}},$$

$$e^{3iI(\frac{1}{3}\omega_2)} = \left\{\frac{\Theta(u-\frac{1}{3}K'i)}{\Theta(u+\frac{1}{3}K'i)}\right\}^3$$

$$= \frac{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}+i(s-2+c-c^2)\sqrt{(s-s_3)}}{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}-i(s-2+c-c^2)\sqrt{(s-s_3)}},$$

and so on.

1. Taking $Oz$ as the vertical axis of revolution, and $r$, $\psi$ the polar coordinates of the projection on a horizontal plane of a point $P$ at a height $z$, then the statical equations of equilibrium of the chain are

$$T = w(z-h) \text{ or } w(h-z)$$
$$= w(z \sim h), \tag{1}$$

and
$$Tr^2 \frac{d\psi}{ds'} = H, \tag{2}$$

connecting $s'$ the arc, $T$ the tension, and $w$ the weight per unit length of the chain.

Eliminating $T$, we obtain a general equation, of the form

$$(z \sim h)\, r^2 \frac{d\psi}{ds'} = A, \tag{3}$$

where
$$H = wA. \tag{4}$$

Thence
$$\frac{ds'^2}{dz^2} = 1 + \frac{dr^2}{dz^2} + r^2 \frac{d\psi^2}{dz^2} = \frac{(z-h)^2}{A^2} r^4 \frac{d\psi^2}{dz^2},$$

or
$$\frac{d\psi^2}{dz^2} = A^2 \frac{1 + \frac{dr^2}{dz^2}}{r^2 Z}, \tag{5}$$

where
$$Z = r^2 (z-h)^2 - A^2. \tag{6}$$

*The Catenary on the Paraboloid.*

2. In the paraboloid of revolution, we put

$$r^2 = 4az, \tag{7}$$

so that
$$\frac{d\psi^2}{dz^2} = \frac{A^2}{4a} \frac{z+a}{z^2 Z}, \tag{8}$$

where
$$Z = 4az(z-h)^2 - A^2; \tag{9}$$

so that, putting
$$A^2 = 4a^2 k^2, \tag{10}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2} k \frac{\sqrt{(z+a)}}{z\sqrt{\{z(z-h)^2 - ak^2\}}}. \tag{11}$$

Then, if $p$ denotes the perpendicular from the origin upon the tangent in the projection of the catenary on a horizontal plane,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \frac{dr^2}{d\psi^2}$$
$$= \frac{(z-h)^2 + k^2}{4ak^2(z+a)}, \tag{12}$$

which can be written

$$4a\frac{k^2}{p^2} = z+a-2(a+h)+\frac{(a+h)^2+k^2}{z+a}$$
$$= \left[\sqrt{(z+a)}-\frac{\sqrt{\{(a+h)^2+k^2\}}}{\sqrt{(z+a)}}\right]^2+2\sqrt{\{(a+h)^2+k^2\}}-2(a+h) \quad (13)$$

showing that $p$ is stationary, and there is a point of inflexion, where

$$z+a = \sqrt{\{(a+h)^2+k^2\}}, \quad (14)$$

and then
$$p^2 = 2a\left[\sqrt{\{(a+h)^2+k^2\}}+(a+h)\right], \quad (15)$$
$$r^2 = 4a\sqrt{\{(a+h)^2+k^2\}}-4a^2. \quad (16)$$

If the paraboloid is spinning about its axis with uniform angular velocity $n$, then equation (1) must be replaced by

$$T = w(z\sim h)-\frac{wn^2r^2}{g}+C, \quad (17)$$

but, as this equation can be written in the form

$$T = w'(z\sim h'), \quad (18)$$

where
$$w' = w\pm\frac{4wn^2a}{g}, \quad (19)$$

the equations of equilibrium remain essentially the same as before.

It is also immaterial whether we suppose the vertex of the paraboloid to be its highest or lowest point, as an alteration of the direction of gravity with respect to the surface merely changes the sign of $T$.

With $r^2$ for independent variable, equation (11) must be written

$$\psi = \tfrac{1}{2}\int\frac{8a^2k\sqrt{(r^2+4a^2)}}{r^2\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}dr^2. \quad (20)$$

The arc $s'$ is given by the equation

$$\frac{ds'}{dz} = \frac{ds}{r^2d\psi}\frac{r^2d\psi}{dz} = \frac{z-h}{A}\tfrac{1}{2}k\frac{r^2}{z}\frac{\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}$$
$$= \frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}, \quad (21)$$

or
$$s' = \int\frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}} \quad (22)$$
$$= \int\frac{(r^2-4ah)\sqrt{(r^2+4a^2)}}{\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}dr^2. \quad (23)$$

3. Equations (11) and (15) are elliptic integrals of the third kind, with a single pole at the vertex of the paraboloid, so that they can be compared immediately with the standard form

$$I(v) = \tfrac{1}{2}\int \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{(s-\sigma)\sqrt{S}}\,ds, \tag{A}$$

where
$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2, \tag{B}$$

$$M^2(s-\sigma) = \wp u - \wp v, \tag{C}$$

$$\int \frac{ds}{\sqrt{S}} = Mu, \tag{D}$$

$$e^{iI(v)} = \frac{\sigma(u+v)}{\sigma(u-v)}\, e^{(MP-\zeta v)u}. \tag{E}$$

Writing equation (11)

$$\psi = \tfrac{1}{2}k\int \frac{z+a}{z\sqrt{Z}}\,dz, \tag{24}$$

where
$$Z = (z+a)\{z(z-h)^2 - ak^2\}$$
$$= z^4 + (a-2h)z^3 - h(2a-h)z^2 + a(h^2-k^2)z - a^2k^2, \tag{25}$$

the comparison with (A) is made by putting

$$\frac{Nz}{z+a} = \wp u - \wp v = M^2(s-\sigma), \tag{26}$$

making $u = 0$, $v$, $c$, correspond to $z = -a$, $0$, $\infty$; and then we find

$$\wp u = \frac{\frac{1}{6}(a+h)(3a+h)z - \frac{1}{3}ah(a+h) - \frac{1}{2}ak^2}{2(z+a)}, \tag{27}$$

$$\wp' u = -a\frac{\frac{1}{4}(a+h)^2 + \frac{1}{4}k^2}{(z+a)^2}\sqrt{Z}. \tag{28}$$

Denoting the roots of the cubic factor of $Z$,

$$z(z-h)^2 - ak^2 = 0, \tag{29}$$

by $z_1$, $z_2$, $z_3$, these roots are essentially positive; and they may be written in the order

$$\infty > z > z_1 > z_2 > z > z_3 > 0 > -a > -\infty, \tag{30}$$

so that we may put

$$v = f\omega_3, \quad c = f'\omega_1, \tag{31}$$

and the determination of the arc $s'$, depending on the parameter $c$, will be found to lead to an analysis of the same nature as that already developed for the Spherical Catenary.

Equation (24) may be written

$$\psi = \tfrac{1}{2}\int \frac{Nk}{\wp u - \wp v}\,\frac{dz}{\sqrt{Z}}$$

$$= \tfrac{1}{2}\int \frac{i\wp' v}{\wp u - \wp v}\,du,$$

or
$$\psi i = -u\zeta v + \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}. \qquad (32)$$

Expressed as a function of $s$ and $\sqrt{S}$, as in the standard form (A),

$$\psi = \tfrac{1}{2}\int \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}P\int \frac{ds}{\sqrt{S}} - I(v)$$

$$= \tfrac{1}{2}PMu - I(v), \qquad (33)$$

and
$$u = \int \frac{dz}{\sqrt{Z}} = \frac{2F\phi}{\sqrt{(z_1 - z_3 . z_2 + a)}}, \qquad (34)$$

where

| | In the open branch. | In the limited branch. | |
|---|---|---|---|
| $\sin^2\phi =$ | $\dfrac{z_2+a}{z_1+a}\,\dfrac{z-z_1}{z-z_2}$, | $\dfrac{z_2+a}{z_2-z_3}\,\dfrac{z-z_3}{z+a}$, | (35) |
| $\cos^2\phi =$ | $\dfrac{z_1-z_2}{z_1+a}\,\dfrac{z+a}{z-z_2}$, | $\dfrac{z_3+a}{z_2-z_3}\,\dfrac{z_2-z}{z+a}$, | (36) |
| $\Delta^2\phi =$ | $\dfrac{z_1-z_2}{z_1-z_3}\,\dfrac{z-z_3}{z-z_2}$, | $\dfrac{z_3+a}{z_1-z_3}\,\dfrac{z_1-z}{z+a}$, | (37) |

taking $z$, $\psi$, and $\phi$ as increasing together, to avoid ambiguities of sign.

Thus, in the pseudo-elliptic cases, the catenary on the paraboloid cannot be an algebraical curve unless we can make $P(v)$ vanish.

When $P(v) = 0$, the discriminant of $S$ is negative, and the catenary has one open branch only; and now, with $z_1$ and $z_2$ imaginary,

$$u = \int \frac{dz}{\sqrt{Z}} = \frac{F\phi}{\sqrt{(HH')}}, \qquad (38)$$

where
$$H^2 = z_3 - z_1 . z_3 - z_2, \quad H'^2 = a + z_1 . a + z_2, \qquad (39)$$

and
$$\tan^2\tfrac{1}{2}\phi = \frac{H}{H'}\,\frac{z-z_3}{z+a}. \qquad (40)$$

4. To express the curve made by the projection of the chain on a horizontal plane, we must return to the standard integral (A) of the circular form; and now, putting

$$s-\sigma = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{41}$$

where $\omega$ is the angle the normal makes with the axis, then

$$\frac{r^2}{4a^2} = \frac{z}{a} = \frac{s-\sigma}{Q-s+\sigma}. \tag{42}$$

Denoting the roots of (B) by $s_1$, $s_2$, $s_3$, and replacing

$$s_1-\sigma,\ s_2-\sigma,\ s_3-\sigma \text{ by } \sigma_1,\ \sigma_2,\ \sigma_3,$$

the roots of
$$s^3+3s^2\wp v+\tfrac{1}{2}s\wp''v+\tfrac{1}{4}\wp'^2v=0, \tag{43}$$

then
$$\frac{z_1}{a}=\frac{\sigma_1}{Q-\sigma_1},\ \frac{z_2}{a}=\frac{\sigma_2}{Q-\sigma_2},\ \frac{z_3}{a}=\frac{\sigma_3}{Q-\sigma_3}; \tag{44}$$

and, since $z_1$, $z_2$, $z_3$ are the roots of the cubic equation

$$z(z-h)^2-ak^2=0, \tag{45}$$

therefore
$$\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}=0, \tag{46}$$

or
$$\sqrt{\left(\frac{\sigma_1}{Q-\sigma_1}\right)}+\sqrt{\left(\frac{\sigma_2}{Q-\sigma_2}\right)}+\sqrt{\left(\frac{\sigma_3}{Q-\sigma_3}\right)}=0, \tag{47}$$

an equation for determining $Q$; rationalizing this equation,

$$\sigma_1^2(Q-\sigma_2)^2(Q-\sigma_3)^2+\ldots-2\sigma_2\sigma_3(Q-\sigma_1)^2(Q-\sigma_2)(Q-\sigma_3)-\ldots=0, \tag{48}$$

and putting
$$\sigma_1+\sigma_2+\sigma_3=S_1=-3\wp v, \tag{49}$$

$$\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2=S_2=\tfrac{1}{2}\wp''v, \tag{50}$$

$$\sigma_1\sigma_2\sigma_3=S_3=-\tfrac{1}{4}\wp'^2v, \tag{51}$$

$$(S_1^2-4S_2)Q^4+12S_3Q^3-6S_1S_3Q^2+4S_2S_3Q-3S_3^2=0, \tag{52}$$

a quartic equation in which the quadrinvariant vanishes; which can therefore be resolved, and it has two real and two imaginary roots.

Putting
$$Q=\frac{3S_3}{R+S_2}, \tag{53}$$

$$R^4-6aR^2-8bR-3a^2=0, \tag{54}$$

where
$$a=S_2^2-3S_1S_3=\tfrac{1}{4}\wp''^2-\tfrac{9}{4}\wp\wp'^2, \tag{55}$$

$$\begin{aligned}2b&=2S_2^3+27S_3^2-9S_1S_2S_3\\&=\tfrac{1}{4}\wp''^3+\tfrac{27}{16}\wp'^4-\tfrac{27}{8}\wp\wp'^2\wp'',\end{aligned} \tag{56}$$

and (54) is a Jacobian quartic the roots of which, on putting

$$b^2-a^3=c^3, \tag{57}$$

can be exhibited in the form

$$R=\surd(a+c)+\surd\{2a-c+2\surd(a^2-ac+c^2)\},$$

or

$$R=\surd(a+c)+\surd(a+\omega c)+\surd(a+\omega^2 c), \tag{58}$$

where $\omega$, $\omega^2$ denote the imaginary cube roots of unity.

In the sequel it will be sufficient, in general, to work with the parameter

$$v=\frac{2\omega_3}{\mu}, \tag{59}$$

and thus take

$$\sigma=-x, \quad S_1=\tfrac{1}{4}(y+1)^2+x, \quad S_2=\tfrac{1}{2}x(y+1), \quad S_3=\tfrac{1}{4}x^2, \tag{60}$$

when

$$a=\tfrac{1}{16}x^2\{(y+1)^2-12x\}, \tag{61}$$

$$b=-\tfrac{1}{64}x^3\{(y+1)^3+36x(y+1)-54x\}, \tag{62}$$

$$c^3=\frac{27x^4\Delta}{1024}, \tag{63}$$

where $\Delta$ denotes the discriminant of the cubic $S$ in (B).

5. In a transformation of even order we may assume that a root of the cubic $S=0$, say $s=s_3$, and at the same time that a root of the cubic (29) is known, say $z_3$ or $r_3$, in the form

$$\frac{z_3}{a}=\frac{r_3^2}{4a^2}=4\gamma^2, \tag{64}$$

and then the other two roots will be given in the form

$$\frac{z_1,\ z_2}{a}=\frac{r_1^2,\ r_2^2}{4a^2}=(\beta\pm\gamma)^2, \tag{65}$$

and, with $\beta>3\gamma$, we have

$$r_1=r_2+r_3, \tag{66}$$

and the roots arranged in the order

$$r_1>r_2>r_3.$$

Now, from the general relation (41), or

$$\frac{a}{z}=\frac{4a^2}{r^2}=\cot^2\omega=\frac{Q}{s-\sigma}-1, \tag{67}$$

$$\frac{1}{(\beta+\gamma)^2} = \frac{Q}{s_1-\sigma} - 1, \tag{68}$$

$$\frac{1}{(\beta-\gamma)^2} = \frac{Q}{s_2-\sigma} - 1, \tag{69}$$

$$\frac{1}{4\gamma^2} = \frac{Q}{s_3-\sigma} - 1, \tag{70}$$

and then

$$\frac{(\beta-\gamma)(\beta+3\gamma)}{4\gamma^2(\beta+\gamma)^2} = Q\,\frac{s_1-s_3}{(s_1-\sigma)(s_3-\sigma)}, \tag{71}$$

$$\frac{(\beta+\gamma)(\beta-3\gamma)}{4\gamma^2(\beta-\gamma)^2} = Q\,\frac{s_2-s_3}{(s_2-\sigma)(s_3-\sigma)}, \tag{72}$$

$$\left(\frac{\beta-\gamma}{\beta+\gamma}\right)^3 \frac{\beta+3\gamma}{\beta-3\gamma} = \frac{s_1-s_3}{s_2-s_3}\,\frac{s_2-\sigma}{s_1-\sigma}$$

$$= \frac{s_1-\sigma'}{s_2-\sigma'} = \mathrm{dn}^2(1-f)\,K', \tag{73}$$

where $\sigma'$ is the value of $s$ corresponding to

$$u = (1-f)\,\omega_3\,; \tag{74}$$

thence $Q$ can be determined when $s_1, s_2, s_3, \sigma, \sigma'$ are given, and when the ratio $\beta/\gamma$ has been found, by means of the Jacobian quartic equation, implied in (73).

6. In the degenerate case of the catenary on a paraboloid, when

$$\beta = 3\gamma, \tag{75}$$

then, putting

$$4a\gamma^2 = a, \quad h = 3a, \quad ak^2 = 4a^3, \tag{76}$$

$$Z = (z+a)(z-a)^2(z-4a), \tag{77}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2}a\sqrt{\frac{a}{a}}\,\frac{z+a}{z(z-a)\sqrt{\{(z+a)(z-4a)\}}}$$

$$= \tfrac{1}{2}\left(\frac{-\sqrt{(aa)}}{z} + \sqrt{\frac{a}{a}}\,\frac{a+a}{z-a}\right)\frac{1}{\sqrt{\{(z+a)(z-4a)\}}}, \tag{78}$$

$$\psi = \tfrac{1}{2}\sin^{-1}2\sqrt{\left(\frac{a(z+a)}{(a+4a)z}\right)} - \sqrt{\left(\frac{a+a}{3a}\right)}\sin^{-1}\sqrt{\left(\frac{3a}{a+4a}\,\frac{z+a}{z-4a}\right)}, \tag{79}$$

or, with

$$r^2 = 4az, \quad b^2 = 4aa, \tag{80}$$

$$\psi = \tfrac{1}{2}\sin^{-1}\frac{b}{r}\sqrt{\left(\frac{r^2+4a^2}{a^2+b^2}\right)} - \tfrac{1}{2}\sqrt{\left(\frac{b^2+4a^2}{3a^2}\right)}\sin^{-1}\tfrac{1}{2}\sqrt{\left(\frac{3b^2}{a^2+b^2}\,\frac{r^2+4a^2}{r^2-b^2}\right)}. \tag{81}$$

Also
$$\frac{ds'}{dz} = \frac{(z-3a)\sqrt{(z+a)}}{(z-a)\sqrt{(z-4a)}}, \tag{82}$$

so that the catenary is rectifiable in the form

$$s' = \sqrt{(z+a.z-4a)} + a\,\mathrm{th}^{-1}\sqrt{\left(\frac{z-4a}{z+a}\right)} - 4\sqrt{\left(\frac{a.a+a}{3}\right)}\tan^{-1}\sqrt{\left(\frac{3a}{a+a}\,\frac{z+a}{z-4a}\right)}. \tag{83}$$

*The Whirling Spherical Catenary.*

7. In the catenary curve on a sphere due to centrifugal whirling (*L.M.S.* XXVII, p. 181),
$$\psi = \tfrac{1}{2}\int \frac{A\,dr^2}{r^2\sqrt{R}}, \tag{1}$$

where
$$R = (1-r^2)\left\{r^2(r^2-b)^2 - A^2\right\}$$
$$= (1-r^2)(r^2-r_1^2)(r^2-r_2^2)(r^2-r_3^2), \tag{2}$$

suppose; in which we can take
$$r_1 = r_2 + r_3, \tag{3}$$

in consequence of which relation we encounter the same Jacobian quartic (54) as with the catenary on the paraboloid.

We reduce (1) to the standard form (A), by putting
$$s-\sigma = s+x = \frac{Q'r^2}{1-r^2} = Q'\tan^2\theta, \tag{4}$$

where $\theta$ denotes the co-latitude of a point; and now
$$S = 4\,\frac{(Q'+x)\,r^2-x}{1-r^2}\,\frac{Q'^2r^4}{(1-r^2)^2} - \left\{(1+y)\,\frac{Q'r^2}{1-r^2} - x\right\}^2$$

has to assume the form
$$S = \frac{r^2(r^2-b)^2-A^2}{N^2(1-r^2)^3}, \tag{5}$$

and this leads to the Jacobian quartic for the determination of $Q'$, namely,
$$\left[\left\{(1+y)^2+4x\right\}Q'^2 + 4x(1+y)Q' + 3x^2\right]^2 - 4\left\{2x(1+y)Q' + 3x^2\right\}\left[4Q'^3 + \left\{(1+y)^2+4x\right\}Q'^2 + 2x(1+y)Q' + x^2\right] = 0. \tag{6}$$

Next,
$$ds = \frac{Q'dr^2}{(1-r^2)^2}, \tag{7}$$

$$\frac{ds}{\sqrt{S}} = \frac{NQ'dr^2}{\sqrt{R}}, \tag{8}$$

and, putting $r = 0$,

$$x = \sqrt{(-\Sigma)} = \frac{A}{N}, \tag{9}$$

$$\frac{\sqrt{(-\Sigma)}}{s-\sigma} = \frac{A}{NQ'}\ \frac{1-r^2}{r^2}, \tag{10}$$

so that
$$I = \tfrac{1}{2}PNQ'\int \frac{dr^2}{\sqrt{R}} - \tfrac{1}{2}A\int \frac{1-r^2}{r^2}\ \frac{dr^2}{\sqrt{R}}$$

$$= \tfrac{1}{2}N(PQ'+x)\int \frac{dr^2}{\sqrt{R}} - \psi. \tag{11}$$

To cancel the secular term, we must therefore introduce the condition that
$$PQ'+x = 0, \quad Q' = -\frac{x}{P}, \tag{12}$$

and now
$$s+x = -\frac{x}{P}\ \frac{r^2}{1-r^2}, \tag{13}$$

$$s = -\frac{x}{P}\ \frac{(1-P)r^2+P}{1-r^2}. \tag{14}$$

In the projection of the catenary on a plane perpendicular to the axis of rotation, if $\phi$ denotes the radial angle,
$$\cot\phi = \frac{dr}{rd\theta} = \frac{\sqrt{R}}{A}, \tag{15}$$

$$\sin\phi = \frac{A}{r\sqrt{\{(1-r^2)(r^2-b)^2+A^2\}}}, \tag{16}$$

$$p^2 = \frac{A^2}{(1-r^2)(r^2-b)^2+A^2}, \tag{17}$$

or
$$\frac{A^2}{p^2} = (1-r^2)(r^2-b)^2+A^2. \tag{18}$$

The projection will have points of inflexion where $p$ is stationary, corresponding to the maximum of $(1-r^2)(r^2-b)^2$, and therefore when
$$r^2 = \tfrac{1}{3}(b+2), \tag{19}$$

$$\frac{A^2}{p^2} = \tfrac{4}{27}(1-b)^3+A^2. \tag{20}$$

8. To illustrate the preceding theory, begin with the simplest parameter obtained by the bisection of a period, and put

$$v = \tfrac{1}{2}\omega_3. \tag{21}$$

We have to put, for a catenary on a paraboloid,

$$s+x = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{22}$$

in the associated pseudo-elliptic integral

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{-\frac{1}{2}(s+x)+x}{(s+x)\sqrt{S}}\,ds \\ &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{s}}{2(s+x)} = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{\{4(s+x)^2-s\}}}{2(s+x)}, \end{aligned} \tag{23}$$

in which $$S = 4s(s+x)^2-s^2, \tag{24}$$

obtained from the general case by putting

$$y = 0. \tag{25}$$

Then, substituting from (22),

$$(z+a)^3 S = 4\{(Q-x)z-xa\}Q^2z^2-(z+a)\{(Q-x)z-xa\}^2,$$

and this, in consequence of (29), § 3, must assume a form

$$= z(Az-B)^2-x^2a^3, \tag{26}$$

and thus

$$\begin{aligned} A^2 &= (Q-x)(4Q^2-Q+x), \\ 2AB &= 4Q^2x+(Q-x)(Q-3x), \\ B^2 &= 2Qx-3x^2, \end{aligned} \tag{27}$$

$$4(Q-x)(4Q^2-Q+x)(2Qx-3x^2) = \{(1+4x)Q^2-4Qx+3x^2\}^2,$$

$$(1-24x+16x^2)Q^4+48x^3Q^3-6x^2(1+4x)Q^2+8x^3Q-3x^4 = 0, \tag{28}$$

obtainable from the general case of (52), § 4, by putting $y = 0$.

Then, putting

$$Q = \frac{3x}{R+2}, \tag{29}$$

$$R^4-6(1-12x)R^2+8(1-18x)R-3(1-12x)^2 = 0, \tag{30}$$

a Jacobian quartic, with

$$\begin{aligned} a &= 1-12x, \quad b = 1-18x, \\ c^3 &= b^2-a^3 = -108x^2(1-16x). \end{aligned} \tag{31}$$

Writing this equation as a quadratic in $x$,

$$(R-1)^3(R+3)+72(R-1)^2x-432x^2=0, \tag{32}$$

so that, putting

$$12x=(R-1)^2q, \tag{33}$$

$$R+3+6(R-1)q-3(R-1)q^2=0, \tag{34}$$

$$R-1=\frac{4}{3(q-1)^2-4}. \tag{35}$$

Put

$$q-1=\tfrac{2}{3}r, \tag{36}$$

$$R-1=\frac{3}{r^2-3}, \tag{37}$$

$$x=\frac{2r+3}{4(r^2-3)^2}, \tag{38}$$

$$Q=\frac{2r+3}{4(r^2-2)(r^2-3)}, \tag{39}$$

$$1-16x=\frac{(r+1)^3(r-3)}{(r^2-3)^2}. \tag{40}$$

It is convenient to put

$$\frac{r-3}{r+1}=p^2 \quad \text{or} \quad -p^2, \tag{41}$$

according as the discriminant

$$\Delta=x^4(1-16x) \tag{42}$$

is positive or negative; and then

$$1-16x=\frac{64p^2}{(p^4-6p^2-3)^2} \quad \text{or} \quad \frac{-64p^2}{(p^4+6p^2-3)^2}; \tag{43}$$

$$x=\frac{(p^2-1)^3(p^2-9)}{16(p^4-6p^2-3)^2} \quad \text{or} \quad \frac{(p^2+1)^3(p^2+9)}{16(p^4+6p^2-3)^2}; \tag{44}$$

$$Q=\frac{(p^2-1)^3(p^2-9)}{8(p^4-6p^2-3)(p^4-10p^2-7)} \quad \text{or} \quad \frac{(p^2+1)^3(p^2+9)}{8(p^4+6p^2-3)(p^4+10p^2-7)}, \tag{45}$$

$$\kappa^2=\left(\frac{p+1}{p-1}\right)^6\left(\frac{p-3}{p+3}\right)^2 \quad \text{or} \quad \frac{(p^4+6p^2-3)^2}{(p^2+1)^3(p^2+9)}=\frac{1}{16x}, \tag{46}$$

$$\kappa'^2=\frac{32p(p^4-6p^2-3)}{(p-1)^6(p+3)^2} \quad \text{or} \quad \frac{64p^2}{(p^2+1)^3(p^2+9)}. \tag{47}$$

The roots of the cubic (B) are given by $s_3 = 0$, and

$$s_1, s_2 = \frac{1-8x \pm \sqrt{(1-16x)}}{8}$$

$$= \left\{ \frac{1 \pm \sqrt{(1-16x)}}{4} \right\}^2, \tag{48}$$

$$= \frac{(p \mp 1)^6 (p \pm 3)^2}{16 (p^4-6p^2-3)^2} \quad \text{or} \quad \frac{(ip \mp 1)^6 (ip \pm 3)^2}{16 (p^4+6p^2-3)^2}; \tag{49}$$

and then $4\gamma^2 = \frac{z_3}{a} = \frac{x}{Q-x} = \frac{p^4-10p^2-7}{(p^2-1)^2}$ or $\frac{p^4+10p^2-7}{(p^2+1)^2}$, (50)

while $(\beta \pm \gamma)^2 = \frac{z_1, z_2}{a} = \frac{p^4-10p^2-7}{4(p \mp 1)^2}$ or $\frac{p^4+10p^2-7}{4(ip \mp 1)^2}$; (51)

so that $$\frac{a}{z_1} + \frac{a}{z_2} = \frac{a}{z_3} - 1, \quad \text{and} \quad \frac{\beta}{\gamma} = p \text{ or } ip. \tag{52}$$

The constants $h$ and $k$ of the problem are determined by

$$\frac{h}{a} = \frac{B}{A} = \frac{4Q^2x + (Q-x)(Q-3x)}{2(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^2+3)(p^4-10p^2-7)}{4(p^2-1)^2}, \tag{53}$$

$$\frac{k^2}{a^2} = \frac{x^2}{A^2} = \frac{x^2}{(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^4-10p^2-7)^3}{(p^2-1)^4}. \tag{54}$$

Now we shall find that, in the closed branch of the catenary between $z_3$ and $z_2$, $$\psi = \tfrac{1}{2}(1+\kappa) F\phi - I, \tag{56}$$

where $$I = \tfrac{1}{2} \sin^{-1} \frac{\sqrt{(p^4-10p^2-7)}}{\sqrt{(p^2-1 . p^2-9)}} \frac{\sqrt{(z+a . z-z_3)}}{z}$$

$$= \tfrac{1}{2} \cos^{-1} \frac{4}{\sqrt{(p^2-1 . p^2-9)}} \frac{\sqrt{(z-z_1 . z-z_2)}}{z}, \tag{57}$$

and $$\sin^2 \phi = \frac{z_2+a}{z_2-z_3} \frac{z-z_3}{z+a}, \text{ \&c.} \tag{58}$$

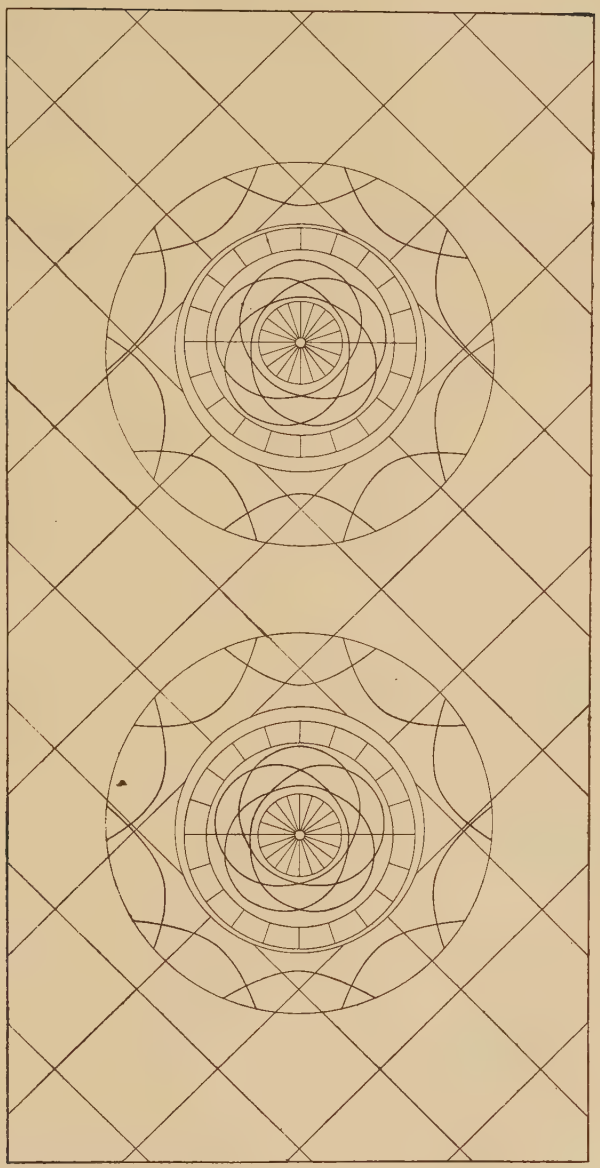
In the open infinite branch, extending from $z_1$ to infinity,

$$\psi = \tfrac{1}{2}(1+\kappa) F\phi + I, \tag{59}$$

but in this $I$ the $\sin^{-1}$ and $\cos^{-1}$ in (57) must be interchanged; also

$$\sin^2 \phi = \frac{z_2+a}{z_1+a} \frac{z-z_1}{z-z_2}, \text{ \&c.} \tag{60}$$

In the stereoscopic diagram, drawn by Mr. T. I. Dewar, we have taken $\kappa = 0{\cdot}77384$, so as to make the apsidal angles 144° and 54°.



Catenary on a Paraboloid.

9. In the associated case of the whirling spherical catenary, we put

$$s+x = \frac{Q'r^2}{1-r^2} = Q' \tan^2\theta, \tag{61}$$

agreeing with the case of the catenary on the paraboloid, if we take

$$r^2 = -\frac{z}{a}. \tag{62}$$

thus utilizing the imaginary part of the catenary, and the negative values of $Q$, given by the quartic (28), and making

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{-\frac{1}{2}\frac{Qr^2}{1-r^2}+x}{\frac{Qr^2}{1-r^2}\,\frac{A\sqrt{R}}{(1-r^2)^{\frac{3}{2}}}}\,\frac{Q\,dr^2}{(1-r^2)^2} \\ &= \tfrac{1}{2}\int \frac{-(\frac{1}{2}Q+x)\,r^2+x}{Ar^2\sqrt{(1-r^2)}\,\sqrt{R}}\,dr^2 \\ &= -\tfrac{1}{2}\,\frac{\frac{1}{2}Q+x}{A}\int \frac{dr^2}{\sqrt{(1-r^2)}\,\sqrt{R}} + \psi. \end{aligned} \tag{63}$$

If we should try to cancel the secular term by putting

$$\tfrac{1}{2}Q+x = 0, \tag{64}$$

then $x$ must be negative to make $Q$ positive.

Substituting in the expression for $S$,

$$S\,\frac{(1-r^2)^3}{x^2} = (1-16x)\,r^4+(1-16x)\,r^4-r^2-1, \tag{65}$$

and it is not possible to construct a real case which shall make this assume the form

$$\frac{r^2\,(r^2-b)^2-A^2}{H^2}. \tag{66}$$

10. For a parameter obtained by the trisection of a period, take

$$v = \tfrac{2}{3}\omega_3, \tag{67}$$

and build up solutions on the pseudo-elliptic integral

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{\frac{1}{3}s+m}{s\sqrt{S}}\,ds \\ &= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \end{aligned} \tag{68}$$

where

$$S = 4s^3-(s+m)^2. \tag{69}$$

Putting $$s = t^2,$$

$$I(v) = \tfrac{2}{3}\sin^{-1}\frac{\surd(2t^3-t^2-m)}{2t^{\frac{3}{2}}} = \tfrac{2}{3}\cos^{-1}\frac{\surd(2t^3+t^2+m)}{2t^{\frac{3}{2}}}, \quad (70)$$

and the associated catenaries on a paraboloid, or on a whirling sphere, are obtained by putting

$$t^2 = s = Q\frac{z}{z+a} = Q\sin^2\omega, \quad t = \surd Q\sin\omega, \quad (71)$$

or $$t^2 = s = Q'\frac{r^2}{1-r^2} = Q'\tan^2\theta, \quad t = \surd Q'\tan\theta; \quad (72)$$

and $Q$, $Q'$ are determined by the condition that

$$\tan\omega_3 = \tan\omega_1 \pm \tan\omega_2, \quad (73)$$

or $$\sin\theta_3 = \sin\theta_1 \pm \sin\theta_2. \quad (74)$$

Then $$\frac{s_1}{s} = \frac{z_1}{z}\frac{z+a}{z_1+a} \text{ or } \frac{r_1^2}{r^2}\frac{1-r^2}{1-r_1^2}, \quad (75)$$

$$1-\frac{s_1}{s} = \frac{a}{z}\frac{z-z_1}{z_1+a} \text{ or } \frac{1}{r^2}\frac{r^2-r_1^2}{1-r_1^2}, \quad (76)$$

$$\psi+\tfrac{1}{6}qu = I(v) = \tfrac{1}{3}\sin^{-1}\surd\left(1-\frac{s_1}{s}.1-\frac{s_2}{s}.1-\frac{s_3}{s}\right)$$

$$= \tfrac{1}{3}\sin^{-1}\surd\left(\frac{a^3}{z^3}\ \frac{z-z_1.z-z_2.z-z_3}{z_1+a.z_2+a.z_3+a}\right)$$

$$\text{or } \tfrac{1}{3}\sin^{-1}\surd\left(\frac{1}{r^6}\ \frac{r^2-r_1^2.r^2-r_2^2.r^2-r_3^2}{1-r_1^2.1-r_2^2.1-r_3^2}\right)$$

$$= \tfrac{1}{3}\sin^{-1}\surd\left\{\frac{a^3}{z^3}\ \frac{z(z-h)^2-ak^2}{a(a+h)^2+ak^2}\right\}$$

$$\text{or } \tfrac{1}{3}\sin^{-1}\frac{1}{r^3}\surd\left\{\frac{r^2(r^2-b^2)^2-A^2}{(1-b^2)^2-A^2}\right\}. \quad (77)$$

Again, since $$\psi+\tfrac{1}{6}u = I(v) = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \quad (78)$$

therefore it assumes the form

$$\tfrac{1}{3}\cos^{-1}\frac{(Hz-K)\surd(z+a)}{z^{\frac{3}{2}}\surd\{(a+h)^2+k^2\}} \text{ or } \tfrac{1}{3}\cos^{-1}\frac{(Hr^2-K)\surd(1-r^2)}{r^3\surd\{(1-b^2)^2-A^2\}}, \quad (79)$$

where
$$H^2 = 2ah + h^2 + k^2 \quad \text{or} \quad A^2 + 2b^2 - b^4, \tag{80}$$
$$2HK = a(h^2 + k^2) \quad \text{or} \quad -A^2, \tag{81}$$
$$K^2 = a^2 k^2 \quad \text{or} \quad A^2 - b^4, \tag{82}$$

and therefore
$$a^2(h^2+k^2)^2 - 4a^2k^2(2ah + h^2 + k^2) = 0,$$
$$(h^2+k^2)(h^2-3k^2) = 8ahk^2, \tag{83}$$

or
$$4(A^2+2b^2-b^4)(A^2-b^4) - A^4 = 0,$$
$$3A^4 + 8A^2b^2(1-b^2) - 8b^6 + 4b^8. \tag{84}$$

Otherwise, the equation (52), § 4, becomes in this case
$$(1+32m)Q^4 + 48m^2Q^3 - 6m^2Q^2 - 8m^3Q - 3m^4 = 0; \tag{85}$$

or, on putting
$$Q = \frac{3m}{R-2}, \tag{86}$$
$$R^4 - 6R^2 - 8(1+54m)R - 3 = 0, \tag{87}$$

so that
$$m = \frac{(R+1)^3(R-3)}{432R}, \tag{88}$$
$$Q = \frac{(R+1)^3(R-3)}{144R(R-2)}, \tag{89}$$

and then, in the catenary on the paraboloid, we find
$$\frac{h}{a} = -\frac{2R(R-2)}{R^2-1}, \tag{90}$$
$$\frac{k^2}{a^2} = -\frac{4R(R-2)^3}{3(R^2-1)^2}. \tag{91}$$

If, in (83), we had put
$$k^2 = h^2 p, \tag{92}$$

then
$$\frac{h}{a} = \frac{8p}{(1+p)(1-3p)}, \tag{93}$$

agreeing with (90), when
$$p = -\frac{R-2}{3R}. \tag{94}$$

But there does not appear to be any simple explicit relation of $x$ in terms of $R$, or $R$ in terms of $x$.

In the whirling catenary the secular term would be cancelled by putting
$$Q' = -Q = 3m, \tag{95}$$

but this makes $R = 1$, and $h$ and $k$ infinite.

11. Other trisection formulas, corresponding to parameters

$$v = \tfrac{1}{3}\omega_3 \quad \text{or} \quad \tfrac{2}{3}\omega_3, \tag{96}$$

can be obtained by taking (*L.M.S.*, xxv, p. 216)

$$\gamma_6 = 0 \quad \text{or} \quad x = y - y^2, \tag{97}$$

and then

$$\begin{aligned} I(\tfrac{1}{3}\omega_3) &= \tfrac{1}{2}\int \frac{\frac{2}{3}(s+y-y^2)-y(1-y)}{s+y-y^2}\,\frac{ds}{\sqrt{S}} \\ &= \tfrac{1}{3}\sin^{-1}\frac{s-\frac{1}{2}(1-y)(1-2y)}{(s+y-y^2)^{\frac{3}{2}}}\sqrt{(s-y^2)} \\ &= \tfrac{1}{3}\cos^{-1}\frac{\sqrt{\{s^2-\frac{1}{4}(1-y)(1-sy)s+\frac{1}{4}y^2(1-y)^2\}}}{(s+y-y^2)^{\frac{3}{2}}} \end{aligned} \tag{98}$$

or

$$\begin{aligned} I(\tfrac{2}{3}\omega_3) &= \tfrac{1}{2}\int \frac{\frac{1}{3}(1-3y)s-y^2(1-y)}{s}\,\frac{ds}{\sqrt{S}} \\ &= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} \\ &= \tfrac{1}{3}\cos^{-1}\frac{(1-3y)s-y^2(1-y)}{2s^{\frac{3}{2}}}. \end{aligned} \tag{99}$$

To construct an algebraical catenary on a paraboloid by utilizing the integral (99), put

$$1-3y = 0, \quad y = \tfrac{1}{3}, \quad x = \tfrac{2}{9}. \tag{100}$$

But now

$$S = 4s^3 - {}_{7}^{4}{}_{2}{}_{9}, \tag{101}$$

and, putting

$$s = \frac{Qz}{z+a}, \tag{102}$$

$$\begin{aligned} (z+a)^3 S &= 4Q^3z^3 - {}_{7}^{4}{}_{2}{}_{9}(z+a)^3 \\ &= z(Bz-Ca)^2 - {}_{7}^{4}{}_{2}{}_{9}a^3, \end{aligned} \tag{103}$$

the requisite form, provided that

$$B^2 = 4Q^3 - {}_{7}^{4}{}_{2}{}_{9}, \quad 2BC = -{}_{2}^{4}{}_{4}{}_{3}, \quad C^2 = -{}_{2}^{4}{}_{4}{}_{3}, \tag{104}$$

leading to the value

$$Q = \frac{1}{9\sqrt[3]{4}}; \tag{105}$$

and therefore

$$(z+a)^3 S = -\frac{z}{243}(z+2a)^2 - {}_{7}^{4}{}_{2}{}_{9}a^3, \tag{106}$$

so that the catenary is imaginary.

In the same way, in attempting to construct an algebraical case of the whirling spherical catenary, by putting, in (98) or (99),

$$s+y-y^2 \quad \text{or} \quad s=\frac{Q'r^2}{1-r^2}, \tag{107}$$

where $$Q'=-\tfrac{3}{2}y\,(1-y) \quad \text{or} \quad -\frac{3y^2\,(1-y)}{1-3y}, \tag{108}$$

the results obtained are again imaginary.

12. An algebraical case of a catenary on a paraboloid, corresponding to a parameter

$$v=\tfrac{1}{4}\omega_3, \tag{109}$$

can be constructed by taking the pseudo-elliptic integral on p. 228, *L.M.S.*, xxv, and putting

$$z=\tfrac{3}{4}; \tag{110}$$

this makes $$x=-\tfrac{3}{8}, \quad y=-\tfrac{3}{2},$$

$$S_1=-\tfrac{5}{16}, \quad S_2=\tfrac{3}{32}, \quad S_3=\tfrac{9}{256},$$

$$a=\frac{3^2\times 19}{2^{12}}, \quad b=\frac{3^3\times 215}{2^{18}}, \quad c=\frac{3^5\sqrt[3]{2}}{2^{12}}; \tag{111}$$

whence the equation for $Q$ can be constructed, when we put

$$s-\tfrac{3}{8}=\frac{Qz}{z+a}=\frac{Qr^2}{r^2+4a^2}, \tag{112}$$

and now the catenary is given by putting this expression for $s$ in

$$\psi=\tfrac{1}{4}\sin^{-1}\frac{\sqrt{(s-\frac{9}{16})}}{4\,(s-\frac{3}{8})^2}$$

$$=\tfrac{1}{4}\cos^{-1}\frac{(s-\frac{5}{8})\sqrt{(16s^2-4s+\frac{9}{4})}}{4\,(s-\frac{3}{8})^2}. \tag{113}$$

13. To form the corresponding algebraical whirling spherical catenary, we start with the pseudo-elliptic integral (*L.M.S.*, xxv, p. 228)

$$I(\tfrac{1}{4}\omega_3)=\int\frac{\frac{1}{4}\dfrac{3-4z}{1-z}\,(s+z-2z^2)-z+2z^2}{s+z-2z^2}\,\frac{ds}{\sqrt{S}}$$

$$=\tfrac{1}{4}\cos^{-1}\frac{(s-1+2z-2z^2)\sqrt{\left\{4s^2-\dfrac{(1-2z)^4}{(1-z)^2}\,s+\dfrac{z^2\,(1-2z)^4}{(1-z)^2}\right\}}}{2\,(s+z-2z^2)^2}$$

$$= \tfrac{1}{4}\sin^{-1}\frac{\dfrac{(3-4z)\,s-(1-2z)^3}{1-z}\sqrt{(s-z^2)}}{2\,(s+z-2z^2)^2}, \tag{114}$$

and put
$$s+z-2z^2 = \frac{Q'r^2}{1-r^2} \tag{115}$$

with
$$Q' = -\frac{4z\,(1-z)\,(1-2z)}{3-4z}, \tag{116}$$

thereby cancelling the secular term.

Then
$$s-z^2 = -\frac{z\,(1-z)}{3-4z}\cdot\frac{(1-4z)\,r^2+3-4z}{1-r^2}, \tag{117}$$

so that
$$r_1^2 = -\frac{3-4z}{1-4z}. \tag{118}$$

Also
$$4s^2-\frac{(1-2z)^4}{(1-z)^2}s+\frac{z^2\,(1-2z)^4}{(1-z)^2} = \frac{z\,(1-2z)^2}{(1-z)\,(3-4z)^2\,(1-r^2)^2}$$
$$\times\left\{-(3-8z+8z^2)(1-8z+8z^2)\,r^4-2\,(3-4z)(1-8z+8z^2)\,r^2+(3-4z)^2\right\}, \tag{119}$$

so that
$$r_2^2+r_3^2 = -2\frac{3-4z}{3-8z+8z^2}, \tag{120}$$

$$r_2^2\,r_3^2 = -\frac{(3-4z)^2}{(3-8z+8z^2)\,(1-8z+8z^2)}, \tag{121}$$

$$r_1^2-r_2^2-r_3^2 = -\frac{(3-4z)\,(1+8z^2)}{(1-4z)\,(3-8z+8z^2)}. \tag{122}$$

Then, from the conditions
$$(r_1^2-r_2^2-r_3^2)^2 = 4r_2^2r_3^2, \tag{123}$$

we obtain the equation
$$(1+8z^2)^2\,(1-8z+8z^2)+4\,(1-4z)^2\,(3-8z+8z^2) = 0 \tag{124}$$

or
$$512z^6-512z^5+704z^4-896z^3+504z^2-136z+13 = 0, \tag{125}$$

a sextic equation for the determination of $z$.

Putting $z = \frac{1}{4}p$, this becomes
$$p^6-4p^5+22p^4-112p^3+252p^2-272p+104 = 0, \tag{126}$$

the roots of which are
$$p = 2{\cdot}77764,\ 0{\cdot}7604,\ 1{\cdot}22961 \pm 0{\cdot}92181i,\ 0{\cdot}90861 \pm 4{\cdot}45039i, \tag{127}$$

as calculated by Mr. T. I. Dewar.

Choosing the root $p = 2{\cdot}77764$, this makes

$$z = 0{\cdot}6944, \tag{128}$$

$$r_1^2 = 0{\cdot}1251, \quad r_1 = \sin 20^\circ\, 42', \tag{129}$$

$$Q' = 0{\cdot}1386, \tag{130}$$

$$1 - 8z + 8z^2 = -0{\cdot}68, \tag{131}$$

$$3 - 8z + 8z^2 = 1{\cdot}32, \tag{132}$$

$$b = \tfrac{1}{2}(r_1^2 + r_2^2 + r_3^2) = -0{\cdot}1096, \tag{133}$$

$$A^2 = r_1^2 r_2^2 r_3 = 0{\cdot}008. \tag{134}$$

At the point of inflexion on the projection,

$$r^2 = \tfrac{1}{3}(2 + b) = 0{\cdot}63, \quad r = \sin 52^\circ\, 32'. \tag{135}$$

Substituting for $s$ in terms of $r^2$ from (117) in (114), we shall obtain equations of the form

$$\psi = \tfrac{1}{4}\sin^{-1}\frac{(Hr^2 + H_1)\sqrt{(1 - r^2 \cdot r^2 - r_1^2)}}{r^4}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(Kr^2 + K_1)\sqrt{(r^2 - r_2^2 \cdot r^2 - r_3^2)}}{r^2}, \tag{136}$$

$$H = \frac{(3 - 4z \cdot 4z - 1)^{\frac{3}{2}}}{32\,(2z - 1)\,(z - z^2)^{\frac{3}{2}}}, \quad H_1 = \frac{(3 - 4z)^{\frac{3}{2}}\,(4z - 1)^{\frac{1}{2}}}{32\,(2z - 1)\,(z + z^2)^{\frac{3}{2}}}, \tag{137}$$

$$K = \frac{(3 - 8z + 8z^2)^{\frac{3}{2}}\,(-1 + 8z - 8z^2)^{\frac{1}{2}}}{32\,(2z - 1)\,(z - z^2)^{\frac{3}{2}}\,(3 - 4z)^{\frac{3}{2}}}, \quad K_1 = -\frac{(3 - 8z + 8z^2)^{\frac{1}{2}}\,(-1 + 8z - 8z^2)^{\frac{1}{2}}}{32\,(2z - 1)\,(z - z^2 \cdot 3 - 4z)^{\frac{3}{2}}}, \tag{138}$$

giving the form of the whirling spherical catenary shown in the stereoscopic diagram annexed, drawn by Mr. T. I. Dewar.

14. In the case of quinquisection, with a parameter

$$v = \tfrac{2}{5}\omega_3, \tag{139}$$

the relation $\gamma_5 = 0$ is satisfied by $y = x$, and now

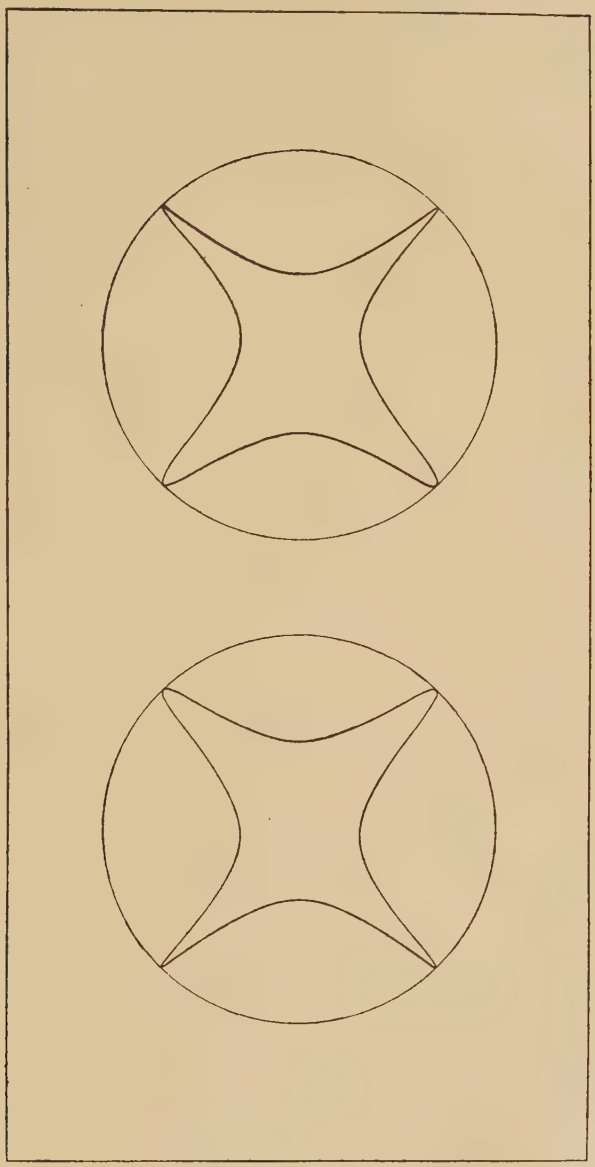
$$S_1 = \tfrac{1}{4}(x^2 + 6x + 1), \quad S_2 = \tfrac{1}{2}(x^2 + x), \quad S_3 = \tfrac{1}{4}x^2; \tag{140}$$

and

$$a = \tfrac{1}{16}x^2\,(x^2 - 10x + 1), \tag{141}$$

$$b = -\tfrac{1}{64}x^3\,(x^3 + 39x^2 - 15x + 1), \tag{142}$$

$$c^3 = \frac{27x^9}{1024}(x^2 + 11x - 1). \tag{143}$$



Catenary on a Whirling Sphere.

The associated pseudo-elliptic integral is

$$I(v) = \tfrac{1}{2}\int \frac{\tfrac{1}{5}(x+3)(s+x)-x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(s+x-1)\sqrt{S}}{2(s+x)^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\cos^{-1}\frac{(x+3)s^2+(2x^2+4x-1)s+x^3+x^2}{2(s+x)^{\frac{5}{2}}}, \qquad (144)$$

and the corresponding catenary on the paraboloid or on the whirling sphere is obtained by putting

$$s+x = \frac{Qz}{z+a} = Q\sin^2\omega, \qquad (145)$$

or

$$s+x = \frac{Q'r^2}{1-r^2} = Q'\tan^2\theta, \qquad (146)$$

where $Q$ is determined by the quartic (52), § 4; and then

$$\psi = \frac{x+3}{10}\int\frac{ds}{\sqrt{S}} - I(v) \qquad (147)$$

or

$$\psi = \tfrac{1}{2}\left(\frac{x+3}{5}Q'+x\right)\int\frac{ds}{\sqrt{S}} - I(v). \qquad (148)$$

If we put

$$s+x = t^2, \qquad (149)$$

then we find that we can put

$$I(v) = \tfrac{2}{5}\cos^{-1}\frac{(t+1)\sqrt{\{2t^3-(1-x)t^2-2xt+x\}}}{2t^{\frac{5}{2}}}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{(t-1)\sqrt{\{2t^3+(1-x)t^2-2xt-x\}}}{2t^{\frac{5}{2}}}, \qquad (150)$$

and, to obtain the catenaries, we put

$$t = \sqrt{Q}\sin\omega \quad \text{or} \quad \sqrt{Q'}\tan\theta. \qquad (151)$$

To construct an algebraical catenary on the paraboloid, put

$$x+3 = 0, \qquad (152)$$

and now the cubic $2t^3-(1-x)t^2-2xt+x = 0$

becomes

$$2t^3-4t^2+6t-3 = 0, \qquad (153)$$

which has one real root

$$t_1 = \frac{3}{\sqrt[3]{(10)}+2}. \qquad (154)$$

Also
$$S_1 = -2, \quad S_2 = 3, \quad S_3 = \tfrac{9}{4};$$
$$a = \frac{45}{2} = 22{\cdot}5, \quad b = \frac{27 \times 185}{32};$$
$$c^3 = \frac{3^{12} \times 5^2}{2^{10}}, \quad c = \frac{81\sqrt[3]{(100)}}{16} = 23{\cdot}5; \tag{155}$$

and the real roots of (54), § 4, are

$$R = +\sqrt{\left\{\frac{45}{2} + \frac{81\sqrt[3]{(100)}}{16}\right\}}$$
$$\pm\sqrt{\left[45 - \frac{81\sqrt[3]{(100)}}{16} + \tfrac{9}{8}\sqrt{(10)}\sqrt{\{160 - 36\sqrt[3]{(100)} + 6561\sqrt[3]{(10)}\}}\right]},$$

or, approximately, $R = 15$ and $-1{\cdot}437$,

and
$$Q = 0{\cdot}375 \quad \text{or} \quad 4{\cdot}5. \tag{156}$$

Denoting by $\omega_1$ the value of $\omega$ corresponding to $t = t_1$, so that

$$\sin\omega_1 = \frac{t_1}{\sqrt{Q}}, \tag{157}$$

the first value of $Q$ makes $\sin\omega_1$ greater than unity, and must be rejected, but the second makes

$$\sin\omega_1 = 0{\cdot}3405, \quad \omega_1 = 19^\circ\,55';$$

and
$$\frac{r_1}{2a} = \tan\omega_1 = 0{\cdot}3623. \tag{158}$$

When
$$t = \sqrt{\tfrac{3}{5}}, \quad \omega = 21^\circ\,25', \quad \psi = 18^\circ;$$
$$t = 1, \quad \omega = 28^\circ\,8', \quad \psi = 36^\circ;$$
$$t = \sqrt{Q}, \quad \omega = 90^\circ, \quad \psi = 51^\circ\,24'. \tag{159}$$

15. To make the associated catenary on a whirling sphere into an algebraical curve, we put

$$Q' = -\frac{5x}{x+3} \tag{160}$$

and
$$s + x = -\frac{5x}{x+r}\frac{r^2}{1-r^2}, \tag{161}$$

which makes
$$S = 4s(s+x)^2 - \{(1+x)(s+x) - x\}^2$$
$$= \frac{x^2}{(x+3)^3}\frac{r^2(Br^2 - C)^2 - (x+3)^3}{(1-r^2)^3}, \tag{162}$$

where
$$\left.\begin{aligned} B^2 &= 4(4x-3)(x^2+11x-1) \\ 2BC &= 8(x+3)(x^2+11x-1) \\ C^2 &= -(x+3)^2(7x+1) \end{aligned}\right\}; \tag{163}$$

and therefore leads to the equation for $x$,

$$32x^2+27x-7=0,$$

$$x=\frac{-27\pm5\sqrt{(65)}}{64}, \tag{164}$$

of which the positive value makes $Q'$ negative, and must therefore be rejected.

Taking, then, $$x+3=\frac{165-5\sqrt{(65)}}{64}, \tag{165}$$

$$Q'=\frac{5\sqrt{(65)}+27}{33-\sqrt{(65)}}$$

$$=\frac{3\{7\sqrt5+3\sqrt{(13)}\}^2}{1024}, \tag{166}$$

$$q=\frac{1}{\sqrt{Q}}=\frac{32}{\sqrt3\{7\sqrt5+3\sqrt{(13)}\}}$$

$$=\frac{7\sqrt5-3\sqrt{(13)}}{4\sqrt3}; \tag{167}$$

and, now, with $$t=\frac{\tan\theta}{q}, \tag{168}$$

the equation of the catenary can be written in either of the forms

$$(\tan\theta)^{\frac{5}{2}}\sin\tfrac{5}{2}\psi=(\tan\theta+q)\sqrt{\{\tfrac{1}{2}\tan^3\theta-\tfrac{1}{4}(1-x)q\tan^2\theta}$$

$$-\tfrac{1}{2}xq^2\tan\theta+\tfrac{1}{4}xq^3\}, \tag{169}$$

$$(\tan\theta)^{\frac{5}{2}}\cos\tfrac{5}{2}\psi=(\tan\theta-q)\sqrt{\{\tfrac{1}{2}\tan^3\theta+\tfrac{1}{4}(1-x)q\tan^2\theta}$$

$$-\tfrac{1}{2}xq^2\tan\theta-\tfrac{1}{4}xq^3\}. \tag{170}$$

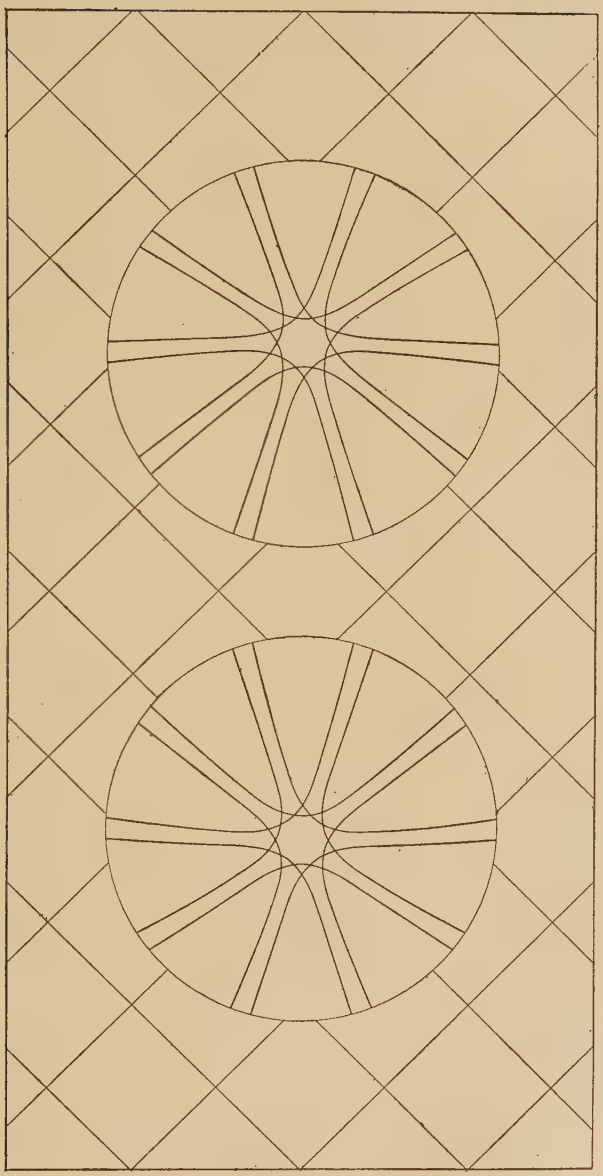
So also the equation of the algebraical catenary on the paraboloid may be written

$$(\sin\omega)^{\frac{5}{2}}\sin\tfrac{5}{2}\psi=(\sin\omega+q)\sqrt{(\tfrac{1}{2}\sin^3\omega-q\sin^2\omega+\tfrac{3}{2}q^2\sin\omega-\tfrac{3}{4}q^3)}, \tag{171}$$

$$(\sin\omega)^{\frac{5}{2}}\cos\tfrac{5}{2}\psi=(\sin\omega-q)\sqrt{(\tfrac{1}{2}\sin^3\omega+q\sin^2\omega+\tfrac{3}{2}q^2\sin\omega+\tfrac{3}{4}q^3)}, \tag{172}$$

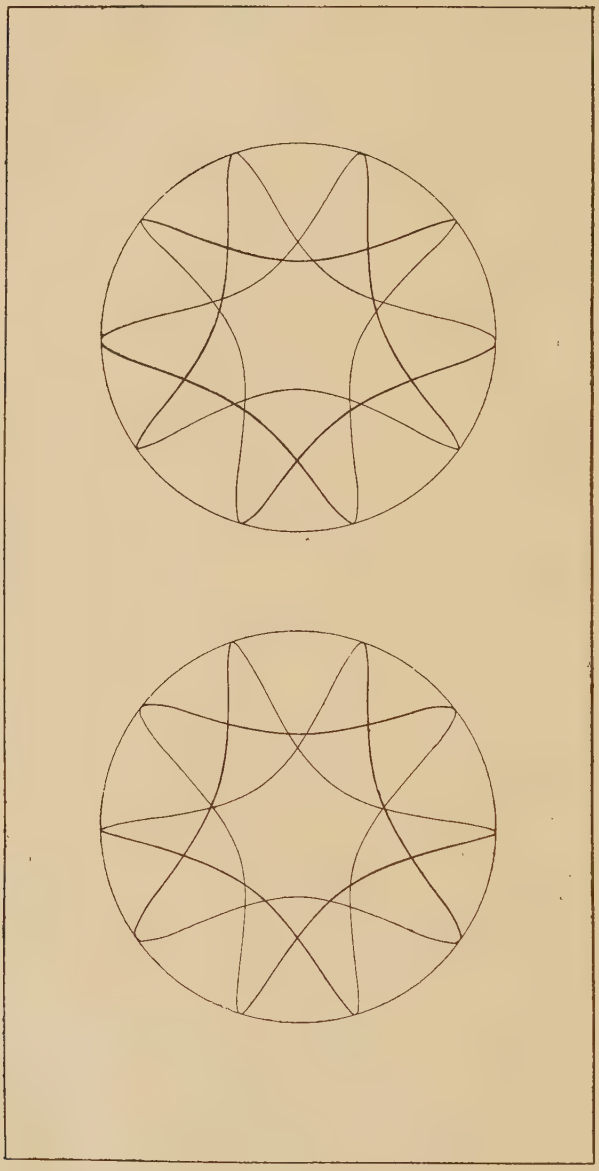
where $$q=\frac{1}{\sqrt{(4.5)}}=0{\cdot}4714.$$

These two catenaries are represented in the annexed stereoscopic diagrams, drawn by Mr. T. I. Dewar.



Catenary on a Paraboloid.

Catenary on a Whirling Sphere.



16. With a parameter $v = \frac{2}{7}\omega_3$, (173)

the associated pseudo-elliptic integral (*L.M.S.*, XXVI, p. 225)

$$I(v) = \tfrac{1}{2}\int \frac{\frac{1}{7}(5-z-z^2)(s+x)-x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{7}\sin^{-1}\frac{s^2+Cs+D}{2(s+x)^{\frac{7}{2}}}\sqrt{S}, \quad \&c., \tag{174}$$

in which $x = z(1-z)^2, \quad y = z(1-z),$ (175)

and, again putting $s+x = t^2$, the result can be expressed by

$$I(v) = \tfrac{2}{7}\sin^{-1}\frac{(t^2-t-1+z)\sqrt{\{2t^3-(1-z-z^2)t^2-2z(1-z)t+z(1-z)^2\}}}{2t^{\frac{7}{2}}}$$

$$= \tfrac{2}{7}\cos^{-1}\frac{(t^2+t-1+z)\sqrt{\{2t^3+(1-z-z^2)t^2-2z(1-z)t-z(1-z)^2\}}}{2t^{\frac{7}{2}}}, \tag{176}$$

and, to obtain the corresponding catenaries, we put

$$t = \sqrt{Q}\sin\omega \quad \text{or} \quad \sqrt{Q'}\tan\theta, \tag{177}$$

and determine $Q$ from the condition that $S$ assumes the form

$$M\frac{r^2(Br^2-C)^2-E}{(r^2+4a^2)^3} \text{ or } \frac{}{(1-r^2)^3}. \tag{178}$$

To make the catenary on the paraboloid algebraical, put

$$5-z-z^2 = 0, \quad z = \frac{-1\pm\sqrt{(21)}}{2}. \tag{179}$$

But, to obtain an algebraical whirling spherical catenary, put

$$Q' = \frac{-7z(1-z)^2}{5-z-z^2}, \tag{180}$$

and now it will be found that

$$\left.\begin{aligned} B^2 &= 4(1-8z+5z^2+z^3)(5-19z+11z^2-9z^3)\\ 2BC &= -8(5-z-z^2)(2-3z)(1-8z+5z^2+z^3)\\ C^2 &= (5-z-z^2)^2(1-17z+11z^2)\end{aligned}\right\}, \tag{181}$$

thus leading to the equation

$$4(2-3z)^2(1-8z+5z^2+z^3)-(1-17z+11z^2)(5-19z+11z^2-9z^3) = 0$$

or $$(5z-1)(27z^4-23z^3-26z^2+17z-11) = 0. \tag{182}$$

The value $z = \frac{1}{5}$ makes $Q'$ negative; of the other four roots

$$+1{\cdot}376, \quad -1{\cdot}03, \quad \text{and} \quad 0{\cdot}253\pm 0{\cdot}464i, \tag{183}$$

the negative root will make $Q'$ positive, and give a real case.

*The Trajectory of a Particle sliding on a smooth Paraboloid.*

17. The path of a particle on a smooth paraboloid whose axis is vertical is of a similar analytical character to the catenary curve. (Züge, *Grunert's Archiv*, 70, 1884.)

The equations for the conservation of momentum and energy are

$$r^2 \frac{d\psi}{dt} = H, \tag{1}$$

$$\tfrac{1}{2}\frac{dz^2}{dt^2} + \tfrac{1}{2}\frac{dr^2}{dt^2} + \tfrac{1}{2}r^2\frac{d\psi^2}{dt^2} = g\,(z \sim h)\,; \tag{2}$$

with
$$r^2 = 4az, \tag{3}$$

$$\frac{dr}{dt} = \sqrt{\frac{a}{z}}\,\frac{dz}{dt} \tag{4}$$

Thence
$$\left(1 + \frac{a}{z}\right)\frac{dz^2}{dt^2} = 2g\,(z \sim h) - \frac{H^2}{4az}, \tag{5}$$

or
$$\frac{dz^2}{dt^2} = \frac{2gz\,(z \sim h) - \dfrac{H^2}{4a}}{z + a}$$

$$= 2g\,\frac{z\,(z \sim h) - \frac{1}{4}k_2^2}{z + a}\,; \tag{6}$$

putting
$$H^2 = 2gak^2 \quad \text{and} \quad 4az\frac{d\psi}{dt} = H = \sqrt{(2gak^2)}, \tag{8}$$

so that, dividing (8) by (6),

$$\frac{d\psi}{dz} = \tfrac{1}{2}\frac{k}{\sqrt{a}}\,\frac{\sqrt{(z+a)}}{z\sqrt{\{4z\,(z \sim h) - k^2\}}}$$

$$= \tfrac{1}{2}\frac{k}{\sqrt{a}}\,\frac{z+a}{z\sqrt{Z}}, \tag{9}$$

where
$$Z = (z+a)\,\{4z\,(z \sim h) - k^2\}. \tag{10}$$

Putting
$$g = 2an^2, \tag{11}$$

$$n\frac{dt}{dz} = \frac{z+a}{\sqrt{(aZ)}}, \tag{12}$$

so that the time is given by Legendre's $E\phi$ and $F\phi$.

Taking $r^2$ as independent variable,

$$\psi = \tfrac{1}{2}k\int \frac{(r^2 + 4a^2)\,dr^2}{r^2\sqrt{R}}, \tag{13}$$

where
$$R = (r^2 + 4a^2)\,\{r^2\,(r^2 \sim 4ah) - 4a^2k^2\}, \tag{14}$$

so that the projection of the catenary is of herpolhode character.

(i.) When the vertex of the paraboloid is upwards,

$$Z = (z+a)\left\{4z(z-h)-k^2\right\}$$
$$= 4(z-z_3)(z-z_2)(z-z_1), \tag{15}$$

where $z_1 = \frac{1}{2}h+\frac{1}{2}\sqrt{(h^2+k^2)}$, (16)

$z_2 = \frac{1}{2}h-\frac{1}{2}\sqrt{(h^2+k^2)}$, or $-a$, a negative quantity, (17)

$z_3 = -a$, or $\frac{1}{2}h-\frac{1}{2}\sqrt{(h^2+k^2)}$; (18)

so that $\infty > z > z_1 > 0 > z_2 > z_3 > -\infty$,

and, putting $z = M^2(s-\sigma) = \wp u-\wp v$, (19)

then $u = p\omega_1$, (20)

for real parts of the path of the particle, while

$$v = \omega_1+f\omega_3. \tag{21}$$

Then $z+a = M^2(s-s_3)$ or $M^2(s-s_2)$, (22)

$a = M^2(\sigma-s_3)$ or $M^2(\sigma-s_2)$, (23)

$z_1+z_{2,3} = h = M^2(s_1+s_{2,3}-2\sigma)$, (24)

$z_1-z_{2,3} = \sqrt{(h^2+k^2)} = M^2(s_1-s_{2,3})$, (25)

$$k\sqrt{a} = M^3\sqrt{(-\Sigma)}, \tag{26}$$

$$\frac{k}{\sqrt{a}} = M\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}, \tag{27}$$

$$\sqrt{Z} = M^3\sqrt{S}, \tag{28}$$

$$dz = M^2 ds, \tag{29}$$

$$\frac{dz}{\sqrt{Z}} = \frac{1}{M}\frac{ds}{\sqrt{S}}; \tag{30}$$

and, introducing these substitutions into (9),

$$\psi = \tfrac{1}{2}k\int_{z_1}^{z}\frac{z+a}{z\sqrt{(aZ)}}dz$$
$$= \tfrac{1}{2}\int_{s_1}^{s}\left\{\frac{\sqrt{(-\Sigma)}}{s-\sigma}+\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\frac{ds}{\sqrt{S}}$$
$$= I(v)-\tfrac{1}{2}\left\{P(v)-\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\int\frac{ds}{\sqrt{S}}$$
$$= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\int\frac{dz}{\sqrt{Z}}$$
$$= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\frac{F\phi}{\sqrt{(z_1-z_3)}}, \tag{31}$$

in which $$\sin^2\phi = \frac{z-z_1}{z-z_2}, \quad \&c., \tag{32}$$

and now, in (12),

$$ndt = \frac{z-z_{2,3}}{\sqrt{(aZ)}}\,dz$$

$$= \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\Delta\phi\,d\phi}{\kappa'\cos^2\phi} \quad \text{or} \quad \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\kappa'^2 d\phi}{\cos^2\phi\,\Delta\phi}, \tag{33}$$

so that (Legendre, I, p. 257)

$$nt = \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{1}{\kappa'}\,(\tan\phi\,\Delta\phi + F\phi - E\phi)$$

or $$\sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,(\tan\phi\,\Delta\phi + \kappa'^2 F\phi - E\phi). \tag{34}$$

18. As the secular term cannot be cancelled, it will be sufficient to consider the single special case of a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3; \tag{35}$$

and now, with $$z+a = M^2(s-s_3), \quad z = M^2(s-\sigma), \tag{36}$$

we shall find (*L.M.S.*, xxv, p. 212)

$$I = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{(hz+ah)}}{z} = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z^2-hz-ah)}}{z}, \tag{37}$$

with $$k^2 = 4ah = \frac{4a^2}{c+c^2}, \tag{38}$$

$$z_1,\ z_2 = \tfrac{1}{2}h \pm \sqrt{(\tfrac{1}{4}h^2+ah)} = \frac{a}{c} \quad \text{or} \quad -\frac{a}{1+c}. \tag{39}$$

Differentiating (37), $$\frac{dI}{dz} = \sqrt{h}\,\frac{\frac{1}{2}z+a}{z\sqrt{Z}}$$

$$= \frac{d\psi}{dz} - \tfrac{1}{2}\frac{\sqrt{h}}{\sqrt{Z}}. \tag{40}$$

Also $$\kappa = \sqrt{\left(\frac{z_2-z_3}{z_1-z_3}\right)} = \frac{\sqrt{(4a+h)}-\sqrt{h}}{\sqrt{(4a+h)}+\sqrt{h}}, \tag{41}$$

and $$\tfrac{1}{2}\int_{z_1}^{z}\frac{\sqrt{h}\,dz}{\sqrt{Z}} = \tfrac{1}{2}\sqrt{\left(\frac{h}{z_1-z_3}\right)}\,F\phi$$

$$= \frac{\sqrt{h}F\phi}{\sqrt{(4a+h)}+\sqrt{h}} = \tfrac{1}{2}(1-\kappa)\,F\phi, \tag{42}$$

with $$\sin^2\phi = \frac{z-z_1}{z-z_2} = \frac{z-\dfrac{a}{c}}{z+\dfrac{a}{1+c}} = \frac{z-a\dfrac{1-\kappa}{\kappa}}{z+a\,(1-\kappa)}, \quad \&c., \tag{43}$$

so that the equation of the trajectory is

$$\psi = I + \tfrac{1}{2}(1-\kappa)\,F\varphi. \tag{44}$$

Expressed in terms of $r^2$, this may be written in either of the forms

$$r^2\cos\{2\psi-(1-\kappa)\,F\varphi\} = \surd\{4ah\,(r^2+4a^2)\}, \tag{45}$$

$$r^2\sin\{2\psi-(1-\kappa)\,F\varphi\} = \surd(r^4-4ahr^2-16a^3h)$$

$$= \sqrt{\left[\left(r^2-4a^2\frac{1-\kappa}{\kappa}\right)\{r^2+4a^2(1-\kappa)\}\right]}; \tag{46}$$

and, while $r^2$ ranges from $4a^2\dfrac{1-\kappa}{\kappa}$ to infinity, the angle $\psi$ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1-\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right\}. \tag{47}$$

With the same parameter, but with

$$z+a = M^2(s-s_2) \quad = M^2(s-c^2), \tag{48}$$

$$z = M^2(s-\sigma) \quad = M^2(s-c-c^2), \tag{49}$$

$$a = M^2c, \tag{50}$$

$$h = z_1+z_3 \quad = M^2(1-c^2), \tag{51}$$

$$\surd(h^2+k^2) = z_1-z_3 \quad = M^2(1+c^2), \tag{52}$$

$$z_1 = M^2(1+c) \quad = a\frac{1+c}{c}, \tag{53}$$

$$z_2 = -M^2c \quad = -a, \tag{54}$$

$$z_3 = -M^2(c+c^2) = -a\,(1+c), \tag{55}$$

$$I = \tfrac{1}{2}\sin^{-1}\frac{\surd(z-z_1\,.\,z-z_2)}{z} \quad = \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\,\frac{\surd(z-z_3)}{z}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\left(z-a\dfrac{1+c}{c}\,.\,z+a\right)}}{z} = \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\,\frac{\surd\{z+a(1+c)\}}{z}, \tag{56}$$

and, by differentiation, we find

$$\frac{dI}{dz} = \frac{d\psi}{dz} - \frac{1+2c}{2\surd c}\,\frac{\surd a}{\surd Z}. \tag{57}$$

To reduce the secular term to Legendre's standard form, put

$$\sin^2\phi = \frac{z-a\dfrac{1+c}{c}}{z+a}, \quad \cos^2\phi = \frac{a\dfrac{1+2c}{c}}{z+a}, \quad \Delta^2\phi = \frac{1+2c}{(1+c)^2}\,\frac{z+a(1+c)}{z+a}, \tag{58}$$

and then

$$\frac{2+c}{2\sqrt{c}}\,\frac{\sqrt{a}\,dz}{\sqrt{Z}} = \frac{1+2c}{2+2c}\,\frac{d\phi}{\Delta\phi} = \tfrac{1}{2}(1+\kappa)\,\frac{d\phi}{\Delta\phi}, \tag{59}$$

so that

$$\psi - \tfrac{1}{2}(1+\kappa)\,F\phi = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z-z_1\,.\,z-z_2)}}{z}, \quad \&c., \tag{60}$$

and the equation of the projection of the catenary may be written, as before:

$$r^2\sin\left\{2\psi-(1+\kappa)\,F\phi\right\} = \sqrt{\left\{\left(r^2-\frac{4a^2}{\kappa}\right)(r^2+4a^2)\right\}}, \tag{61}$$

$$r^2\cos\left\{2\psi-(1+\kappa)\,F\phi\right\} = \frac{2a}{\sqrt{c}}\sqrt{\left(r^2+\frac{4a^2}{1-\kappa}\right)}, \tag{62}$$

and, while $r$ ranges from $2a/\sqrt{\kappa}$ to infinity, $\psi$ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1+\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right\}. \tag{63}$$

Thus, for instance, with $c=1$, $h=0$, $k=4a$, $\kappa=\frac{1}{2}=\sin 30°$.

19. To find the pressure on the surface, we notice that the cosine of the angle between the normal and the axis of the paraboloid is

$$\sqrt{\left(\frac{a}{z+a}\right)}, \tag{64}$$

so that, denoting the pressure between the particle and the outside surface by $R$ in dynes, the mass of the particle by $m$ in grammes, and resolving vertically,

$$m\frac{d^2z}{dt^2} = mg - R\sqrt{\left(\frac{a}{z+a}\right)}. \tag{65}$$

But, since

$$\frac{dz^2}{dt^2} = 2g\,\frac{z^2-hz-\frac{1}{4}k^2}{z+a}, \tag{66}$$

$$\frac{d^2z}{dt^2} = g\,\frac{z^2+2az-ah+\frac{1}{4}k^2}{(z+a)^2}, \tag{67}$$

so that
$$R\sqrt{\left(\frac{a}{z+a}\right)} = mg\,\frac{a^2+ah-\frac{1}{4}k^2}{(z+a)^2}, \tag{68}$$

$$\frac{R}{mg} = \frac{a^2+ah-\frac{1}{4}k^2}{\sqrt{a}\,(z+a)^{\frac{3}{2}}}. \tag{69}$$

Then, if
$$a^2+ah = \tfrac{1}{4}k^2, \quad z_2 = z_3 = -a, \tag{70}$$

$$R = 0, \tag{71}$$

and the particle moves freely in space in a parabolic trajectory.

But, in this case, equation (9) reduces to

$$\frac{d\psi}{dz} = \tfrac{1}{2}\,\frac{\frac{k}{2\sqrt{a}}}{z\sqrt{\left(z-\frac{k^2}{4a}\right)}}. \tag{72}$$

and integrating,

$$\psi = \cos^{-1}\frac{k}{2\sqrt{(az)}} = \cos^{-1}\frac{k}{r}, \tag{73}$$

or
$$r\cos\psi = k, \tag{74}$$

so that the projection of the path on a horizontal plane is a straight line, a verification.

20. When the vertex of the paraboloid is downwards, the particle must now move on the interior of the surface, and

$$\begin{aligned} Z &= (z+a)\left\{-4z\,(z-h)-k^2\right\} \\ &= -4\,(z-z_1)\,(z-z_2)\,(z-z_3), \end{aligned} \tag{75}$$

where
$$z_1 = -a, \tag{76}$$

$$z_2 = \tfrac{1}{2}h-\tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{77}$$

$$z_3 = \tfrac{1}{2}h+\tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{78}$$

and
$$\infty > z_3 > z > z_2 > 0 > z_1 > -\infty,$$

so that the parameter
$$v = \omega_1 + f\omega_3, \tag{79}$$

and we put
$$z = M^2\,(\sigma-s) = \wp v - \wp u, \tag{80}$$

$$z+a = M^2\,(s_1-s), \tag{81}$$

$$a = M^2\,(s_1-\sigma), \tag{82}$$

$$z_3+z_2 = h = M^2\,(2\sigma-s_2-s_3), \tag{83}$$

$$z_3-z_2 = \sqrt{(h^2-k^2)} = M^2\,(s_2-s_3), \tag{84}$$

$$k\sqrt{a} = M^3\sqrt{(-\Sigma)}, \tag{85}$$

$$\frac{k}{\sqrt{a}} = M\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}, \tag{86}$$

$$\frac{dz}{dZ} = -\frac{1}{M}\frac{ds}{\sqrt{S}}, \tag{87}$$

and now
$$\begin{aligned}\psi &= \tfrac{1}{2}k\int_z^{z_3}\frac{z+a}{z\sqrt{(aZ)}}\,dz\\ &= \tfrac{1}{2}\int_{s_3}^{s}\left\{\frac{\sqrt{(-\Sigma)}}{\sigma-s}+\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\right\}\frac{ds}{\sqrt{S}}\\ &= \tfrac{1}{2}\left\{P(v)+\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\right\}\int_{s_3}\frac{ds}{\sqrt{S}}-I(v)\\ &= \tfrac{1}{2}MP(v-\omega_1)\int\frac{ds}{\sqrt{Z}}-I(v)\\ &= \tfrac{1}{2}MP(f\omega_3)\frac{F\phi}{\sqrt{(z_3-z_1)}}-I(v),\end{aligned} \tag{88}$$

in which
$$\sin^2\phi = \frac{z_3-z}{z_3-z_2},\quad \cos^2\phi = \frac{z-z_2}{z_3-z_2},\quad \Delta^2\phi = \frac{z+a}{z_3-z_1}, \tag{89}$$

and now the pseudo-elliptic cases can be written down as before.

In the expression of the time

$$\begin{aligned}nt &= \int_z^{z_3}\frac{z+a}{\sqrt{(aZ)}}\,dz\\ &= \sqrt{\left(\frac{z_3-z_1}{a}\right)}\int_0^{\phi}\Delta\phi\,d\phi\\ &= \sqrt[4]{\left(\frac{h^3-k^2}{a^2}\right)}\frac{E\phi}{\kappa}.\end{aligned} \tag{90}$$

When $h=k$, the particle describes a horizontal circle, at a height

$$z_3 = \tfrac{1}{2}h, \tag{91}$$

and, if $T$ denotes the time of a small oscillation,

$$\begin{aligned}nT &= \sqrt{\left(\frac{z_3-z_1}{a}\right)}\pi,\\ T &= \pi\sqrt{\left(\frac{2a+2z_3}{g}\right)},\end{aligned} \tag{92}$$

so that the oscillations synchronize with a simple pendulum of length

$$2a+2z_3 = GT, \tag{93}$$

$PG$ and $PT$ being the normal and tangent drawn from a point on the circle to meet the axis.

21. Thus, with the parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{94}$$

we put

$$z = M^2(c + c^2 - s), \tag{95}$$

$$z + a = M^2\{(1+c)^2 - s\}, \tag{96}$$

$$a = M^2(1+c), \tag{97}$$

$$z_1 = -M^2(1+c) = -a, \tag{98}$$

$$z_2 = M^2 c = a\frac{c}{1+c}, \tag{99}$$

$$z_3 = M^2(c+c^2) = ac, \tag{100}$$

$$h = z_3 + z_2 = M^2(2c+c^2), \tag{101}$$

$$\sqrt{(h^2-k^2)} = z_3 - z_2 = M^2 c^2, \tag{102}$$

$$\frac{h}{a} = \frac{2c+c^2}{1+c}, \tag{103}$$

$$\frac{k^2}{a^2} = \frac{4c^2}{1+c}. \tag{104}$$

Then $I = \frac{1}{2}\cos^{-1}\dfrac{\sqrt{(z-z_1 . z-z_2)}}{z} = \frac{1}{2}\sin^{-1}\dfrac{M\sqrt{(z_3-z)}}{z}$

$$= \tfrac{1}{2}\cos^{-1}\frac{\sqrt{\left(z+a\,.\,z-\dfrac{ac}{1+c}\right)}}{z} = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{a}\sqrt{(ac-z)}}{\sqrt{(1+c)}\,z}, \tag{105}$$

and, by differentiation,

$$\frac{dI}{dz} = \tfrac{1}{2}\frac{1+2c}{\sqrt{(1+c)}}\frac{\sqrt{a}}{\sqrt{Z}} - \tfrac{1}{2}k\frac{z+a}{2\sqrt{(aZ)}}, \tag{106}$$

so that

$$\psi = \tfrac{1}{2}k\int_z^{z_3}\frac{z+a}{z\sqrt{(aZ)}}\,dz$$

$$= \tfrac{1}{2}\frac{1+2c}{\sqrt{(1+c)}}\int_z^{z_3}\frac{\sqrt{a}\,dz}{\sqrt{Z}} + I. \tag{107}$$

To reduce the secular term to Legendre's standard form, put

$$ac - z = \frac{c^2}{1+c}a\sin^2\phi,\quad z - \frac{c}{1+c}a = \frac{c^2}{1+c}a\cos^2\phi,\quad z+a = a(1+c)a\Delta^2\phi, \tag{108}$$

and then $\tfrac{1}{2}\dfrac{1+2c}{\sqrt{(1+c)}}\dfrac{\sqrt{a}\,dz}{\sqrt{Z}} = -\tfrac{1}{2}\dfrac{1+2c}{1+c}\dfrac{d\phi}{\Delta\phi} = -\tfrac{1}{2}(1+\kappa)\dfrac{d\phi}{\Delta\phi}$, (109)

so that

$$\psi = \tfrac{1}{2}(1+\kappa)F\phi + I, \tag{110}$$

and the equation of the trajectory may be written

$$r^2 \cos\{2\psi-(1+\kappa)F\phi\} = \sqrt{(r^2+4a^2 . r^2-4\kappa a^2)} \quad (111)$$

or
$$r^2 \sin\{2\psi-(1+\kappa)F\phi\} = 2a\sqrt{(1-\kappa)}\sqrt{\left(\frac{4\kappa}{1-\kappa}\kappa^2-r^2\right)}$$
$$= 2a\sqrt{\{4\kappa a^2-(1-\kappa)r^2\}}. \quad (112)$$

Thus, as $r$ diminishes from $2a\sqrt{\left(\frac{\kappa}{1-\kappa}\right)}$ to $2a\sqrt{\kappa}$, the angle $\Psi$ increases by

$$\Psi = \tfrac{1}{4}\pi\left\{(1+\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \quad (113)$$

We can make the apsidal angle $\Psi$ very nearly $\frac{4}{5}\pi$ or 144°, by trial and error, by taking, as in Fig. 1, p. 599,

$$\kappa = 0{\cdot}77384 = \sin 50° 42',$$

and
$$\frac{r_2}{2a} = \sqrt{\kappa} = 0{\cdot}8796, \quad \frac{r_3}{2a} = \sqrt{\left(\frac{\kappa}{1-\kappa}\right)} = 1{\cdot}849. \quad (114)$$

We thus obtain a trajectory which very nearly closes upon itself.

The pressure $R$ is again obtained by resolving in the vertical direction

$$m\frac{d^2z}{dt^2} = R\sqrt{\left(\frac{a}{z+a}\right)} - mg, \quad (115)$$

and
$$\frac{d^2z}{dt^2} = g\,\frac{-z^2-2az+ah+\frac{1}{4}k^2}{(z+a)^2}, \quad (116)$$

so that
$$\frac{R}{mg} = \frac{a^2+ah+\frac{1}{4}k^2}{\sqrt{a}\,(z+a)^{\frac{3}{2}}}, \quad (117)$$

so that $R$ does not change sign.

### *The Catenary on a Vertical Cone.*

22. Taking the semi-vertical angle of the cone as $\alpha$, then, along a generator,

$$x = z\tan\alpha, \quad (1)$$

and the general equation (5), in § 1, for the curve formed by a chain wrapped on a vertical cone of revolution becomes

$$\frac{d\psi}{dz} = \frac{A\sec\alpha}{z\tan\alpha\sqrt{\{z^2(z-h)^2\tan^2\alpha-A^2\}}}$$

or
$$\sin\alpha\,\frac{d\psi}{dz} = \frac{k^2}{z\sqrt{\{z^2(z-h)^2-k^4\}}}, \quad (2)$$

on putting
$$A = k^2\tan\alpha. \quad (3)$$

If $r$, $\theta$ denote the polar coordinates when the surface of the cone is developed into a plane, we can put

$$\psi \sin \alpha = \theta \quad \text{and} \quad r \cos \alpha = z, \tag{4}$$

so that

$$\frac{d\theta}{dr} = \frac{b^2}{r\sqrt{\{r^2(r-2a)^2 - b^4\}}} = \frac{b^2}{r\sqrt{R}}, \tag{5}$$

involving only one parameter $b/a$; and

$$b = k \sec \alpha, \quad 2a = h \sec \alpha, \tag{6}$$

$$R = r^2(r-2a)^2 - b^4 = (r^2 - 2ar + b^2)(r^2 - 2ar - b^2). \tag{7}$$

This proves that in the developed catenary (Routh, *Analytical Statics*, I, p. 361)

$$\frac{1}{p} = \alpha + \beta r, \tag{8}$$

where $p$ denotes the perpendicular from the origin on the tangent, and $\alpha$, $\beta$ are constants; for, $\phi$ denoting the *radial* angle,

$$r\frac{d\theta}{dr} = \tan \phi = \frac{b^2}{\sqrt{\{r^2(r-2a)^2 - b^4\}}}, \tag{9}$$

$$\sin \phi = \frac{b^2}{r(r-2a)}, \tag{10}$$

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right) = \sqrt{\left(\frac{r^2 - 2ar - b^2}{r^2 - 2ar + b^2}\right)}, \tag{11}$$

and

$$p = r \sin \phi = \frac{b^2}{r-2a}, \tag{12}$$

$$p(r-2a) = b^2. \tag{13}$$

Thus $h = 0$, or $a = 0$, gives

$$pr = b^2, \tag{14}$$

a rectangular hyperbola.

23. Put

$$r - a = x; \tag{15}$$

then

$$\theta = \int \frac{b^2\,dx}{(x+a)\sqrt{X}}, \tag{16}$$

where

$$X = (x^2 - a^2 + b^2)(x^2 - a^2 - b^2). \tag{17}$$

We can now employ the Jacobian notation.

When $a^2 - b^2$ is positive, $= \beta^2$, suppose, and

$$a^2 + b^2 = \alpha^2, \quad a^2 = \tfrac{1}{2}(\alpha^2 + \beta^2);$$

then

$$X = x^2 - \alpha^2 . x^2 - \beta^2. \tag{18}$$

On the open branch of the curve we put

$$x = \frac{a}{\operatorname{sn} u}, \tag{19}$$

and on the closed branch $x = \beta \operatorname{sn} u,$ (20)

with $\kappa = \frac{\beta}{a},$ (21)

so that increasing $u$ by $2K'i$ changes from the closed to the open branch of the curve.

We put $a = \beta \operatorname{sn} v,$ (22)

and then $\operatorname{cn} v = \frac{ib}{\beta}, \quad \operatorname{dn} v = \frac{b}{a},$ (23)

and $v = K + fK'i;$ (24)

so that we can write equation (16)

$$\theta = \int \frac{b^2 x\, dx}{(x^2 - a^2)\sqrt{X}} - \int \frac{ab^2\, dx}{(x^2 - a^2)\sqrt{X}}$$

$$= \sin^{-1} \sqrt{\left(\tfrac{1}{2} \frac{x^2 - a^2 - b^2}{x^2 - a^2}\right)} + \int \frac{i \operatorname{cn} v \operatorname{dn} v}{\operatorname{sn}^2 v - \operatorname{sn}^2 u}\, du, \tag{25}$$

involving a Jacobian Elliptic Integral of the Third Kind, in a standard form.

24. It follows, from the preceding relations (23), that

$$\operatorname{sn}^2 (1-f) K'i = -1; \tag{26}$$

and, to the complementary modulus $K'$,

$$\operatorname{tn}^2 (1-f) K' = 1, \quad (1-f) K' = F(\tfrac{1}{4}\pi), \tag{27}$$

so that, in the pseudo-elliptic applications the results are restricted to special numerical cases.

In equation (25),

$$\sin^{-1} \sqrt{\left(\tfrac{1}{2} \frac{x^2 - a^2 - b^2}{x^2 - a^2}\right)} = \cos^{-1} \sqrt{\left(\tfrac{1}{2} \frac{x^2 - a^2 + b^2}{x^2 - a^2}\right)}$$

$$= \tfrac{1}{2} \cos^{-1} \frac{b^2}{x^2 - a^2} = \tfrac{1}{2} \cos^{-1} \frac{b^2}{r(r - 2a)}$$

$$= \tfrac{1}{2} (\tfrac{1}{2}\pi - \phi); \tag{28}$$

from (10), § 22, so that (25) may be written

$$\theta+\tfrac{1}{2}\phi-\tfrac{1}{4}\pi=\int\frac{i\,\dfrac{\operatorname{cn}v\,\operatorname{dn}v}{\operatorname{sn}v}}{\operatorname{sn}^2v-\operatorname{sn}^2u}\,du. \tag{29}$$

Thus in the rectangular hyperbola

$$\theta=\tfrac{1}{4}\pi-\tfrac{1}{2}\phi. \tag{30}$$

24. Also, if $s'$ denotes the length of the arc

$$\frac{ds'}{dz}=\frac{ds}{r\,d\psi}\,\frac{r^2\,d\psi}{dz}=\frac{z-h}{A}\,\frac{r^2A\sec a}{z\tan^2a\,\sqrt{Z}}$$

$$=\frac{z\,(z-h)\sec a}{\sqrt{Z}}, \tag{31}$$

$$\frac{ds'}{dr}=\frac{r\,(r-2a)}{\sqrt{R}}, \tag{32}$$

$$\frac{ds'}{dx}=\frac{x^2-a^2}{\sqrt{X}}, \tag{33}$$

so that $s'$ is given by Elliptic Integrals of the First and Second Kind.

25. Denoting by $\sigma'$ the value of $s$ corresponding to

$$u=(1-f)\,K'i, \tag{34}$$

then

$$\operatorname{sn}^2(1-f)K'i=\frac{s_1-s_3}{\sigma'-s_3}, \tag{35}$$

so that, from (26),

$$s_1-s_3=s_3-\sigma'. \tag{36}$$

Thus, for $f=\frac{1}{2}$,

$$\sigma'=-c-c^2,\quad s_1=(1+c)^2,\quad s_2=c^2,\quad s_3=0; \tag{37}$$

so that

$$c=\infty,\quad b=0, \tag{38}$$

which must be rejected, as representing a plane curve, a straight line along a generator.

If

$$f=\tfrac{1}{3},\quad \sigma'=0,$$

$$s_1=(1-c)^2,\quad s_2=c^2,\quad s_3=(c-c^2)^2, \tag{39}$$

so that

$$1-2c^2=0,\quad c=\tfrac{1}{2}\sqrt{2}=0{\cdot}707, \tag{40}$$

which must be rejected, as greater than 0·5.

If

$$f=\tfrac{2}{3},\quad \sigma'=-2c+2c^2, \tag{41}$$

leading to

$$2c^2-4c+1=0,$$

$$c=1-\tfrac{1}{2}\sqrt{2}=0{\cdot}293; \tag{42}$$

$$\frac{\beta^2}{a^2} = \kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{(\sqrt{2}-1)^2}{2\sqrt{2}-1} = \frac{4\sqrt{2}-5}{7} = \frac{a^2-b^2}{a^2+b^2}, \tag{43}$$

$$\kappa'^2 = \frac{12-4\sqrt{2}}{7}, \tag{44}$$

$$\frac{b^2}{a^2} = 2(\sqrt{2}-1), \tag{45}$$

$$\frac{a^2}{a^2} = 2\sqrt{2}-1, \tag{46}$$

$$\frac{\beta^2}{a^2} = (\sqrt{2}-1)^2. \tag{47}$$

The equation of the catenary can now be written

$$\theta = \sin^{-1}\sqrt{\left(\tfrac{1}{2}\,\frac{x^2-(2\sqrt{2}-1)\,a^2}{x^2-a^2}\right)} + I + \tfrac{1}{3}\sqrt{2}\int \frac{a\,dx}{\sqrt{X}}, \tag{48}$$

where
$$I = \tfrac{1}{3}\cos^{-1}\frac{\{x^2-(\sqrt{2}+1)\,a^2\}\sqrt{\{x^2-(\sqrt{2}-1)^2\,a^2\}}}{(x^2-a^2)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{2}\,\frac{ax\sqrt{\{x^2-(2\sqrt{2}-1)\,a^2\}}}{(x^2-a^2)^{\frac{3}{2}}}, \tag{49}$$

and
$$\tfrac{1}{3}\sqrt{2}\int\frac{a\,dx}{\sqrt{X}} = \tfrac{1}{3}\sqrt{2}\,\frac{a}{\alpha}F\phi, \tag{50}$$

where
$$\sin\phi = \frac{x}{\beta} \text{ in the limited branch,}$$

$$\sin\phi = \frac{\alpha}{x} \text{ in the unlimited branches.}$$

26. When $a^2-b^2$ is negative, $= -\beta^2$, and $a^2+b^2 = \alpha^2$,

$$X = x^2+\beta^2.x^2-\alpha^2, \tag{51}$$

and we put
$$x = \frac{\alpha}{\operatorname{cn} u}, \quad \kappa = \frac{\beta}{\sqrt{(\alpha^2+\beta^2)}}, \tag{52}$$

and the curve consists of a single open branch.

To reduce the integral to the standard form, we put

$$\operatorname{cn} u = \frac{\alpha}{x}, \quad \operatorname{cn} v = \frac{\alpha}{a}, \tag{53}$$

so that
$$u = pK, \quad v = fK'i, \tag{54}$$

and then
$$\operatorname{sn}^2 v = -\frac{b^2}{a^2}, \quad \operatorname{dn}^2 v = \frac{a^2}{2a^2}, \tag{55}$$

so that
$$\mathrm{cn}^2 v - 2\,\mathrm{dn}^2 v = 0, \tag{56}$$

$$\mathrm{cn}\left\{(1-f)K', \kappa'\right\} = \frac{\kappa}{\kappa'}. \tag{57}$$

Then, from (25),

$$\begin{aligned}\theta &= \int \frac{b^2 x\,dx}{(x^2-a^2)\sqrt{X}} - \int \frac{ab^2 dx}{(x^2-a^2)\sqrt{X}} \\ &= \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2}{x^2-\alpha^2}\right)} - \frac{ab^2}{a^2\sqrt{(a^2+\beta^2)}}\int \frac{du}{\mathrm{nc}^2 u - \mathrm{nc}^2 v} \\ &= \tfrac{1}{2}(\tfrac{1}{2}\pi-\phi) - \frac{i\,\mathrm{sn}\,v\,\mathrm{dn}\,v}{\mathrm{cn}^3 v}\int \frac{du}{\mathrm{nc}^2 u - \mathrm{nc}^2 v},\end{aligned} \tag{58}$$

depending upon a Jacobian elliptic integral, in a standard form.

To change from the Jacobian form to that given in (A), § 3, we put

| $a^2-b^2$ positive | $a^2-b^2$ negative |
|---|---|
| $a^2 = \frac{1}{2}(a^2+\beta^2)$ | $a^2 = \frac{1}{2}(a^2-\beta^2)$ |
| $b^2 = \frac{1}{2}(a^2-\beta^2)$ | $b^2 = \frac{1}{2}(a^2+\beta^2)$ |
| $x^2 = m^2(s-s_3)$ | $x^2 = m^2(s-s_2)$ |
| $x^2-\alpha^2 = m^2(s-s_1)$ | $x^2-\alpha^2 = m^2(s-s_1)$ |
| $x^2-\beta^2 = m^2(s-s_2)$ | $x^2+\beta^2 = m^2(s-s_3)$ |
| $x^2-a^2 = m^2(s-\sigma)$ | $x^2-a^2 = m^2(s-\sigma)$ |

(59)

so that

| | |
|---|---|
| $\sigma = \frac{1}{2}(s_1+s_2)$ | $\sigma = \frac{1}{2}(s_1+s_3)$ |
| $\alpha^2 = m^2(s_1-s_3)$ | $\alpha^2 = m^2(s_1-s_2)$ |
| $\beta^2 = m^2(s_2-s_3)$ | $\beta^2 = m^2(s_2-s_3)$ |
| $m^3 S = x\sqrt{(x^2-\alpha^2.x^2-\beta^2)}$ | $m^3 S = \sqrt{(x^2-\alpha^2.x^2+\beta^2)}$ |

Now
$$\frac{ds}{\sqrt{S}} = m\frac{dx}{\sqrt{X}}, \tag{60}$$

and
$$m^3\sqrt{(-\Sigma)} = 2ab^2, \tag{61}$$

$$\tfrac{1}{2}\frac{m\sqrt{(-\Sigma)}}{s-\sigma} = \frac{ab^2}{x^2-a^2}, \tag{62}$$

so that
$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - \tfrac{1}{2}\int \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}; \tag{63}$$

and, therefore, from (A), § 3,

$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - I + \tfrac{1}{2}mP\int \frac{dx}{\sqrt{X}}. \tag{64}$$

27. Try $$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{65}$$

with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{66}$$

then (*L.M.S.*, xxv, p. 212)

$$c + c^2 = \tfrac{1}{2}(1+c)^2 + \tfrac{1}{2}c^2, \tag{67}$$

giving $c = \infty$, which must be rejected.

But, with $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{68}$$

$$c + c^2 = \tfrac{1}{2}(1+c)^2, \tag{69}$$

and $$c = 1, \quad \kappa = \tfrac{1}{2}, \tag{70}$$

$$\frac{a^2}{\beta^2} = \frac{s_1 - s_2}{s_2 - s_3} = 3, \quad a^2 = 3\beta^2 = 3a^2, \tag{71}$$

$$b^2 = 2a^2, \tag{72}$$

and we find that $$I = \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{(x^2-3a^2)}}{x^2-a^2}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{a\sqrt{(x^2+a^2)}}{x^2-a^2}, \tag{73}$$

with $$P = 1, \quad m = a, \text{ in (64).} \tag{74}$$

Try $$v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{75}$$

with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{76}$$

then (*L.M.S.*, xxv, p. 218)

$$2c(1-c)^2 = \tfrac{1}{2}(1 - 2c + 2c^2), \tag{77}$$

$$(2c-1)(2c^2 - 4c + 1) = 0, \tag{78}$$

of which the root $$c = 1 - \tfrac{1}{2}\sqrt{2} = 0{\cdot}293 \tag{79}$$

must be taken, as before, in (42), § 25. (80)

With $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{81}$$

we have $$(c-1)^2(c^2 - 4c + 1) = 0, \tag{82}$$

of which the root $$c = 2 - \sqrt{3} \tag{83}$$

makes $$\kappa = \sin 15^\circ. \tag{84}$$

With $$v = \omega_1 + \tfrac{1}{4}\omega_3, \tag{85}$$

$$\sigma = \tfrac{1}{2}(s_1 + s_2) \text{ or } \tfrac{1}{2}(s_1 + s_3), \tag{86}$$

we take (*L.M.S.*, xxv, p. 226)

$$-z(1-2z) = \frac{(1-2z)^4}{8(1-z)^2}, \tag{87}$$

reducing to $$(2z-1)(4z^2-2z-1) = 0, \tag{88}$$

so that $$z = \tfrac{1}{2}, \quad -\sin 18^\circ, \quad \sin 54^\circ. \tag{89}$$

The root $z = -\sin 18^\circ$ makes

$$1-8z+8z^2 = 2+\sqrt{5}, \tag{90}$$

and must therefore be chosen.

Having determined in this manner the numerical value of the modulus, &c., the corresponding value of $I$ as a function of $x$ is calculated, and the secular term $\int \frac{dx}{\sqrt{X}}$ is reduced to Legendre's form $F\phi$ by the appropriate substitutions; so that the equation of the catenary is expressed in functions which have been tabulated numerically.

28. Although the theory of the catenary on a cone is now analytically complete, still we shall find it more convenient, because of the occurrence of the term $\frac{1}{2}(\frac{1}{2}\pi-\phi)$, to adopt another mode of reduction of the integral, equivalent to Landen's quadric transformation performed on the first substitution.

Suppose $R$ then to be resolved into linear factors, so that

$$R = r-r_0.r-r_3.r-r_2.r-r_1, \tag{1}$$

where $$\infty > r > r_0 > 2a > r_3 > r > r_2 > 0 > r_1 > r > -\infty. \tag{2}$$

To agree with the notation employed in the *Applications of Elliptic Functions*, p. 154, we replace $r_0$, $r_3$, $r_2$, $r_1$ by $\alpha$, $\beta$, $\gamma$, $\delta$; and put

$$\wp u - e_1 = M^2(s-s_1) = \tfrac{1}{4}(\alpha-\beta)(\alpha-\gamma)\frac{r-\delta}{r-\alpha}, \tag{3}$$

$$\wp u - e_2 = M^2(s-s_2) = \tfrac{1}{4}(\alpha-\delta)(\alpha-\beta)\frac{r-\gamma}{r-\alpha}, \tag{4}$$

$$\wp u - e_3 = M^2(s-s_3) = \tfrac{1}{4}(\alpha-\gamma)(\alpha-\delta)\frac{r-\beta}{r-\alpha}, \tag{5}$$

$$\kappa^2 = \frac{e_2-e_3}{e_1-e_3} = \frac{s_2-s_3}{s_1-s_3} = \frac{\beta-\gamma.\alpha-\delta}{\alpha-\gamma.\beta-\delta}, \tag{6}$$

$$\kappa'^2 = \frac{e_1-e_2}{e_1-e_3} = \frac{s_1-s_2}{s_1-s_3} = \frac{\alpha-\beta.\gamma-\delta}{\alpha-\gamma.\beta-\delta}. \tag{7}$$

We can put $a=1$, without loss of generality; also we put

$$b^4 = 1-e^2, \tag{8}$$

$$R = r^2(r-2)^2 - 1 + e^2, \tag{9}$$

so that

$$\alpha = r_0 = 1+\sqrt{\left(\frac{1+e}{2}\right)}+\sqrt{\left(\frac{1-e}{2}\right)}, \tag{10}$$

$$\beta = r_3 = 1+\sqrt{\left(\frac{1+e}{2}\right)}-\sqrt{\left(\frac{1-e}{2}\right)}, \tag{11}$$

$$\gamma = r_2 = 1-\sqrt{\left(\frac{1+e}{2}\right)}+\sqrt{\left(\frac{1-e}{2}\right)}, \tag{12}$$

$$\delta = r_1 = 1-\sqrt{\left(\frac{1+e}{2}\right)}-\sqrt{\left(\frac{1-e}{2}\right)}, \tag{13}$$

and

$$\alpha-\gamma = \beta-\delta, \quad \alpha-\beta = \gamma-\delta, \quad \kappa' = \frac{\alpha-\beta}{\alpha-\gamma}; \tag{14}$$

and this makes

$$\kappa^2 = \frac{2e}{1+e}, \quad \kappa'^2 = \frac{1-e}{1+e}. \tag{15}$$

But, denoting by $\lambda$, $\lambda'$ the moduli introduced in the original substitution in § 23,

$$\lambda = \frac{\beta}{\alpha} = \sqrt{\left(\frac{1-b^2}{1+b^2}\right)} = \sqrt{\left\{\frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}\right\}}$$

$$= \frac{\sqrt{(1+e)}-\sqrt{(1-e)}}{\sqrt{(1+e)}+\sqrt{(1-e)}} = \frac{1-\kappa'}{1+\kappa'}, \tag{16}$$

$$\lambda' = \frac{2\sqrt{\kappa'}}{1+\kappa'}, \tag{17}$$

as in one of Landen's transformations.

The elliptic argument

$$u = \int \frac{dr}{\sqrt{R}} = \frac{2F\phi}{\sqrt{(\alpha-\gamma.\beta-\delta)}} = \sqrt{\left(\frac{2}{1+e}\right)} F\phi = \sqrt{(2-\kappa^2)}\, F\phi, \tag{18}$$

and, in the region of the limited branch of the catenary

$$r_3 > r > r_2, \tag{19}$$

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{\beta-\delta\,.\,r-\gamma}{\beta-\gamma\,.\,r-\delta}\right)} = \&c. \tag{20}$$

or

$$\sin^2\phi = \frac{\beta-\delta\,.\,r-\gamma}{\beta-\gamma\,.\,r-\delta}, \ \&c.\ ; \tag{21}$$

while, in the two regions $\infty > r > r_0$ and $r_1 > r > -\infty$, (22)

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{\beta-\delta.r-\alpha}{\alpha-\delta.r-\beta}\right)}, \quad \&c., \tag{23}$$

$$\sin^2\phi = \frac{\beta-\delta.r-\alpha}{\alpha-\delta.r-\beta}, \quad \&c. \tag{24}$$

Calculating the invariants $g_2$ and $g_3$ of the quartic $R$ in (9), then

$$3g_2 = 1+3e^2, \tag{25}$$

$$27g_3 = 1-9e^2, \tag{26}$$

so that the discriminating cubic

$$4x^3 - g_2x - g_3 = 0 \tag{27}$$

breaks up into the factors

$$(x-\tfrac{1}{3})\left\{(2x+\tfrac{1}{3})^2-e^2\right\} = 0, \tag{28}$$

and thus, arranged in descending order,

$$e_1 = \tfrac{1}{3}, \quad e_2 = -\tfrac{1}{6}+\tfrac{1}{2}e, \quad e_3 = -\tfrac{1}{6}-\tfrac{1}{2}e. \tag{29}$$

Calculating the Hessian $H$ of the quartic $R$,

$$H = -\tfrac{1}{3}R - (1-e^2)(r-1)^2, \tag{30}$$

and, employing Hermite's formula,

$$\wp 2u = -\frac{H}{R} = \tfrac{1}{3} + \frac{(1-e^2)(r-1)^2}{R}, \tag{31}$$

$$\wp 2u - e_1 = \frac{(1-e^2)(r-1)^2}{R}. \tag{32}$$

Since $r = \infty$, and $r = 1$, make

$$\wp 2u - e_1 = 0; \tag{33}$$

it follows that, if $w_1$ and $w_2$ denote the corresponding arguments,

$$w_1 = \tfrac{1}{2}\omega_1, \quad w_2 = \omega_3 + \tfrac{1}{2}\omega_1. \tag{34}$$

Thus, if $A$ denotes the sectorial area of the developed catenary,

$$dA = \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}b^2\,\frac{r\,dr}{\sqrt{R}}, \tag{35}$$

so that $A$ is always given by a pseudo-elliptic integral of the third kind; in fact

$$A = \tfrac{1}{2}b^2\,\mathrm{ch}^{-1}\,\frac{\sqrt{(b^2+2r-r^2)}}{b\sqrt{2}} - \tfrac{1}{2}b^2u, \tag{36}$$

$$= \tfrac{1}{2}b^2\,\mathrm{sh}^{-1}\,\frac{\sqrt{(-b^2+2r-r^2)}}{b\sqrt{2}} - \tfrac{1}{2}b^2u, \tag{37}$$

for the limited branch; but

$$A = \tfrac{1}{2}b^2 \begin{matrix}\mathrm{ch}^{-1}\\ \mathrm{sh}^{-1}\end{matrix} \frac{\sqrt{(r^2-2r+b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2u, \tag{38}$$

$$= \tfrac{1}{2}b^2 \begin{matrix}\mathrm{sh}^{-1}\\ \mathrm{ch}^{-1}\end{matrix} \frac{\sqrt{(r^2-2r-b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2u, \tag{39}$$

for the unlimited branches; both forms being included in

$$A = \tfrac{1}{4}b^2 \,\mathrm{sh}^{-1} \frac{\sqrt{R}}{b^2} \mp \tfrac{1}{2}b^2u. \tag{40}$$

If $v_1$, $v_2$ denote the elliptic arguments corresponding to $r = 0$ and $r = 2$, then

$$\wp 2v_1 = \wp 2v_2 = -\tfrac{2}{3}, \tag{41}$$

so that we may put

$$v_1 = \omega_1 + f\omega_3, \quad v_2 = f\omega_3, \tag{42}$$

$f$ denoting a fraction; also

$$i\wp' 2v_1 = i\wp' 2v_2 = \sqrt{(1-e^2)} = b^2. \tag{43}$$

Also

$$\wp'' 2v_1 = 6\wp^2 - \tfrac{1}{2}g_2$$

$$= \tfrac{5}{2} - \tfrac{1}{2}e^2, \tag{44}$$

so that, from the formula

$$\wp 4v + 2\wp 2v = \tfrac{1}{4}\left(\frac{\wp'' 2v}{\wp' 2v}\right)^2, \tag{45}$$

$$\wp 4v_1 = -\frac{1}{1-e^2} + \tfrac{5}{6} - \frac{1-e^2}{16}. \tag{46}$$

Thus, for instance, with

$$v_1 = \omega_1 + \tfrac{1}{4}\omega_3, \tag{47}$$

$$4v_1 = 4\omega_1 + \omega_3, \tag{48}$$

$$\wp 4v_1 = e_3 = -\tfrac{1}{6} - \tfrac{1}{2}e, \tag{49}$$

leading to the equation

$$(e^2-1)^2 + 8(e+2)(e^2-1) + 16 = 0 \tag{50}$$

or

$$(e^2+4e-1)^2 = 0, \tag{51}$$

$$e = \sqrt{5}-2, \tag{52}$$

$$b^4 = 4(\sqrt{5}-2). \tag{53}$$

The other root

$$e = -\sqrt{5}-2 \tag{54}$$

would correspond to the case when $R$ has four imaginary roots.

Making use of Weierstrass's formula,

$$\wp(v_1 \pm v_2) = \frac{\frac{8}{3} - b^4 \mp b^4}{8}$$

$$= \tfrac{1}{12} + \tfrac{1}{4}e^2 \text{ or } \tfrac{1}{3}; \tag{55}$$

agreeing in making $v_1 - v_2 = \omega_1$, (56)

and putting $v_1 + v_2 = v$, (57)

then $12\wp v = 1 + 3e^2$, (58)

while $12g_2 = 4 + 12e^2$

$$= (1+3e^2)^2 + 24\frac{(1-e^2)(1+3e^2)}{8}, \tag{59}$$

$$216g_3 = 8 - 72e^2$$

$$= -(1+3e^2)^3,$$

$$1728\Delta = (12g_2)^3 - (216g_3)^2 = 1728e^2(1-e^2)^2. \tag{60}$$

Forming the invariants of the cubic $S$ in (B), § 3, we can make

$$\frac{12g_2}{M^4} = \{(y+1)^2 + 4x\}^2 - 24x(y+1), \tag{61}$$

$$\frac{216g_3}{M^6} = \{(y+1)^2 + 4x\}^3 - 36x(y+1)\{(y+1)^2 + 4x\} + 216x^2 \tag{62}$$

(*L.M.S.*, xxvii, p. 129), so that, on comparison,

$$-M^2\{(y+1)^2 + 4x\} = 1 + 3e^2, \tag{63}$$

$$-M^4 x(y+1) = \tfrac{1}{8}(1-e^2)(1+3e^2), \tag{64}$$

$$16M^6x^2 = (1-e^2)^3; \tag{65}$$

and therefore, from (64) and (65),

$$4M^2(y+1)^2 = \frac{(1+3e^2)^2}{1-e^2}; \tag{66}$$

and, from (63), $4M^2x = -\frac{1}{4}\frac{(1+3e^2)^2}{1-e^2} - 1 - 3e^2$

$$= -\frac{(5-e^2)(1+3e^2)}{4(1-e^2)}, \tag{67}$$

$$M^2 = \frac{16(1-e^2)^5}{(5-e^2)^2(1+3e^2)^2}, \tag{68}$$

$$x = -\frac{(5-e^2)^3(1+3e^2)^3}{256(1-e^2)^6}, \tag{69}$$

$$y+1 = \frac{(5-e^2)(1+3e^2)^2}{8(1-e^2)^3}, \tag{70}$$

$$y = -\frac{(3+e^2)(1-18e^2+e^4)}{8(1-e^2)^3}; \tag{71}$$

whence an equation for $e^2$ can be formed, when we put Halphen's

$$\gamma_n = 0. \tag{72}$$

29. If we turn to the values of Halphen's $x$ and $y$ in terms of $a$ and $m$, as given in the *L.M.S.*, XXVII, p. 449, we shall find that

$$a = \frac{3+e^2}{4(1-e^2)}, \tag{73}$$

$$m = \frac{(5-e^2)(1+3e^2)}{4(1-e^2)(3+e^2)}, \tag{74}$$

$$a-m = \frac{1-e^2}{3+e^2}, \tag{75}$$

and thence $$4a(a-m) = 1, \tag{76}$$

$$\frac{a}{a-m} = \frac{(3+e^2)^2}{4(1-e^2)^2}, \tag{77}$$

$$ma = \frac{(5-e^2)(1+3e^2)}{16(1-e^2)^2}, \tag{78}$$

$$s_\gamma = \frac{(5-e^2)^2(1+3e^2)^2(3+e^2)^2}{256(1-e^2)^6}, \tag{79}$$

and $$s_\gamma = s_1, \text{ always.}$$

Also, with $$s_\alpha = s_2, \quad s_\beta = s_3,$$

$$(s_1-s_2)(s_1-s_3) = \frac{(5-e^2)^4(1+3e^2)^4}{1024(1-e^2)^9}, \tag{80}$$

$$(s_2-s_3)^2 = \frac{e^2(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{81}$$

$$(s_1-s_2+s_1-s_3)^2 = \frac{(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{82}$$

$$s_1-s_2,\ s_1-s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\mp e)}{32(1-e^2)^{10}}, \tag{83}$$

$$s_2,\ s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\pm 4e-e^2)^2}{256(1-e^2)^6}. \tag{84}$$

Also
$$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{2c}{1+c}, \tag{85}$$

$$\kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3} = \frac{1-c}{1+c}, \tag{86}$$

as before, thus serving as a verification.

Having fixed upon some pseudo-elliptic form of the integral $I(v_1)$ in (A), we shall obtain an equation for $e$, giving it a certain numerical value; and then we must express the variable $s$ in terms of $r$ by means of the relations in (3), (4), (5) of § 28.

We can also employ the formula

$$\wp u = \tfrac{1}{24} R''(r_0) + \tfrac{1}{4} \frac{R'(r_0)}{r - r_0}, \tag{87}$$

where
$$R(r) = r^4 - 2r^3 + r^2 + 0 - b^4, \tag{88}$$

$$R'(r) = 4r^3 - 6r^2 + 2r, \tag{89}$$

$$R''(r) = 12r^2 - 12r + 2, \tag{90}$$

and now
$$\wp u - \wp v_1 = M^2 (s - \sigma) = \tfrac{1}{2} (2r_0^2 - 3r_0 + 1) \frac{r}{r - r_0}. \tag{91}$$

Another simple relation is

$$\frac{s - s_1}{\sqrt{(s_1 - s_2 . s_1 - s_3)}} = \sqrt{\left(\frac{\alpha - \beta . \alpha - \gamma}{\beta - \delta . \gamma - \delta}\right)} \frac{r - r_1}{r - r_0} = \frac{r - r_1}{r - r_0}, \tag{92}$$

because
$$\alpha + \delta = \beta + \gamma, \quad \alpha - \beta = \gamma - \delta, \quad \alpha - \gamma = \beta - \delta, \tag{93}$$

and then
$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{s - s\tfrac{1}{2}\omega_1} = \frac{r - r_0}{r_0 - r_1}, \tag{94}$$

because
$$s\tfrac{1}{2}\omega_1 = s_1 + \sqrt{(s_1 - s_2 . s_1 - s_3)}. \tag{95}$$

Putting
$$r = 0, \quad u = v_1, \quad s = \sigma,$$

$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{\sigma - s\tfrac{1}{2}\omega_1} = \frac{-r_0}{r_0 - r_1}, \tag{96}$$

$$\frac{\sigma - s\tfrac{1}{2}\omega_1}{s - s\tfrac{1}{2}\omega_1} = 1 - \frac{r}{r_0}, \tag{97}$$

so that
$$\frac{s - \sigma}{s - s\tfrac{1}{2}\omega_1} = \frac{r}{r_0}, \tag{98}$$

$$\frac{s_a - \sigma}{s_a - s\tfrac{1}{2}\omega_1} = \frac{r_a}{r_0}. \tag{99}$$

The relation between the roots

$$r_0 r_1 + r_2 r_3 = 0 \tag{100}$$

thus leads to

$$\frac{s_1 - \sigma}{s_1 - s\tfrac{1}{2}\omega_1} + \frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_2 - s\tfrac{1}{2}\omega_1 \,.\, s_3 - s\tfrac{1}{2}\omega_1} = 0, \tag{101}$$

an equation which may be employed to settle the numerical value of a parameter.

From (3), (4), (5) of § 28,

$$-\wp' u = M^3 \sqrt{S} = \tfrac{1}{4}(r_0 - r_1 \,.\, r_0 - r_2 \,.\, r_0 - r_3) \frac{\sqrt{R}}{(r - r_0)^2}$$

$$= \tfrac{1}{4} R'(r_0) \frac{\sqrt{R}}{(r - r_0)^2}, \tag{102}$$

and

$$M^2 ds = \tfrac{1}{4} R'(r_0) \frac{-dr}{(r - r_0)^2}; \tag{103}$$

therefore

$$\frac{ds}{\sqrt{S}} = -M \frac{dr}{\sqrt{R}}. \tag{104}$$

Also

$$M^3 \sqrt{(-\Sigma)} = \tfrac{1}{4} R'(R_0) \frac{b^2}{r_0^2}, \tag{105}$$

$$\frac{M\sqrt{(-\Sigma)}}{s - \sigma} = b^2 \frac{r - r_0}{r_0 r}, \tag{106}$$

so that

$$d\theta = \frac{b^2 dr}{r\sqrt{R}} = -\frac{r_0}{r - r_0} \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= -\frac{s - s\tfrac{1}{2}\omega_1}{s\tfrac{1}{2}\omega_1 - \sigma} \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= -\frac{\sqrt{(-\Sigma)}}{s\tfrac{1}{2}\omega_1 - \sigma} \frac{ds}{\sqrt{S}} + \frac{\sqrt{(-\Sigma)}}{s - \sigma} \frac{ds}{\sqrt{S}}$$

$$= 2dI(v_1) - \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\tfrac{1}{2}\omega_1 - \sigma} \right\} \frac{ds}{\sqrt{S}}. \tag{107}$$

$$\theta = 2I(v_1) + M \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\tfrac{1}{2}\omega_1 - \sigma} \right\} \int \frac{dr}{\sqrt{R}}$$

$$= 2I(v_1) + M \left\{ P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\tfrac{1}{2}\omega_1 - \sigma} \right\} u, \tag{108}$$

and it is the coefficient of the secular term $u$ which is apt to be very baffling; so, to make sure of it in the applications, having constructed the pseudo-elliptic term $I(v_1)$, a differentiation will be employed for the verification.

It will not be necessary, and it would occupy too much space, to go through all the details by which $I(v_1)$ is converted from being a function of $s$, as worked out in the paper on "Pseudo-Elliptic Integrals," *L.M.S.*, XXV, into a function of $r$, and in which $r_0$, $r_1$, $r_2$, $r_3$ have certain numerical values depending upon the solution of an associated equation; so now we proceed to the simplest cases.

Having drawn some branches of the curve on a sheet of paper, we can bend the paper into a conical shape, and, as the vertical angle of the cone is at our disposal, we can, if we like, choose it so as to make the branches close and overlap each other, after an assigned number of convolutions, and thus have a figure suitable for stereoscopic representation.

The stereoscopic diagram on p. 638, drawn by Mr. Dewar, shows the catenary on a cone given by equation (147) in § 30, when the closed branch is made to have an apsidal angle of 180°, so as to form a single loop.

30. With $$v_1 = \omega_1 + \tfrac{1}{2}\omega_3, \quad v_2 = \tfrac{1}{2}\omega_3, \quad v = \omega_1 + \omega_3, \tag{109}$$

$$\wp v = e_2 \quad \text{or} \quad \tfrac{1}{12} + \tfrac{1}{4}e^2 = -\tfrac{1}{6} + \tfrac{1}{2}e, \tag{110}$$

so that $$e = 1, \quad b = 0, \tag{111}$$

and the catenary degenerates into a generating line of the cone.

Next, with $$v_1 = \omega_1 + \tfrac{1}{3}\omega_3, \quad v_2 = \tfrac{1}{3}\omega_3, \quad v = \omega_1 + \tfrac{2}{3}\omega_3, \tag{112}$$

and $$\wp 3v = e_1, \tag{113}$$

so that [*L.M.S.*, XXVII, (326), p. 450]

$$m = 1, \tag{114}$$

and therefore, from (74),

$$e^4 + 22e^2 - 7 = 0, \tag{115}$$

$$e^2 = 8\sqrt{2} - 11 = (\sqrt{2}-1)^2(2\sqrt{2}-1), \tag{116}$$

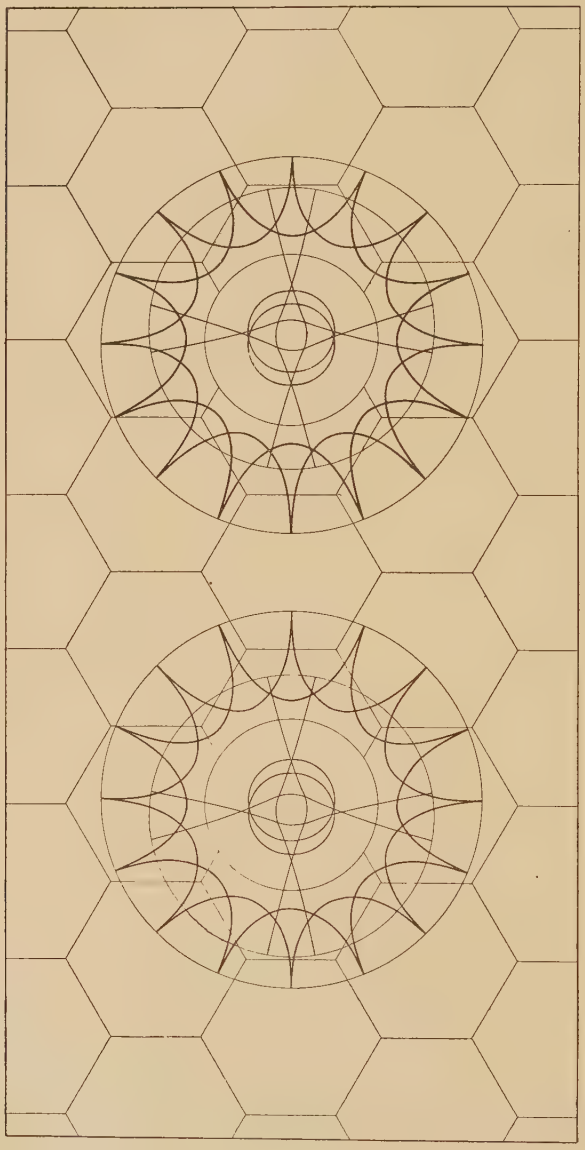
$$b^4 = 12 - 8\sqrt{2}, \quad b^2 = 2\sqrt{2} - 2, \tag{117}$$

$$\kappa = \tfrac{1}{2}\sqrt[4]{(2\sqrt{2}-1)}\left\{\sqrt{(2\sqrt{2}-1)} - \sqrt{2}+1\right\}. \tag{118}$$

The $c$ employed in *L.M.S.*, XXV, p. 217, is now given by

$$(1-c)^2 = \frac{s_3^2}{s_2^2} = \left(\frac{1-4e-e^2}{1+4e-e^2}\right)^2, \tag{119}$$

$$1 - c = \pm\frac{1-4e-e^2}{1+4e-e^2}, \tag{120}$$



Catenary on a Cone.

in which the negative sign must be taken; so that

$$c = 1 + \frac{1-4e-e^2}{1+4e-e^2} = 2\,\frac{1-e^2}{1+4e-e^2} = \frac{\sqrt{(2\sqrt{2}-1)} - \sqrt{2}+1}{2}. \quad (121)$$

In the catenary curve the corresponding value of $I(v_1)$ for the infinite branches is found to be given by

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r-r_0.r-r_1.r-r_3)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{(r+r_3)\sqrt{(r-r_2)}}{r^{\frac{3}{2}}}, \quad (122)$$

with
$$r_0 = 1+\sqrt{(2\sqrt{2}-1)}, \quad (123)$$
$$r_3 = \sqrt{2}, \quad (124)$$
$$r_2 = 2-\sqrt{2}, \quad (125)$$
$$r_1 = 1-\sqrt{(2\sqrt{2}-1)}. \quad (126)$$

Differentiating, we find

$$\frac{dI}{dr} = \tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \quad (127)$$

so that
$$\theta = \tfrac{1}{3}\sqrt{2}\int_{r_0}^{r}\frac{dr}{\sqrt{R}} - 2I$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi - 2I. \quad (128)$$

When
$$r = \infty, \quad F\phi = \tfrac{1}{2}K,$$

and
$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)} = \frac{\pi}{24}, \quad (129)$$

so that $\Theta$, the corresponding value of $\theta$, is given by

$$\Theta = \tfrac{1}{6}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{K} - \frac{\pi}{12}$$

$$= \frac{\pi}{12}\left\{\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{\mathrm{K}}{\frac{1}{2}\pi} - 1\right\}. \quad (130)$$

In the branch of the catenary extending from $-\infty$ to $r_1$, in which $r$ is negative, we must write

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_0-r.r_1-r.r_3-r)}}{(-r)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_2+r)}\sqrt{(r_2-r)}}{(-r)^{\frac{3}{2}}}. \quad (131)$$

In the closed branch of the catenary, we take

$$I = \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r_0-r\,.\,r_3-r\,.\,r-r_1)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{(r+r_2)\sqrt{(r-r_2)}}{r^{\frac{3}{2}}}, \qquad (132)$$

leading to $$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \qquad (133)$$

and then $$\theta - 2I = \tfrac{1}{3}\sqrt{2}\int_{r_2}\frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1\,.\,r-r_2}{r_3-r_2\,.\,r-r_1}\right)},\ \&c., \qquad (134)$$

and, as $r$ ranges from $r_2$ to $r_3$, the apsidal angle is given by

$$\Theta = \frac{\pi}{3}\left\{\tfrac{1}{2}\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right\}. \qquad (135)$$

With the parameters

$$v_1 = \omega_1 + \tfrac{1}{4}\omega_3, \quad v = \omega_1 + \tfrac{1}{2}\omega_3, \qquad (136)$$

we have already shown that

$$e = \sqrt{5}-2 = \left(\frac{\sqrt{5}-1}{2}\right)^3, \qquad (137)$$

$$b^4 = 4\sqrt{5}-8, \qquad (138)$$

and now $$\kappa = \tfrac{1}{2}(\sqrt{5}-1) = \sin 38^\circ 10'. \qquad (139)$$

Now $\kappa = \kappa'^2$, $e = \kappa'^6$; and we find that

$$r_0 = 1+\kappa'+\kappa'^2, \qquad (140)$$

$$r_3 = 1+\kappa'-\kappa'^2, \qquad (141)$$

$$r_2 = 1-\kappa'+\kappa'^2, \qquad (142)$$

$$r_1 = 1-\kappa'-\kappa'^2; \qquad (143)$$

and it is also found that

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1-\kappa'^3)\sqrt{(r-r_0\,.\,r-r_3)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1+\kappa'^3)\sqrt{(r-r_1\,.\,r-r_2)}}{r^2\sqrt{2}}, \qquad (144)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\kappa'(1+\kappa')\,\frac{1}{\sqrt{R}} - \frac{\kappa'^3}{r\sqrt{R}}; \qquad (145)$$

so that, in the infinite branch from $r_0$ to $\infty$,

$$\theta+2I = \tfrac{1}{2}\kappa'(1+\kappa'^2)\int_{r_0}\frac{dr}{\sqrt{R}} = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_0}{r_0-r_1.r-r_3}\right)},\ \&c. \qquad (146)$$

We can also write the relations

$$\left.\begin{aligned} r^2\sin\left(\frac{F\phi}{\kappa}-2\theta\right) &= \frac{1}{\sqrt{2}}(r+1-\kappa'^3)\sqrt{\{r^2-2(1+\kappa')\,r+2\kappa'(1+\kappa')\}} \\ r^2\cos\left(\frac{F\phi}{\kappa}-2\theta\right) &= \frac{1}{\sqrt{2}}(r+1+\kappa'^3)\sqrt{\{r^2-2(1-\kappa')\,r-2\kappa'(1-\kappa')\}} \end{aligned}\right\}. \qquad (147)$$

In passing from $r_0$ to $\infty$,

$$\Theta = \frac{\mathrm{K}}{4\kappa}-\frac{\pi}{8} = \frac{\pi}{8}\left(\frac{1}{\kappa}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}-1\right) = \frac{\pi}{8}\times 0{\cdot}8185. \qquad (148)$$

In the infinite branch from $-\infty$ to $r_1$, it is preferable to change the sign of the $r$'s, and to write

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r-1-\kappa'^3)\sqrt{\{r^2+2(1-\kappa')\,r-2\kappa'(1-\kappa')\}}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r-1+\kappa'^3)\sqrt{\{r^2+2(1+\kappa')\,r+2\kappa'(1+\kappa')\}}}{r^2\sqrt{2}}, \qquad (149)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\kappa'(1+\kappa')\frac{1}{\sqrt{R}}+\frac{\kappa'^3}{r\sqrt{R}}, \qquad (150)$$

so that $$2I-\theta = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_2-r_0.r-r_1}{r_1-r_0.r-r_2}\right)},\ \&c., \qquad (151)$$

and, as $r$ increases from $r_1$ to $\infty$, $4I$ increases from 0 to $\frac{5}{4}\pi$, and $F\phi$ from 0 to $\frac{1}{2}\mathrm{K}$, so that

$$\Theta = \tfrac{5}{8}\pi-\frac{\mathrm{K}}{4\kappa}$$

$$= \frac{\pi}{8}\left(5-\frac{1}{\kappa}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}\right) = \frac{\pi}{8}\times 3{\cdot}1815. \qquad (152)$$

In the limited branch, take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1+\kappa'^3)\sqrt{(r-r_2.r-r_1)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1-\kappa'^3)\sqrt{(r_0-r.r_3-r)}}{r^2\sqrt{2}}, \qquad (153)$$

and then $$\frac{dI}{dr} = -\tfrac{1}{4}\kappa'(1+\kappa'^2)\frac{1}{\sqrt{R}} + \frac{\kappa'^3}{r\sqrt{R}}, \qquad (154)$$

so that $$\theta - 2I = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_2}{r_3-r_2.r-r_1}\right)}, \text{ &c.}, \qquad (155)$$

or, as it may be written

$$\left.\begin{aligned} r^2\sin\left(2\theta - \frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1+\kappa'^3)\sqrt{(r-r_2.r-r_1)} \\ r^2\cos\left(2\theta - \frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1-\kappa'^3)\sqrt{(r_0-r.r_3-r)} \end{aligned}\right\}. \qquad (156)$$

As $r$ grows from $r_2$ to $r_3$, the apsidal angle

$$\Theta = \frac{\pi}{4}\left(\frac{1}{\kappa}\frac{\mathrm{K}}{\frac{1}{2}\pi} + 1\right) = \frac{\pi}{4}\times 2{\cdot}8185. \qquad (157)$$

With $$v_1 = \omega_1 + \tfrac{1}{5}\omega_3, \quad v = \omega_1 + \tfrac{2}{5}\omega_3, \qquad (158)$$

$$\wp 5v = e_1, \qquad (159)$$

so that, from (330), p. 450, *L.M.S.*, XXVII,

$$(1-2m)\,\alpha = m^2(1-m), \qquad (160)$$

and making use of the values of $\alpha$ and $m$ in (73) and (74), § 29, and putting $e^2 = c$, there results the sextic equation

$$c^6 + 194c^5 - 745c^4 + 6908c^3 + 2015c^2 - 4318c + 41 = 0, \qquad (161)$$

requiring solution.

The corresponding form of $I(v_1)$ must be

$$I = \tfrac{1}{5}\sin^{-1}\frac{(Hr+H_1)\sqrt{(r-r_0.r-r_1.r-r_3)}}{r^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\cos^{-1}\frac{(Kr^2+K_1r+K_2)\sqrt{(r-r_2)}}{r^{\frac{5}{2}}}, \qquad (162)$$

in which the determination of the $H$'s and $K$'s will give some trouble.

If we try $$v_1 = \omega_1 + \tfrac{1}{6}\omega_3, \quad v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{163}$$

then $$\wp 3v = e_2; \tag{164}$$

and, from (326), p. 450, *L.M.S.*, XXVII,

$$s_1 - s_2 = \frac{(1-m)^2\,a}{a-m}, \tag{165}$$

leading to the equation

$$(e-1)(e^4-8e^3-18e^2-24e+1)^2 = 0. \tag{166}$$

The corresponding $I(v_1)$ must be of the form

$$I = \tfrac{1}{6}\sin^{-1}\frac{(Hr^2+H_1r+H_2)\sqrt{(r-r_0.r-r_3)}}{r^3}$$

$$= \tfrac{1}{6}\cos^{-1}\frac{(Kr^2+K_1r+K_2)\sqrt{(r-r_1.r-r_2)}}{r^3}. \tag{167}$$

The parameters $$v_1 = \omega_1 + \tfrac{1}{7}\omega_3, \quad v = \omega_1 + \tfrac{2}{7}\omega_3 \tag{168}$$

make $$s(7v) = s_1, \tag{169}$$

so that, from (332), p. 450, *L.M.S.*, XXVII,

$$(1-2m)^3\,a^2 - m\,(1-m)(1-2m)(2-3m)\,a + m^2(1-m)^4 = 0, \tag{170}$$

giving an equation for $e^2$; and now

$$I = \tfrac{1}{7}\sin^{-1}\frac{(Hr^2+H_1r+H_2)\sqrt{(r-r_0.r-r_1.r-r_3)}}{r^{\frac{7}{2}}}$$

$$= \tfrac{1}{7}\cos^{-1}\frac{(Kr^3+K_1r^2+K_2r+K_3)\sqrt{(r-r_2)}}{r^{\frac{7}{2}}}, \tag{171}$$

and so on, for higher values; but the complexity, as is seen, increases very rapidly.

31. If $b>1$, two roots of $R$ are imaginary; we must now replace $e^2$ by $-e^2$, and $e$ by $ie$ in the preceding, and put

$$R = r^2(r-2)^2 - 1 - c^2$$

$$= r^2(r-2)^2 - b^4, \tag{172}$$

and $$r_0 = 1 + \sqrt{(b^2+1)}, \tag{173}$$

$$r_3 = 1 + i\sqrt{(b^2-1)}, \tag{174}$$

$$r_2 = 1 - i\sqrt{(b^2-1)}, \tag{175}$$

$$r_1 = 1 - \sqrt{(b^2+1)}. \tag{176}$$

Also
$$e_1 = \tfrac{1}{3}, \quad e_2, e_3 = -\tfrac{1}{6} \pm \tfrac{1}{2}ie, \tag{177}$$

$$\kappa^2, \ \kappa'^2 = \tfrac{1}{2}\left(1 \mp \frac{1}{b^2}\right) = \tfrac{1}{2}\left\{1 \mp \frac{1}{\sqrt{(1+e^2)}}\right\}, \tag{178}$$

and
$$u = \int_{r_0}^{r} \frac{dr}{\sqrt{R}} = \frac{F\phi}{b\sqrt{2}} = \sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi, \tag{179}$$

where
$$\tan^2 \tfrac{1}{2}\phi = \frac{r - r_0}{r - r_1}, \tag{180}$$

$$\cos\phi = \frac{r_0 - r_1}{2r - r_0 - r_1}. \tag{181}$$

These catenaries are not so interesting to draw, as the limited branch is absent; but a few specimens are added here.

Thus, for instance, for a parameter
$$v_1 = \tfrac{2}{3}\omega_2', \tag{182}$$

we have, changing the sign of $\sqrt{2}$ in (116), § 30,
$$b^2 = 2\sqrt{2} + 2, \tag{183}$$

$$\kappa^2 = \tfrac{1}{4}(3 - \sqrt{2}), \quad \kappa'^2 = \tfrac{1}{4}(\sqrt{2} + 1), \tag{184}$$

$$r_0 = 2 + \sqrt{2}, \qquad r_1 = -\sqrt{2}, \tag{185}$$

and we must take
$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\ \frac{(r+2+\sqrt{2})\sqrt{(r-2-\sqrt{2})}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\ \frac{\sqrt{(r+\sqrt{2}.r^2 - 2r + 2\sqrt{2}+2)}}{r^{\frac{3}{2}}}; \tag{186}$$

for, differentiating,
$$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{187}$$

so that
$$\theta - 2I = \tfrac{1}{3}\sqrt{2}\int_{r_0}^{r} \frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi. \tag{188}$$

With a parameter
$$v_1 = \tfrac{1}{4}\omega_2', \tag{189}$$

we shall find that
$$b^2 = 2, \tag{190}$$

$$\kappa^2 = \tfrac{1}{4}, \quad \kappa = \tfrac{1}{2} = \sin 30^\circ; \tag{191}$$

and
$$I = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{(r^2 - 2r - 2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1)\sqrt{(r^2 - 2r + 2)}}{r^2}, \tag{192}$$

and $$\frac{dI}{dr} = -\tfrac{1}{4}\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{193}$$

so that $$\theta - 2I = \tfrac{1}{2}\int_{r_0}\frac{dr}{\sqrt{R}} = \tfrac{1}{4}F\phi\,; \tag{194}$$

and this can be replaced by

$$r^2 \sin(2\theta - \tfrac{1}{2}F\phi) = \sqrt{(r^2-2r-2)}, \tag{195}$$

$$r^2 \cos(2\theta - \tfrac{1}{2}F\phi) = (r+1)\sqrt{(r^2-2r+2)}, \tag{196}$$

giving the development of a catenary on a cone, consisting of two sets of separate infinite branches.

Again, with $$v = \tfrac{2}{3}\omega_2',$$

$$I = \tfrac{1}{3}\sin^{-1}\frac{\left\{r^2-(\sqrt{3}-1)\,r-\frac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r-\frac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{\left\{r^2+(\sqrt{3}+1)\,r+\frac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r+\frac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}, \tag{197}$$

$$\frac{dI}{dr} = \frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{R}} - \frac{2}{\sqrt{3}}\,\frac{1}{r\sqrt{R}}, \tag{198}$$

so that, taking $$b^2 = \frac{2}{\sqrt{3}}, \tag{199}$$

we can make $$\theta + I = \frac{1}{\sqrt{3}}\int_{r_0}\frac{dr}{\sqrt{R}} = \frac{F\phi}{2\sqrt[4]{3}}, \tag{200}$$

and $$\kappa^2 = \tfrac{1}{2}(1-\tfrac{1}{2}\sqrt{3}), \quad \kappa = \sin 15°. \tag{201}$$

*Whirling Catenary on a Cone.*

32. If the cone is made to rotate about its axis with sufficient angular velocity to make gravity insensible compared with the centrifugal force, the differential equation of the catenary becomes changed into

$$\frac{d\theta}{dr^2} = \tfrac{1}{2}\frac{b^3}{r^2\sqrt{R}}, \tag{1}$$

where $$R = 4r^2(r^2-a^2)^2 - b^6, \tag{2}$$

giving the relation between $\theta$ and $r$ in the plane development of the catenary.

Now
$$r\frac{d\theta}{dr} = \tan\phi = \frac{b^3}{\sqrt{R}}, \tag{3}$$

$$\sin\phi = \frac{b^3}{2r\,(r^2-a^2)}, \tag{4}$$

so that
$$p\,(r^2-a^2) = \tfrac{1}{2}b^3 \tag{5}$$

in the plane development.

This gives, as a particular case, when the cone itself is a plane, the catenary formed on a whirling horizontal table, or the catenary under a central force varying as the distance.

To reduce the relation (1) to the standard form of (A), put

$$y+1 = 0, \tag{6}$$

$$r^2 = M^2 s, \tag{7}$$

$$r^2-a^2 = M^2\,(s+x), \tag{8}$$

$$b^6 = -M^6\Sigma = M^6 x^2, \tag{9}$$

&c.,

and the analytical development is practically the same as that which follows for the trajectory of a particle on a smooth vertical cone.

*The Trajectory of a Particle on a Cone.*

33. The path of a particle on a smooth vertical cone (which can be imitated by rolling a coin inside a conical cardboard lamp-shade, or by the path of a bicycle on the banked conical turnings of a racing track) is given by

$$x^2\frac{d\psi}{dt} = H, \tag{1}$$

$$\sec^2\alpha\,\frac{dx^2}{dt^2} + x^2\frac{d\psi^2}{dt^2} = 2g\,(z\sim h)$$
$$= 2g\cot\alpha\,(x\sim a), \tag{2}$$

$$\sec^2\alpha\,\frac{dx^2}{dt^2} = 2g\cot\alpha\,(x\sim a) - \frac{H^2}{x^2}, \tag{3}$$

so that
$$\sin^2\alpha\,\frac{x^2\,d\psi}{dx^2} = \frac{H^2}{2g\cot\alpha\,(x\sim a)\,x^2 - H^2}$$
$$= \frac{k^3}{4\,(x\sim a)\,x^2 - k^3}, \tag{4}$$

on putting
$$H^2 = \tfrac{1}{2}gk^3\cot\alpha\,;$$

or
$$\frac{r\,d\theta}{dr} = \frac{b^{\frac{3}{2}}}{\sqrt{\{4(r\sim a)\,r^2 - b^3\}}} = \frac{b^{\frac{3}{2}}}{\sqrt{R}} \tag{5}$$

in the developed curve, where

$$R = 4(r\sim a)\,r^2 - b^3. \tag{6}$$

Then
$$\sin\phi = \frac{b^{\frac{3}{2}}}{2r\sqrt{(r\sim a)}}, \tag{7}$$

so that
$$p = r\sin\phi = \frac{b^{\frac{3}{2}}}{\sqrt{(r\sim a)}}, \tag{8}$$

or
$$p^2(r\sim a) = b^3, \tag{9}$$

is the curve which is the development of the path of the particle on the cone.

Since $p$ has no stationary values, this curve has no points of inflexion; but the projection on a horizontal plane of the trajectory on the cone can have points of inflexion.

In particular, if $a = 0$, the curve becomes

$$2r^{\frac{3}{2}}\cos\tfrac{3}{2}\theta = b^{\frac{3}{2}}, \tag{10}$$

a central orbit described under a constant central force.

The arc $s$ is given by

$$s = \int \frac{r\sqrt{(r\sim a)}\,dr}{\sqrt{\{4(r\sim a)\,r - b^3\}}}, \tag{11}$$

an integral similar to those occurring with the paraboloid.

If the developed curve is described as a central orbit, with velocity $v$,

$$v^2 = \frac{h^2}{p^2} = \frac{h^2}{b^3}(r\sim a), \tag{12}$$

and the central force, given by $d\tfrac{1}{2}v^2/dr$, is constant.

So also, if the developed catenary on the cone is treated as a plane catenary under a central force at the origin, and if $T$ denotes the tension

$$T = \frac{H}{p} = \frac{H}{b^2}(r-2a), \tag{13}$$

and the central force, given by $dT/dr$, is also constant.

But, if this catenary is described as a central orbit, the force varies as $r-2a$; and, if $a = 0$, we obtain the rectangular hyperbola, as before.

With a central force (realisable with an elastic thread)

$$P = ar + b, \tag{14}$$

$$v^2 = ar^2 + 2br + c, \tag{15}$$

$$= \frac{h^2}{p^2} = h^2\left(\frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}\right), \tag{16}$$

so that

$$\frac{h^2}{r^2}\frac{dr^2}{d\theta^2} = ar^4 + 2br^3 + cr^2 + 0 - h^2. \tag{17}$$

Or, more generally, with (Legendre, *F.E.I.*, p. 557)

$$P = ar + b + dr^{-2}, \tag{18}$$

$$\frac{h^2}{r^2}\frac{dr^2}{d\theta^2} = ar^4 + 2br^3 + cr^2 - dr - h^2 = R, \tag{19}$$

suppose; or, with

$$r = 1/u,$$

$$h^2u^2\frac{du^2}{d\theta^2} = -h^2u^4 - du^3 + cu^2 + 2bu + a = U, \tag{20}$$

suppose; so that

$$\theta = \int \frac{hu\,du}{\sqrt{U}}, \tag{21}$$

a form which can be compared immediately with Abel's results.

Generally, if the trajectory on the cone is described under a central force $P$, acting through the vertex of the cone, we find that

$$\int P\,dr = \frac{h^2}{p^2}, \tag{22}$$

where

$$H^2 = h^2\sin^2\alpha, \tag{23}$$

so that $P$ is the same as for the plane development of the trajectory, described as a central orbit, and this explains why the developed curve is described as a central orbit under a constant central force.

Again, for instance, the equation of the development of a plane section of the eight circular cone being of the form

$$\frac{l}{r} = 1 + e\cos k\theta, \tag{24}$$

the central force

$$P = Ar^{-2} + Br^{-3}. \tag{25}$$

34. Comparing equation (5) with the standard form in (A), we must take

$$y+1=0; \tag{26}$$

or, employing the forms given on p. 449, *L.M.S.*, XXVII,

$$y+1=(2a-1)\frac{m}{a-m}, \tag{27}$$

so that either $m=0$, which must be rejected, or

$$a=\tfrac{1}{2}; \tag{28}$$

and this makes

$$x=-\frac{2m^3}{(1-2m)^2}, \tag{29}$$

$$s_\gamma=\frac{m^2}{(1-2m)^2}, \tag{30}$$

$$(s_\alpha-s_\gamma)(s_\beta-s_\gamma)=\frac{m^4(3-2m)}{(1-2m)^3}, \tag{31}$$

$$(s_\alpha-s_\beta)^2=\frac{m^4(8m-3)}{(1-2m)^4}, \tag{32}$$

$$(s_\alpha-s_\gamma+s_\beta-s_\gamma)^2=\frac{m^4(4m-3)^2}{(1-2m)^4}, \tag{33}$$

$$s_\alpha-s_\gamma,\quad s_\beta-s_\gamma=\frac{m^2}{2(1-2m)^2}\{4m-3\pm\sqrt{(8m-3)}\}, \tag{34}$$

$$s_\alpha,\ s_\beta=\frac{m^2}{2(1-2m)^2}\{4m-1\pm\sqrt{(8m-3)}\}. \tag{35}$$

When $8m-3$ is positive, the parameter

$$v=\omega_1+f\omega_3, \tag{36}$$

and there are three cases to distinguish :

(i.) When $1/f$ is an odd integer,

$$s_\alpha>s_\gamma>s_\beta, \tag{37}$$

$$\kappa^2=\frac{s_\gamma-s_\beta}{s_\alpha-s_\beta}=\frac{3-4m+\sqrt{(8m-3)}}{2\sqrt{(8m-3)}}, \tag{38}$$

$$\kappa'^2=\frac{s_\alpha-s_\gamma}{s_\alpha-s_\beta}=\frac{3+4m+\sqrt{(8m-3)}}{2\sqrt{(8m-3)}}, \tag{39}$$

$$s_\gamma-s(v)=\frac{m^2a}{a-m}=\frac{m^2}{1-2m}, \tag{40}$$

and this must be negative; so that

$$m>\tfrac{1}{2}. \tag{41}$$

(ii.) When $1/f$ is an even integer,

$$s_\alpha > s_\beta > s_\gamma, \tag{42}$$

$$\kappa^2 = \frac{s_\beta - s_\gamma}{s_\alpha - s_\gamma} = \frac{4m-3-\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{43}$$

$$\kappa'^2 = \frac{s_\alpha - s_\beta}{s_\alpha - s_\gamma} = \frac{2\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{44}$$

and $s_\gamma - s(v)$ is negative; so that

$$m > \tfrac{3}{4}. \tag{45}$$

(iii.) When $1/f$ is half an odd integer,

$$s_\gamma > s_\alpha > s_\beta,$$

$$\kappa^2 = \frac{s_\alpha - s_\beta}{s_\gamma - s_\beta} = \frac{2\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{46}$$

$$\kappa'^2 = \frac{s_\gamma - s_\alpha}{s_\gamma - s_\beta} = \frac{-4m+3-\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{47}$$

and $s_\gamma - s(v)$ is positive, so that

$$\tfrac{1}{2} > m > \tfrac{3}{8}. \tag{48}$$

When $8m-3$ is negative, the parameter

$$v = f\omega_2', \tag{49}$$

and

$$\kappa^2,\ \kappa'^2 = \tfrac{1}{2}\left\{1 \mp \frac{s_\gamma - \frac{1}{2}(s_\alpha + s_\beta)}{\sqrt{(s_\alpha - s_\gamma . s_\beta - s_\gamma)}}\right\} \tag{50}$$

$$= \tfrac{1}{2}\left\{1 \mp \frac{3-4m}{2\sqrt{(3-2m . 1-2m)}}\right\}. \tag{51}$$

In the upper limited branch of the trajectory, described by the particle sliding on the inside of the upper sheet of the cone, take

$$R = 4(a-r)r^2 - b^3$$

$$= 4(r_1 - r . r_2 - r . r_3 - r), \tag{52}$$

$$\infty > r_3 > r > r_2 > 0 > r_1 > -\infty. \tag{53}$$

Writing $\sigma$ for $s(v)$, then, we put

$$r = -M^2(s+x) = M^2(\sigma - s), \tag{54}$$

$$a - r = M^2 s, \tag{55}$$

$$R = M^6 S,$$

$$a = -M^2 x, \tag{56}$$

$$b^3 = -M^6\Sigma = M^6x^2, \tag{57}$$

$$x = -\frac{a^3}{b^3}, \tag{58}$$

$$M^2 = \frac{b^3}{a^2}, \tag{59}$$

$$M\frac{dr}{\sqrt{R}} = -\frac{ds}{\sqrt{S}}, \tag{60}$$

and now the standard integral (A)

$$I(v) = \tfrac{1}{2}\int_s^{s_2} \frac{P(\sigma - s) - \sqrt{(-\Sigma)}}{\sigma - s}\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_{r_2}^{r}\frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int\frac{b^{\frac{3}{2}}dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\frac{F\phi}{\sqrt{(r_3 - r_1)}} - \tfrac{1}{2}\theta, \tag{61}$$

and
$$F\phi = \mathrm{dn}^{-1}\sqrt{\left(\frac{r_2 - r_1}{r - r_1}\right)}, \tag{62}$$

$$\kappa^2 = \frac{r_3 - r_2}{r_3 - r_1}, \tag{63}$$

so that
$$\Delta^2\phi = \frac{r_2 - r_1}{r - r_1}, \tag{64}$$

$$\sin^2\phi = \frac{r_3 - r_1}{r_3 - r_2}\cdot\frac{r - r_2}{r - r_1}. \tag{65}$$

35. Also, for the time integral,

$$r^2\frac{dr^2}{dt^2} = \tfrac{1}{2}g\cos\alpha\, R, \tag{66}$$

or, putting
$$\tfrac{1}{2}g\cos\alpha = ax^2,$$

$$nt = \int_{r_2}^{r}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_3 - r_1)}}\int_0^{\phi}\left(r_1 + \frac{r_2 - r_1}{\Delta^2\phi}\right)\frac{d\phi}{\Delta\phi},$$

and (Legendre, *Fonctions Elliptiques*, I, p. 256)

$$\int_0^{\phi}\frac{d\phi}{\Delta^3\phi} = \frac{E\phi}{\kappa'^2} - \frac{\kappa^2}{\kappa'^2}\frac{\sin\phi\cos\phi}{\Delta\phi}. \tag{67}$$

The expression for the time is simpler if reckoned from the upper limit of the trajectory; and now

$$I(v) = \tfrac{1}{2}\int_{s_3}^{s} \frac{P(\sigma - s) - \sqrt{(-\Sigma)}}{\sigma - s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_{r}^{r_3}\frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int_{r}^{r_3}\frac{b^2\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\,\frac{F\phi}{\sqrt{(r_3 - r_1)}} - \tfrac{1}{2}\theta, \qquad (68)$$

where $$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r_3 - r}{r_3 - r_2}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\frac{r - r_2}{r_3 - r_2}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\frac{r - r_1}{r_3 - r_1}\right)}, \qquad (69)$$

and

$$nt = \int_{r}^{r_3}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_3 - r_1)}}\int_0^{\phi}\left\{r_1 + (r_3 - r_1)\,\Delta^2\phi\right\}\frac{d\phi}{\Delta\phi}$$

$$= \frac{1}{\sqrt{(a.r_3 - r_1)}}\left\{r_1 F\phi + (r_3 - r_1)\,E\phi\right\}. \qquad (70)$$

If $r_2 = r_3$, as in steady motion in a horizontal circle, then

$$r_1 = -\tfrac{1}{2}r_3,$$

and

$$\kappa = 0, \quad F\phi = E\phi = \phi;$$

so that

$$nt = \frac{r_3\phi}{\sqrt{(a.r_3 - r_1)}}, \qquad (71)$$

and the small oscillations synchronize with a simple pendulum of length $l$, such that

$$n\sqrt{\frac{l}{g}} = \frac{r_3}{\sqrt{(a.r_3 - r_1)}},$$

or

$$l = \frac{g}{n^2}\,\frac{r_3^2}{a\,(r_3 - r_1)} = 2\sec a\,\frac{r_3^2}{r_3 - r_1} = \tfrac{4}{3}OG, \qquad (72)$$

if the normal to the cone meets the axis in $G$.

But, if the particle describes the same circle, suspended from $G$, as a conical or spherical pendulum, then it will be found that the small oscillations synchronize with a pendulum of length

$$\frac{OG}{1 + \frac{1}{4}\cot^2 a}. \qquad (73)$$

36. In the unlimited branch of the trajectory, described by the particle sliding on the outside of the lower sheet of the cone, it is convenient to change the sign of $r$, $r_1$, $r_2$, $r_3$, and to make

$$R = 4(r+a)r^2 - b^3$$
$$= 4.r-r_1.r-r_2.r-r_3, \tag{74}$$
$$\infty > r > r_1 > 0 > r_2 > r_3 > -\infty. \tag{75}$$

We now put
$$r = M^2(s+\alpha) = M^2(s-\sigma), \tag{76}$$
$$r+a = M^2 s, \tag{77}$$
$$R = M^6 S, \tag{78}$$
$$M\frac{dr}{\sqrt{R}} = \frac{ds}{\sqrt{S}}, \tag{79}$$

and now
$$I(v) = \tfrac{1}{2}\int_{s_1}^{s} \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$
$$= \tfrac{1}{2}PM\int_{r_1}^{r}\frac{dr}{\sqrt{R}} + \tfrac{1}{2}\int_{r_1}^{r}\frac{b^3\,dr}{r\sqrt{R}}$$
$$= \tfrac{1}{2}PM\frac{F\phi}{\sqrt{(r_1-r_3)}} + \tfrac{1}{2}\theta, \tag{80}$$

where
$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r-r_2}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r_1-r_3}\,\frac{r-r_3}{r-r_2}\right)}, \tag{81}$$

and, in the expression for the time,
$$nt = \int_{r_1}^{r}\frac{r\,dr}{\sqrt{(aR)}}$$
$$= \frac{1}{\sqrt{(a.r_1-r_3)}}\int_0^{\phi}\left\{r_2+(r_1-r_2)\sec^2\phi\right\}\frac{d\phi}{\Delta\phi}, \tag{82}$$

in which (Legendre)
$$\int_0 \sec^2\phi\,\frac{d\phi}{\Delta\phi} = \tan\phi\,\Delta\phi + \kappa'^2 F\phi - E\phi. \tag{83}$$

We might also have started from $r=\infty$, and then
$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r_1-r_3}{r-r_3}\right)} = \mathrm{cn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_3}\right)} = \mathrm{dn}^{-1}\sqrt{\left(\frac{r-r_2}{r-r_3}\right)}, \tag{84}$$

but now the expression for the time will introduce an infinite constant.

We notice that, while $r=0$ corresponds to the elliptic argument or parameter $u=v=\omega_1+f\omega_3$, the value of $r=a$ in the upper branch, or of $r=-a$ in the lower branch, corresponds to the argument

$$u=2v=2f\omega_3.$$

37. Suppose we begin by taking $f=\frac{1}{2}$; then $2v=\omega_3$, so that $r\mp a$ must be a factor of $R$, which requires $b=0$; and the trajectory lies in a vertical plane through the vertex.

Next take $f=\frac{1}{3}$, when $m=1$, (85)

$$x=-2,\quad y=-1,\quad M^2=\tfrac{1}{2}a,\quad b^3=\tfrac{1}{2}a^3, \tag{86}$$

[*L.M.S.*, XXVII, (364), p. 460]; and, from (38) and (39), § 34,

$$\kappa^2,\ \kappa'^2=\frac{\sqrt{5}\mp1}{2\sqrt{5}}. \tag{87}$$

In the upper limited branch of the trajectory,

$$r=\tfrac{1}{2}a\,(2-s), \tag{88}$$

$$a-r=\tfrac{1}{2}as, \tag{89}$$

$$R=4\,(a-r)\,r^2-\tfrac{1}{2}a^3$$

$$=(r-\tfrac{1}{2}a)(-4r^2+2ar+a^2), \tag{90}$$

$$r_3=\tfrac{1}{4}\,(\sqrt{5}+1)\,a,\quad r_2=\tfrac{1}{2}a,\quad r_1=-\tfrac{1}{4}\,(\sqrt{5}-1)\,a, \tag{91}$$

and the corresponding pseudo-elliptic integral

$$I(\omega_1+\tfrac{1}{3}\omega_3)=\tfrac{1}{3}\sin^{-1}\frac{(r+\frac{1}{2}a)\sqrt{(r-\frac{1}{2}a)}}{r^{\frac{3}{2}}}$$

$$=\tfrac{1}{3}\cos^{-1}\frac{\sqrt{\left\{\frac{1}{2}a\,(-r^2+\frac{1}{2}ar+\frac{1}{4}a^2)\right\}}}{r^{\frac{3}{2}}}, \tag{92}$$

$$\frac{dI}{dr}=-\tfrac{1}{6}\sqrt{2}\,\frac{\sqrt{a}}{\sqrt{R}}+\frac{1}{2\sqrt{2}}\,\frac{a^{\frac{3}{2}}}{r\sqrt{R}}, \tag{93}$$

so that

$$\theta=\tfrac{1}{3}\sqrt{2}\int\frac{\sqrt{a}\,dr}{\sqrt{R}}+2I$$

$$=\frac{2F\phi}{3\sqrt[4]{5}}+2I, \tag{94}$$

$$F\phi=\mathrm{dn}^{-1}\sqrt{\left\{\frac{\frac{1}{4}\,(\sqrt{5}+1)\,a}{r+\frac{1}{4}\,(\sqrt{5}-1)\,a}\right\}},\ \text{&c.},$$

and the apsidal angle

$$\Theta=\frac{\pi}{3}\left(\frac{1}{\sqrt[4]{5}}\,\frac{\mathrm{K}}{\frac{1}{2}\pi}+1\right). \tag{95}$$

Changing the sign of $r$ in the open unlimited branch of the trajectory on the lower half of the cone,

$$r = \tfrac{1}{2}a(s-2), \tag{96}$$

$$r+a = \tfrac{1}{2}as, \tag{97}$$

$$R = 4(r+a)r^2 - \tfrac{1}{2}a^3$$

$$= (r+\tfrac{1}{2}a)(4r^2+2ar-a^2), \tag{98}$$

$$r_1 = \tfrac{1}{4}(\sqrt{5}-1)a, \quad r_2 = -\tfrac{1}{2}a, \quad r_3 = -\tfrac{1}{4}(\sqrt{5}+1)a, \tag{99}$$

and now the corresponding

$$I = \tfrac{1}{3}\sin^{-1}\frac{\sqrt{\{\tfrac{1}{2}a(r^2+\tfrac{1}{2}ar-\tfrac{1}{4}a^2)\}}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{(-r+\tfrac{1}{2}a)\sqrt{(r+\tfrac{1}{2}a)}}{r^{\frac{3}{2}}}, \tag{100}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{6}\sqrt{2}\frac{\sqrt{a}}{\sqrt{R}} + \frac{1}{2\sqrt{2}}\frac{a^{\frac{3}{2}}}{r\sqrt{R}}, \tag{101}$$

so that

$$\theta = -\frac{2F\phi}{3\sqrt[4]{5}} + 2I, \tag{102}$$

$$F\phi = \mathrm{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)}, \text{ &c.}, \tag{103}$$

and now the apsidal angle, in going from $r_1$ to infinity is given by

$$\Theta = -\frac{2\mathrm{K}}{3\sqrt[4]{5}} + \frac{2\pi}{3}$$

$$= \frac{\pi}{3}\left(2 - \frac{1}{\sqrt[4]{5}}\frac{\mathrm{K}}{\frac{1}{2}\pi}\right). \tag{104}$$

With $f = \frac{1}{4}$, and

$$v = \omega_1 + \tfrac{1}{4}\omega_3, \tag{105}$$

the equation obtained from putting in (329) (*L.M.S.*, XXVII, p. 450) $s_7 - s_4 = 0$ and $\alpha = \frac{1}{2}$ gives

$$2m^2 - 4m + 1 = 0 \tag{106}$$

or

$$m = 1 \pm \sqrt{2} \tag{107}$$

of which the upper sign must be taken so as to make $m > \frac{3}{4}$ in (45); and then

$$\frac{b^3}{a^3} = -\frac{1}{x} = 2 - \sqrt{2}. \tag{108}$$

Now, in the upper limited branch of the trajectory,

$$R = 4(a-r)r^2-(2-\sqrt{2})a^3$$
$$= 4\left(\frac{a}{\sqrt{2}}-r\right)\left(r^2-\frac{2-\sqrt{2}}{2}ar-\frac{\sqrt{2}-1}{2}a^2\right), \quad (109)$$

so that $$r_3 = \frac{a}{\sqrt{2}}, \quad r_2,\ r_1 = \frac{\sqrt{2}-1\pm\sqrt{(2\sqrt{2}-1)}}{2\sqrt{2}}a\,; \quad (110)$$

and we shall find that the associated pseudo-elliptic integral is

$$I(\omega_1+\tfrac{1}{4}\omega_3) = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\{r+(\sqrt{2}-1)a\}\sqrt{\left(\frac{a}{\sqrt{2}}-r\right)}}{r^2}$$
$$= \tfrac{1}{4}\cos^{-1}\frac{(r+a)\sqrt{\{r^2-\frac{1}{2}(2-\sqrt{2})ar-\frac{1}{2}(\sqrt{2}-1)a^2\}}}{r^2}, \quad (111)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\sqrt{a}}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \quad (112)$$

so that $$\theta = \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\int_r^{r_3}\frac{\sqrt{a}\,dr}{\sqrt{R}} - 2I$$
$$= \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\sqrt{\left(\frac{a}{r_3-r_1}\right)}F\phi - 2I, \quad (113)$$

where $$\sin^2\phi = \frac{r_3-r}{r_3-r_2}, \text{ \&c.}, \quad (114)$$

and, on reduction,

$$\sqrt{\left(\frac{a}{r_3-r_1}\right)} = \frac{2\sqrt[4]{2}}{\sqrt{(\sqrt{2}+1)}+\sqrt{(\sqrt{2}-1)}\sqrt{(2\sqrt{2}-1)}}, \quad (115)$$

so that $$\theta = mF\phi - 2I, \quad (116)$$

where $$m = \tfrac{1}{4}\{1-(\sqrt{2}-1)\sqrt{(2\sqrt{2}-1)}\}, \quad (117)$$

and the apsidal angle $$\Theta = mK - \tfrac{1}{4}\pi. \quad (118)$$

In the lower unlimited branch, changing the sign of $r$, so that

$$R = 4(r+a)r^2-(2-\sqrt{2})a^3, \quad (119)$$

$$r_1,\ r_2 = \frac{\pm\sqrt{(2\sqrt{2}-1)}-\sqrt{2}+1}{2\sqrt{2}}a, \quad r_3 = -\frac{a}{\sqrt{2}}, \quad (120)$$

and we take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(a-r)\sqrt{(r-r_1 . r-r_2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\{r-(\sqrt{2}-1)\,a\}\sqrt{(r-r_3)}}{r^2}, \qquad (121)$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\sqrt{a}}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \qquad (122)$$

so that
$$\theta = 2I - mF\phi, \qquad (123)$$

where
$$\sin^2\phi = \frac{r-r_1}{r-r_2}, \text{ \&c.}, \qquad (124)$$

and now the apsidal angle, in passing from $r_1$ to infinity, is given by

$$\Theta = \tfrac{3}{4}\pi - m\mathrm{K}. \qquad (125)$$

38. The other value of $m$,

$$m = 2 - \tfrac{1}{2}\sqrt{2}, \qquad (126)$$

makes
$$\frac{b^3}{a^3} = 2+\sqrt{2}, \qquad (127)$$

and $R$ has now two imaginary roots, so that the trajectory consists of a single infinite branch.

As another specimen of a trajectory of this character, we may take the one based upon $I\left(\frac{2}{5}\omega_2'\right)$, in which

$$x = y = -1, \qquad (128)$$

so that
$$S = 4s\,(s-1)^2 - 1, \qquad (129)$$

as in the Transformation of the Eleventh Order (*L.M.S.*, XXVII, p. 424), and in Abel's *Œuvres*, III, p. 142.

We have now to put

$$r = M^2(s-1), \qquad (130)$$

$$r+a = M^2 s, \qquad (131)$$

$$a = M^2, \qquad (132)$$

$$b^3 = a^3, \text{ \&c.}, \qquad (133)$$

in the pseudo-elliptic integral

$$I\left(\tfrac{2}{5}\omega_2'\right) = \tfrac{1}{5}\cos^{-1}\frac{2s^2-3s}{2\,(s-1)^{\frac{1}{2}}}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(s-2)\sqrt{\{4s\,(s-2)^2-1\}}}{2\,(s-1)^{\frac{1}{2}}},$$

so that $$I = \tfrac{1}{5}\cos^{-1}\frac{2r^2+ar-a^2}{2r^{\frac{5}{2}}}\sqrt{a}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(r-a)\sqrt{\{4(r+a)r^2-a^5\}}}{2r^{\frac{5}{2}}}, \tag{134}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{(r^{\frac{1}{2}}-a^{\frac{1}{2}})\sqrt{(2r^{\frac{3}{2}}+2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}+a^{\frac{3}{2}})}}{2r^{\frac{5}{4}}}$$

$$= \tfrac{2}{5}\cos^{-1}\frac{(r^{\frac{1}{2}}+a^{\frac{1}{2}})\sqrt{(2r^{\frac{3}{2}}-2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}-a^{\frac{3}{2}})}}{2r^{\frac{5}{4}}}, \tag{135}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{5}\frac{\sqrt{a}}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{136}$$

so that $$\theta = 2I - \tfrac{1}{5}\int_{r_2}^{r}\frac{\sqrt{a}\,dr}{\sqrt{R}}. \tag{137}$$

To reduce this elliptic integral of the first kind to Legendre's function $F\phi$, we must determine the real root of $R=0$, or $S=0$; or else determine $m$ from the cubic equation

$$2m^3-(2m-1)^2=0. \tag{138}$$

39. If the particle, instead of moving freely under gravity on the cone, is attached by a light thread to another particle, hanging freely at the end of the thread, the thread passing through a smooth hole at the vertex of the cone, there is no material change in the analytical equations; so that the preceding equations can be applied without essential modification. Thus, if the particles balance, the trajectory on the cone develops into a Cotes's spiral.

The cone, for instance, can be replaced by a flat smooth horizontal table, along which the particle is projected, attached to the second particle by a thread passing through a hole in the table.

When the particle is replaced by a ball, spinning and rolling on the cone, the dynamical equations are of the same form (Routh, *Rigid Dynamics*), but we have an additional constant at our disposal, as the restriction of (26), § 34, is no longer required.

If the cone is made to rotate with constant angular velocity about its axis, the function $R$ rises from the third to the fourth degree, and the differential equations are of the same form as those given in § 33.

In a stereoscopic representation we should take a *thalweg* procession of particles spaced at equal time intervals (Hadamard, *Liouville*, 1897).

*The Geodesics on a Quadric of Revolution.**

40. These geodesics have been discussed recently by Prof. Forsyth, in the *Messenger of Mathematics*, 1896; they are introduced here as another illustration of the Elliptic Integral of the Third Kind in the notation employed in this paper.

In the geodesic on the oblate spheroid, generated by the revolution about $Oy$ of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

$$x^2 \frac{d\psi}{ds} = k, \tag{2}$$

or
$$ds^2 = dx^2 + dy^2 + x^2 d\psi^2 = \frac{x^4}{k^2} d\psi^2; \tag{3}$$

and therefore, with
$$y = \frac{b}{a} \sqrt{(a^2 - x^2)}, \tag{4}$$

$$dy^2 = \frac{b^2}{a^2} \frac{x^2 dx^2}{a^2 - x^2}, \tag{5}$$

$$dx^2 \left(1 + \frac{b^2}{a^2} \frac{x^2}{a^2 - x^2}\right) + x^2 d\psi^2 \left(1 - \frac{x^2}{k^2}\right) = 0, \tag{6}$$

or
$$d\psi^2 = \frac{k^2 e^2 \left(\frac{a^2}{e^2} - x^2\right)}{x^2 (x^2 - k^2)(a^2 - x^2)} dx^2. \tag{7}$$

Put
$$x^2 = m^2 (s - \sigma), \tag{8}$$

$$\frac{a^2}{e^2} - x^2 = m^2 (s_1 - s), \tag{9}$$

$$a^2 - x^2 = m^2 (s_2 - s), \tag{10}$$

$$x^2 - k^2 = m^2 (s - s_3), \tag{11}$$

so that
$$a > x > k,$$

and
$$s_2 > s > s_3 > \sigma. \tag{12}$$

Since $s_3 - \sigma$ is positive, the parameter $v$ of the integral is a fraction of the imaginary period, or

$$v = f\omega_3. \tag{13}$$

Then
$$d\psi = \tfrac{1}{2} \frac{ke}{m} \frac{\sqrt{(s_1 - s)}}{(s - \sigma)\sqrt{(s_2 - s . s - s_3)}} ds, \tag{14}$$

* Legendre, *Fonctions Elliptiques*, t. I, § III, 360.

and, putting $x = 0, \quad s = \sigma,$

$$\frac{a^2}{e^2} = m^2(s_1 - \sigma), \tag{15}$$

$$a^2 = m^2(s_2 - \sigma), \tag{16}$$

$$k^2 = m^2(s_3 - \sigma), \tag{17}$$

$$e^2 = \frac{s_2 - \sigma}{s_3 - \sigma}, \tag{18}$$

$$\frac{k^2 e^2}{m^2} = \frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_1 - \sigma}. \tag{19}$$

Then, with $$S = 4(s - s_1)(s - s_2)(s - s_3), \tag{20}$$

$$\Sigma = 4(\sigma - s_1)(\sigma - s_2)(\sigma - s_3), \tag{21}$$

$$\begin{aligned}
\frac{d\psi}{ds} &= \tfrac{1}{2}\sqrt{\left(\frac{s_2 - \sigma \,.\, s_3 - \sigma}{s_1 - \sigma}\right)} \frac{\sqrt{(s_1 - s)}}{(s - \sigma)\sqrt{(s_2 - s \,.\, s - s_3)}} \\
&= \tfrac{1}{2} \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} \frac{s_1 - s}{(s - \sigma)\sqrt{S}} \\
&= \tfrac{1}{2} \frac{\sqrt{'-\Sigma)}}{(s - \sigma)\sqrt{S}} - \tfrac{1}{2} \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} \frac{1}{\sqrt{S}} \\
&= \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right) \frac{1}{\sqrt{S}} - \tfrac{1}{2} \frac{P(s - \sigma) - \sqrt{(-\Sigma)}}{(s - \sigma)\sqrt{S}},
\end{aligned} \tag{22}$$

$$\psi = \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right) u - I(v). \tag{23}$$

But $P$ or $P(v)$ is connected with $P(\omega_1 + v)$ by the relation

$$P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} = P(\omega_1 + v), \tag{24}$$

so that $$\psi = \tfrac{1}{2} \frac{P(\omega_1 + f\omega_3)}{\sqrt{(s_1 - s_3)}} F\phi - I(v), \tag{25}$$

where $$\sin^2\phi = \frac{s - s_3}{s_2 - s_3} = \frac{x^2 - k^2}{a^2 - k^2}, \quad \cos^2\phi = \frac{a^2 - x^2}{a^2 - k^2}, \quad \Delta^2\psi = \frac{a^2 - e^2x^2}{a^2 - e^2k^2}. \tag{26}$$

The first requirement of an algebraical geodesic is that

$$P(\omega_1 + f\omega_3) = 0; \tag{27}$$

but, as pointed out by Halphen (*F.E.*, II, p. 275), this relation implies a negative discriminant for the cubic $S$ and two imaginary roots, which is excluded by the conditions of the problem; the same applies to the geodesics on all the other quadric surfaces.

We shall find that, in Glaisher's notation,

$$\frac{\frac{1}{2}P(\omega_1+f\omega_3)}{\sqrt{(s_1-s_3)}} = Z(f\mathrm{K}', \kappa') = \mathrm{zn}\, f\mathrm{K}', \quad \frac{\frac{1}{2}P(f\omega_3)}{\sqrt{(s_1-s_2)}} = \mathrm{zs}\, f\mathrm{K}', \text{ \&c.} \tag{28}$$

41. On the prolate spheroid, generated by the revolution of the ellipse about $Ox$, we find in a similar manner, from

$$y^2\frac{d\psi}{ds} = k, \tag{29}$$

$$d\psi = \tfrac{1}{2}\frac{kae}{b}\,\frac{y^2+\dfrac{b^4}{a^2e^2}}{y^2\sqrt{\left(b^2-y^2.y^2-k^2.y^2+\dfrac{b^4}{a^2e^2}\right)}}\,dy^2, \tag{30}$$

and we put

$$y^2 = m^2(\sigma - s), \tag{31}$$

$$b^2-y^2 = m^2(s-s_3), \tag{32}$$

$$y^2-k^2 = m^2(s_2-s), \tag{33}$$

$$y^2+\frac{b^4}{a^2e^2} = m^2(s_1-s). \tag{34}$$

Putting $y=0$ or $s=\sigma$,

$$b^2 = m^2(\sigma-s_3), \tag{35}$$

$$k^2 = m^2(\sigma-s_2), \tag{36}$$

$$\frac{b^4}{a^2e^2} = m^2(s_1-\sigma), \tag{37}$$

$$\frac{k^2a^2e^2}{m^2b^2} = \frac{\sigma-s_2\,.\,\sigma-s_3}{s_1-\sigma}, \tag{38}$$

and

$$s_1>\sigma>s_2>s>s_3, \tag{39}$$

so that

$$v=\omega_1+f\omega_3, \quad u=\omega_3+f'\omega_1. \tag{40}$$

We now find

$$\frac{d\psi}{ds} = \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\,\frac{s_1-s}{(\sigma-s)\sqrt{S}}$$

$$= \tfrac{1}{2}\left(\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}+P\right)\frac{1}{\sqrt{S}} - \tfrac{1}{2}\frac{P(\sigma-s)-\sqrt{(-\Sigma)}}{(\sigma-s)\sqrt{S}}, \tag{41}$$

$$\psi = \tfrac{1}{2}\frac{P(f\omega_3)}{\sqrt{(s_1-s_3)}}F\phi - I(v), \tag{42}$$

with $\sin^2\phi = \dfrac{b^2-y^2}{b^2-k^2}$, $\cos^2\phi = \dfrac{y^2-k^2}{b^2-k^2}$, $\Delta^2\phi = 1-e^2+e^2\dfrac{y^2}{b^2}$. (43)

42. In the geodesics on the hyperboloid of one sheet, generated by the revolution about $Oy$ of the hyperbola

$$y = \frac{b}{a}\sqrt{(a^2-x^2)}, \tag{44}$$

$$x^2\frac{d\psi}{ds} = k, \tag{45}$$

$$d\psi = \tfrac{1}{2}ke\frac{x^2-\dfrac{a^2}{e^2}}{x^2\sqrt{X}}\,dx^2, \tag{46}$$

where
$$X = x^2-a^2\,.\,x^2-\frac{a^2}{e^2}\,.\,x^2-k^2, \tag{47}$$

$$e^2 = 1+\frac{b^2}{a^2}, \tag{48}$$

and the parameter
$$v = f\omega_3. \tag{49}$$

When $k^2<a^2$, the geodesic extends over the whole hyperboloid; and

$$\text{(i.)}\quad \infty > x^2 > a^2 > \frac{a^2}{e^2} > k^2, \tag{50}$$

$$s > s_1 > s_2 > s_2 > \sigma, \tag{51}$$

and we put
$$x^2 = m^2(s-\sigma), \tag{52}$$

$$x^2-a^2 = m^2(s-s_1), \tag{53}$$

$$x^2-\frac{a^2}{e^2} = m^2(s-s_2), \tag{54}$$

$$x^2-k^2 = m^2(s-s_3), \tag{55}$$

and find, as before,

$$\psi = \tfrac{1}{2}\frac{P\{\omega_1+(1-f)\,\omega_3\}}{\sqrt{(s_1-s_3)}}F\phi + I(v), \tag{56}$$

where
$$\sin^2\phi = \frac{a^2-k^2}{x^2-k^2},\quad \cos^2\phi = \frac{x^2-a^2}{x^2-k^2},\quad \Delta^2\phi = \frac{x^2-\dfrac{a^2}{e^2}}{x^2-k^2}. \tag{57}$$

The condition
$$k^2 = \frac{a^2}{e^2} \tag{58}$$

makes $s_2 = s_3$, $\kappa = 0$, and gives the generating lines.

$$\text{(ii.)}\quad \infty > x^2 > a^2 > k > \frac{a^2}{e^2}; \tag{59}$$

$s_2$ and $s_3$, $k^2$ and $\dfrac{a^2}{e^2}$ change places, and we find

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}} F\phi + I(v), \tag{60}$$

where
$$\sin^2\phi = \frac{a^2 - \dfrac{a^2}{e^2}}{x^2 - \dfrac{a^2}{e^2}}, \text{ \&c.} \tag{61}$$

$$\text{(iii.)} \quad \infty > x^2 > k^2 > a^2 > \frac{a^2}{e^2}, \tag{62}$$

the geodesic extends from the circle of radius $k$ to infinity, and

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}} F\phi + I(v),$$

where
$$\sin^2\phi = \frac{k^2 - \dfrac{a^2}{e^2}}{x^2 - \dfrac{a^2}{e^2}}, \text{ \&c.} \tag{63}$$

43. On the hyperboloid of two sheets, generated by the revolution of the hyperbola

$$x = \frac{a}{b}\sqrt{(b^2+y^2)} \tag{64}$$

about $Ox$,
$$y^2\frac{d\psi}{ds} = k, \tag{65}$$

leading to
$$d\psi = \tfrac{1}{2}\frac{kae}{b}\,\frac{1}{y^2}\,\frac{\sqrt{\left(y^2 + \dfrac{b^4}{a^2+b^2}\right)}}{\sqrt{(y^2+b^2 \,.\, y^2-k^2)}}\,dy^2. \tag{66}$$

We take
$$y^2 = m^2(s-\sigma), \tag{67}$$

$$y^2 - k^2 = m^2(s-s_1), \tag{68}$$

$$y^2 + \frac{b^4}{a^2+b^2} = m^2(s-s_2), \tag{69}$$

$$y^2 + b^2 = m^2(s-s_3), \tag{70}$$

and
$$s > s_1 > \sigma > s_2 > s_3, \tag{71}$$

so that
$$v = \omega_1 + f\omega_3, \tag{72}$$

and
$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}} + I(v), \tag{73}$$

$$\sin^2\phi = \frac{k^2+b^2}{y^2+b^2}, \quad \cos^2\phi = \frac{y^2-k^2}{y^2+b^2}, \quad \Delta^2\varphi = \frac{y^2 + \dfrac{b^4}{a^2+b^2}}{y^2+b^2}. \tag{74}$$

44. The parameter obtained by the bisection or trisection of a period will serve as an illustration of the general theory; it was employed in Mr. Dewar's stereoscopic diagrams given here.

For the oblate spheroid, and the parameter

$$v = \tfrac{1}{2}\omega_3, \tag{75}$$

we shall find that the projection of the geodesic can be expressed by

$$x^2 \sin\left\{2\psi - (1-\kappa)F\phi\right\} = \sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{(x^2-k^2)}, \tag{76}$$

$$x^2 \cos\left\{2\psi - (1-\kappa)F\phi\right\} = \sqrt{\left(\frac{a^2}{e^2} - x^2.a^2 - x^2\right)}, \tag{77}$$

where
$$e = \kappa, \quad \frac{b^2}{a^2} = 1-\kappa, \quad \frac{k^2}{a^2} = \frac{1}{1+\kappa}; \tag{78}$$

and now the apsidal angle

$$\Psi = \tfrac{1}{4}\pi\left\{(1-\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi} + 1\right\}. \tag{79}$$

By trial it will be found that

$$\Psi = \tfrac{3}{10}\pi, \quad \text{when} \quad \kappa = \sin 58^\circ, \tag{80}$$

so that a closed geodesic can be constructed with these data.

With the parameter
$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{81}$$

we shall find that the equation of the geodesic on the prolate spheroid may be written

$$y^2 \sin\left\{2\psi - (1+\kappa)F\phi\right\} = \sqrt{\left(y^2 - k^2.y^2 + \frac{b^4}{a^2e^2}\right)}, \tag{82}$$

$$y^2 \cos\left\{2\psi - (1+\kappa)F\phi\right\} = \frac{1-\kappa}{\sqrt{\kappa}}\, b\sqrt{(b^2-y^2)}, \tag{83}$$

where
$$\frac{b^2}{a^2} = \frac{k^2}{b^2} = 1-\kappa, \quad e^2 = \kappa, \tag{84}$$

with an apsidal angle

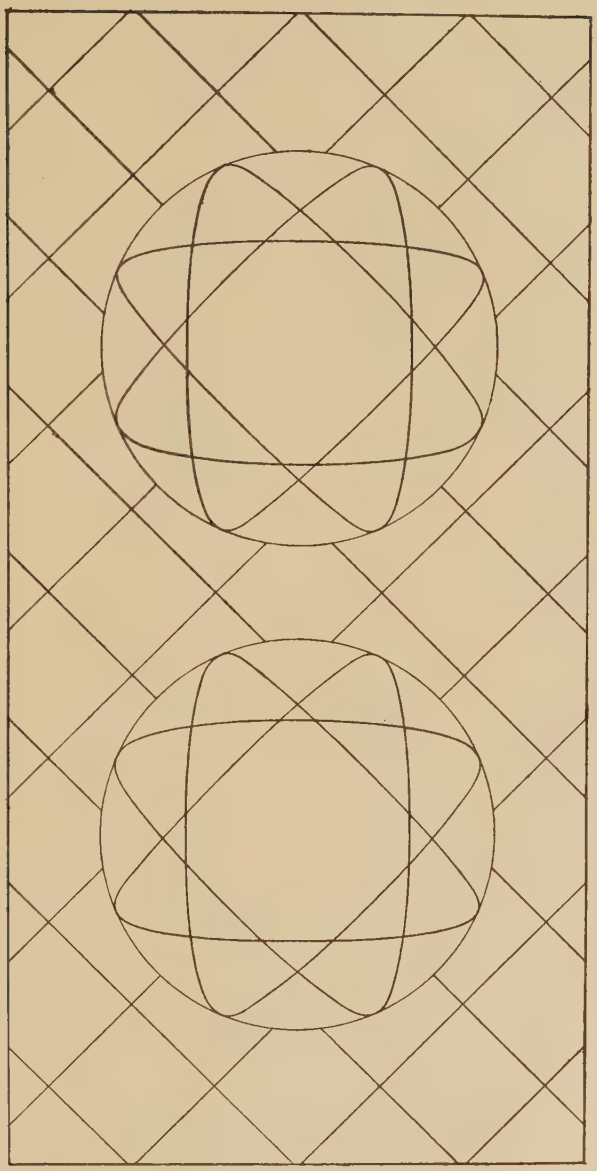
$$\Psi = \tfrac{1}{2}\pi\left\{(1+\kappa)\frac{\mathrm{K}}{\frac{1}{2}\pi} + 1\right\}. \tag{85}$$

By taking, as in § 21,

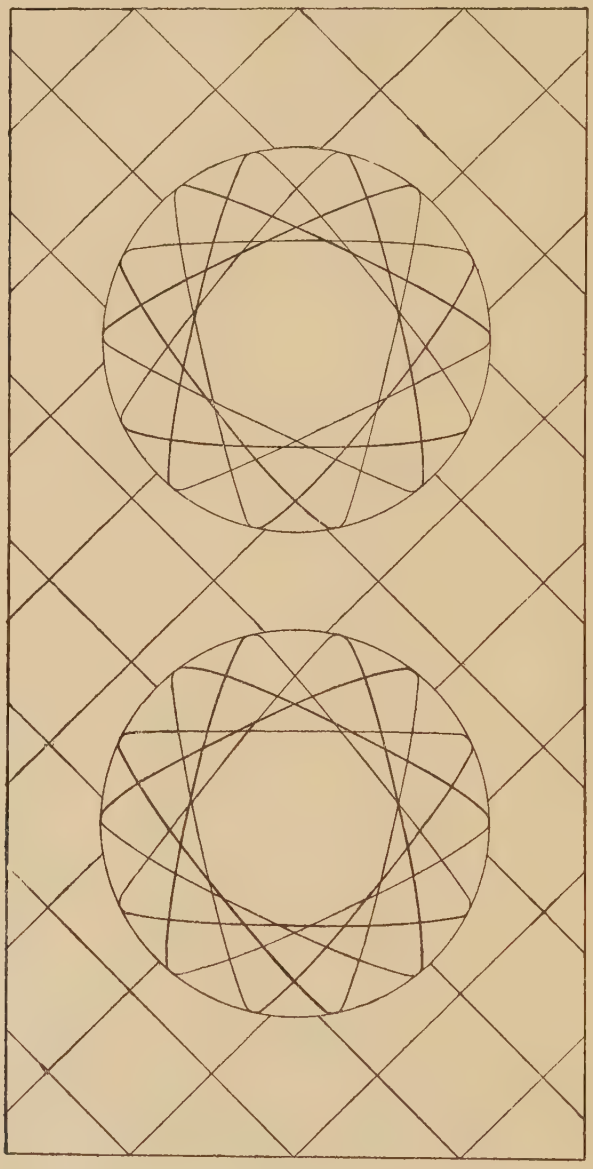
$$\kappa = 0{\cdot}77384 = \sin 50^\circ\, 42', \tag{86}$$

we make
$$\Psi = \tfrac{4}{5}\pi,$$

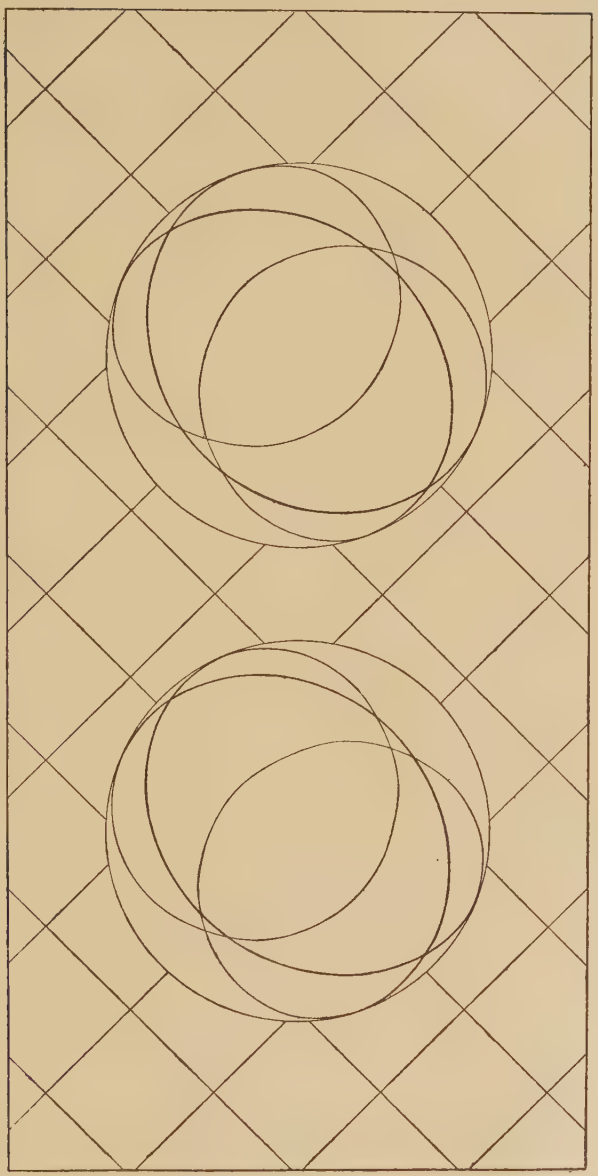
and thus obtain a closed geodesic.



Geodesic on an Oblate Spheroid.



Geodesic on an Oblate Spheroid.



Geodesic on a Prolate Spheroid.

On the hyperboloid of one sheet, with a parameter

$$v = \tfrac{1}{2}\omega_3, \tag{87}$$

the geodesic is given by

$$\text{(i.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(x^2-a^2\,.\,x^2-\frac{a^2}{e^2}\right)}, \tag{88}$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-k^2), \tag{89}$$

and

$$e^2 = \frac{1}{\kappa}, \quad \frac{k^2}{a^2} = \frac{\kappa}{1+\kappa}. \tag{90}$$

$$\text{(ii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-a^2\,.\,x^2-k^2), \tag{91}$$

$$x^2 \cos\{2\psi-(1-\kappa)F\phi\} = a\surd(1+\kappa)\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{92}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \&c. \tag{93}$$

$$\text{(iii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \surd(x^2-a^2\,.\,x^2-k^2),$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = a\sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{94}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \frac{1}{\kappa}. \tag{95}$$

On the hyperboloid of two sheets, with a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{96}$$

we shall find that the geodesic is given by

$$y^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(y^2-k^2\,.\,y^2+\frac{b^4}{a^2e^2}\right)}, \tag{97}$$

$$y^2 \cos\{2\psi-(1+\kappa)F\phi\} = b\,\frac{1-\kappa}{\surd\kappa}\,\surd(y^2+b^2), \tag{98}$$

$$k^2 = a^2, \quad e^2 = \frac{1}{\kappa}. \tag{99}$$

45. On the surface $$r^2 = x^2+y^2 = a^2\,\mathrm{ch}^2\frac{z}{a}, \tag{100}$$

the modified catenoid, the geodesics are given by

$$d\psi = \tfrac{1}{2}k\,\frac{r^2-(a^2-c^2)}{r^2\surd(r^2-a^2\,.\,r^2-k^2\,.\,r^2-a^2+c^2)}\,dr^2, \tag{101}$$

an equation of the same nature as that required for the geodesics on the hyperboloid of one sheet, so that these geodesics are analytically equivalent; this is obvious when we consider that the two surfaces are deformable into each other, without stretching or tearing.

But the catenaries and trajectories are intractable.

Either of these surfaces can also be developed into a skew helicoid, so that the geodesics on this surface are of the same analytical character, which, it may be remarked, is the same as that of Poinsot's herpolhode.

[46. Legendre points out that the integral required in the determination of $\Phi$, the angle of the sector in the development of the surface of the oblique cone on a circular base, is exactly the same as that which gives the angle $\psi$ in the geodesic on a quadric surface of revolution; this is evident from the substitutions

$$m^2(s_2-s) = \frac{f^2 \sin^2\omega}{h^2+(r-f\cos\omega)^2}, \tag{102}$$

or

$$m^2(s-s_2) = \frac{h^2+(r-f\cos\omega)^2}{f^2\sin^2\omega}, \tag{103}$$

in his integral for $\Phi$ on p. 331, t. I, *Fonctions Elliptiques.*

Then, writing $c$ for $\cos\omega$,

$$m^2(s_1-s) = \frac{\frac{1}{2}f^2}{aa'-h^2-r^2+f^2} \frac{\{(a'+a)-(a'-a)\,c\}^2}{h^2+(r-fc)^2}, \tag{104}$$

$$m^2(s-s_3) = \frac{\frac{1}{2}f^2}{aa'+h^2+r^2-f^2} \frac{\{(a'+a)\,c-(a'-a)\}^2}{h^2+(r-fc)^2}, \tag{105}$$

or

$$m^2(s-s_1) = \frac{\{(a'+a)\,c-(a'-a)\}^2}{4f^2(1-c^2)}, \tag{106}$$

$$m^2(s-s_3) = \frac{\{(a'+a)-(a'-a)\,c\}^2}{4f^2(1-c^2)}; \tag{107}$$

and with

$$m^2(\sigma-s) = \frac{h^2+r^2-2rfc+f^2}{h^2+(r-fc)^2}, \tag{108}$$

or

$$m^2(s-\sigma) = \frac{h^2+r^2-2rfc+f^2}{f^2(1-c^2)}, \tag{109}$$

in which $h^2+r^2-2rfc+c^2$ is the squared radius vector, and $h_2+(r-fc)^2$ is the squared perpendicular on the tangent of the development; this makes

$$\Phi = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s} \frac{ds}{\sqrt{S}}, \tag{110}$$

or

$$\Phi = \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_2 - \sigma} \int \frac{s - s_2}{s - \sigma} \frac{ds}{\sqrt{S}}, \tag{111}$$

exactly as for the determination of $\psi$ in the preceding geodesics.]

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*On a Regular Rectangular Configuration of Ten Lines.* *By* F. Morley. Received May 28th, 1898. Read June 9th, 1898.

1. *The Construction.*—I shall say that a straight line is normal to another when they intersect and are perpendicular. Three lines in space form three pairs, and each pair has a common normal. The three lines, with the three normals, form a rectangular hexagon. The three pairs of opposite sides of this hexagon give three more lines—their common normals. It will be shown that *these last three have one common normal.* Thus, if, starting with three lines, we keep on constructing all possible common normals (excluding the common normals of intersecting lines) we get only ten lines in all, forming a regular configuration in the sense that each line has three normals.

2. *The Points at Infinity.*—Taking five points $a, b, c, d, e$ in space and cutting the ten lines such as $ab$ and the ten planes such as $abc$ by an arbitrary plane, we get a well-known configuration (Fig. 1).

[*Cf.* Cayley's *Math. Papers*, Vol. I., p. 318.]

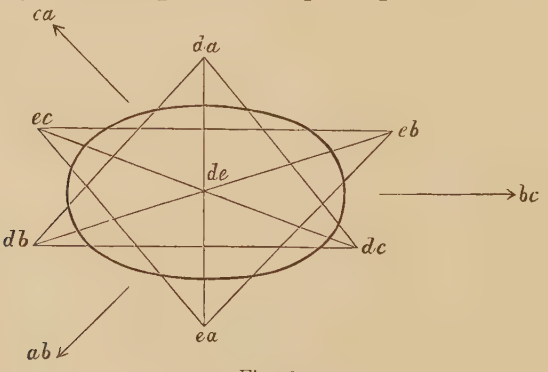


Fig. 1.

Each line *abc* in the plane is the polar of the point *de* with regard to a conic, as appears most easily from the consideration of the space-cubics through *a*, *b*, *c*, *d*, *e* (Reye, *Geometrie der Lage*, Vol. II., p. 273 of the second edition). Three ways of regarding the configuration are to be noticed:—

(i.) Removing a point *de* and its polar *abc*, we have two reciprocal triangles as to the conic, namely, *da*, *db*, *dc* and *ea*, *eb*, *ec*.

(ii.) The complete four-point *ea*, *eb*, *ec*, *ed* contains four of the points and six of the lines; the pairs of lines are conjugate as to the conic. The reciprocal of the four-point as to the conic is the other four lines and the other six points.

(iii.) To a pentagon, such as *ab*, *bc*, *cd*, *de*, *ea*, another *ac*, *ce*, *eb*, *bd*, *da* is both inscribed and circumscribed. The point *ac* is taken arbitrarily on the line *abc*, the second pentagon closing of its own accord.

We are here concerned with the case when the conic is the absolute circle at $\infty$; the five points *a*, *b*, *c*, *d*, *e* are then *orthocentric*, that is, the line through any two is perpendicular to the plane through the other three.

3. *The Proof.*—Returning to the three lines in space, we suppose them to be (Fig. 2)

$$y=\beta,\ z=-\gamma;\quad z=\gamma,\ x=-\alpha;\quad x=\alpha,\ y=-\beta;$$

and the points at $\infty$ on them to be

$$ea,\quad eb,\quad ec.$$

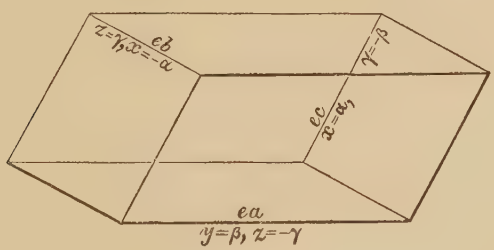


Fig. 2.

The coordinates are oblique; we suppose that the absolute cone is

$$a_1x^2+b_1y^2+c_1z^2+2fyz+2gzx+2hxy=0,$$

and its reciprocal, the absolute circle, is

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0.$$

The coordinates of the ten points at $\infty$ in Fig. 1 are known from the theory of a conic; thus

$$\begin{aligned} da &\text{ is } A, H, G; \\ bc &\text{ is } 0, g, -h; \\ dc &\text{ is } 1/F, 1/G, 1/H. \end{aligned}$$

The problem of writing the common normals is hereby much simplified. Naming each line in space by its point at $\infty$, we have for the equations of $da$, which is the common normal of $eb$ and $ec$,

$$\frac{z-\gamma}{G} = \frac{x+\alpha}{A}, \quad \frac{x-\alpha}{A} = \frac{y+\beta}{H};$$

for we have merely to pick out from the pencils determining $eb$ and $ec$ the planes through the point

$$x/A = y/H = z/G = \infty .$$

Hence, using the same process, the common normal $bc$ of $da$ and $ea$, which passes through the point $bc$, has the equations

$$\frac{A(z-\gamma) - G(x+\alpha)}{-h} = \frac{H(x-\alpha) - A(y+\beta)}{-g},$$

$$\frac{y-\beta}{g} = -\frac{z+\gamma}{h}.$$

Therefore the plane through $bc$ and the point $de$ at $\infty$ is

$$\frac{(hH+gG)x + \alpha(gG-hH) - gA(z-\gamma) - hA(y+\beta)}{(hH+gG)(GH-AF)} = \frac{h(y-\beta)+g(z+\gamma)}{(hH+gG)F},$$

or, since

$$GH - AF = \Delta . f,$$

$$Fx(gG+hH) - Gy . hH - Hz . gG + F\alpha(gG-hH) - G\beta . hH + H\gamma . gG + 2\Delta(hf\beta - fg\gamma) = 0.$$

It is clear that the three such equations, obtained by permuting cyclically $x, y, z$; $f, g, h$; $F, G, H$; and $\alpha, \beta, \gamma$, are consistent, since the left sides have the sum zero. Thus the three planes which should determine the third set of common normals belong to a pencil; which proves the point.

4. *Restatements.*—As in the plane configuration of § 2, the configuration of ten lines in space can be broken up in different ways. Thus we can state our theorem as follows:—

(i.) The common normals of opposite sides of a rectangular hexagon have a common normal.

(ii.) Four lines in space can be regarded as two pairs in three ways. If the common normals of the two pairs are themselves normal in two of the ways, they are so in the third. This statement of the case was suggested by Mr. Richmond.

(iii.) Two rectangular pentagons can be normal to each other; that is, each side of the one can be normal to a side of the other.

There are in the configuration ten rectangular hexagons, five systems of four lines of the kind just mentioned, and six pairs of mutually normal rectangular pentagons.

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*Point-Groups in a Plane, and their effect in determining Algebraic Curves.* *By* F. S. Macaulay, D.Sc. Read and received June 9th, 1898.

## I. Introduction.

The following is a continuation of my former paper on "Point-Groups in relation to Curves" in Vol. xxvi. of the *Proceedings*, p. 519. It deals especially with the reduction of point-groups which supply a known number of conditions for an algebraic curve of any order.

The effect of a group of $N$ points in determining an algebraic curve of order $n$ (called hereafter a $C_n$) need not depend on $N$ and $n$ alone. It may, and often does, happen that the $N$ points do not supply $N$ independent conditions for a $C_n$, but only a smaller number $N-r_n$. In any case, if the point-group $N$ is given, the number $r_n$ has a definite positive* (integral or zero) value. The extreme case is that in which all the $N$ points lie on a straight line; and we then have $r_n = N-(n+1)$ if $n \leqslant N-1$, and $r_n = 0$ if $n \geqslant N-1$.

For the case in which the $N$ points form the complete intersection of two curves, the values of $r_n$ for all values of $n$ have long been known. Thus, if $N$ consists of the complete intersection of a $C_l$ and $C_m$, and if $n$ is less than $l+m$, but not less than $l$ or $m$, then

$$r_n = \tfrac{1}{2}(l+m-n-1)(l+m-n-2),$$

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* In this paper the curves are subject to no other conditions than those of passing through points, *i.e.*, they are general algebraic curves through given point-groups in which the parameters or coefficients enter linearly. For linear systems of curves no value of $r$ could be negative; this would not be true for non-linear systems.

by Cayley's theorem; and the value of $r_n$ for *any* value of $n$ may be written in the form

$$\left[\tfrac{1}{2}(l+m-n-1)(l+m-n-2)\right]-\left[\tfrac{1}{2}(m-n-1)(m-n-2)\right]$$
$$-\left[\tfrac{1}{2}(l-n-1)(l-n-2)\right],$$

each pair of square brackets indicating that the product enclosed is only to be retained so long as its individual factors are positive.

But the case in which $N$ is the complete intersection of two curves is only a very special one. It is easy to give other examples. In general, if two point-groups $N$, $N'$ together make up the complete intersection of two curves $C_l$, $C_m$, having no common factor, and if for $N$ there are values of $r_n$ which do not vanish, then for $N'$ there are corresponding values of $r'_{n'}$ which do not vanish, where $n+n'=l+m-3$.

We name the number $r_n$ the *n-ic excess* of the point-group $N$. It is the number of points which lie on each and every $C_n$ drawn through $N-r_n$ of the $N$ points, *provided these* $N-r_n$ *points are so chosen that their n-ic excess is zero.* Such a choice is evidently always possible,* although it may also be possible to choose $N-r_n$ of the $N$ points whose $n$-ic excess is not zero.

Corresponding to $r_n$ there is a complementary number $q_n$, viz., the degree of freedom of a $C_n$ through $N$, which we name the *n-ic defect* of the point-group $N$.† This is the number of general points in the

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* Such a selection may be made with certainty by choosing the $N-r_n$ points one at a time, each new one being so chosen that a $C_n$ through all those previously chosen does not necessarily pass through it. In this way we must arrive at $N-r_n$ points and no more, since the $n$-ic excess of $N$ is $r_n$, neither more nor less.

† [*Note added October 13th.*—The terminology of the theory of point-groups is extensive. This is owing partly to the number of descriptive terms required to distinguish various kinds of point-groups and their special characteristics, and partly to the fact that there are several essentially different ways of approaching the subject. One branch of the subject has been confined almost entirely to Germany, another to Italy, but the branch to which the present paper belongs—that in which the point-groups themselves are subjected to direct operation—has been developed both by English and Continental mathematicians.

The term *regular* (*regolare*) has been applied by Professor G. Castelnuovo to a system of curves in reference to a given point-group when the point-group possesses no *excess* for the general curve of the system. By a slight inversion we may say that a point-group is regular with respect to a general curve of order $n$ when $r_n = 0$. Point-groups which are regular with respect to all general algebraic curves which can be drawn through them I call *general* point-groups, thereby implying only that they are general in their effect in determining algebraic curves.

A convenient English term for point-groups which are not general in the above sense is desirable. The term *constructional* (instead of *point-group of special form*, used in my former paper) seems not inappropriate, since such point-groups can be constructed as the partial (or total) intersection of algebraic curves. The old use of the term *special* was applied to a point-group on a base-curve $C_m$ for which $R-r<p-1$ ($p$ being the deficiency of $C_m$, $R$ the number of points, and $r$ the multiplicity, of the point-group). The modern use of *special* is, however, applied to any

plane through which a $C_n$ can be drawn which already passes through the $N$ points. Hence, since the $N$ points supply exactly $N-r_n$ conditions for a $C_n$, and the $q_n$ points, being general, $q_n$ more conditions, we have

$$N-r_n+q_n=\tfrac{1}{2}n\,(n+3),$$

or

$$N-r_n+q_n+1=\tfrac{1}{2}\,(n+1)\,(n+2)\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Hence also

$$r_{n-1}-r_n=n+1-(q_n-q_{n-1}).\ \ldots\ldots\ldots\ldots\ldots(2)$$

The effect of a point-group $N$ with respect to algebraic curves of all orders is not known unless all the excesses, or all the defects, are known. As we shall find it convenient to regard the subject chiefly from the point of view of the defects, we may suppose at once that the defects $q$ are all given. If $q_n=0$, one, and only one, $C_n$ can be drawn through $N$. If no $C_n$ can be drawn through $N$, we assign the value $-1$ to $q_n$, the reason for which will be given later (p. 677). It follows that $q_n$ and $r_n$ have definite values for *all* values of $n$, which always satisfy formula (1) above. It is also shown later that, if all the defects are given, the value of the number $N$ can be deduced.

We may say then that a point-group is fully characterized if all the defects, or all the excesses, are given; and that the point-group has a partially or completely assigned *characterization* according as some or all of the defects, or excesses, are assigned. Two of the most general questions that suggest themselves are:—(i.) What is, and what is not, a possible characterization for a point-group? (ii.) What is the quickest method of arriving at or constructing a point-group with an assigned characterization? These and other questions are answered in the following paper.

The above remarks refer to groups of points which coincide at most in pairs. In the first sections of the paper I have dealt only with ordinary point-groups of this type; but in the footnotes on "Multiple Points,"*

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point-group for which $R-r\leqslant p-1$, that is, to any point-group on $C_m$ which lies on an adjoined $C_{m-3}$. (*Cf.* Professor Charlotte Angas Scott, "Intersections of Plane Curves," *Bulletin of the American Mathematical Society*, 2nd series, Vol. IV., March, 1898, p. 267). Miss Scott suggests the convenient term *intraspecial* in place of the old term *special*. The connexion between *intraspecial* and *constructional* is expressed as follows:—"An intraspecial point-group on a $C_m$ to which is added an $(i-1)$-point at each $i$-fold point of $C_m$ is a constructional point-base through which a $C_{m-3}$ can be drawn."

For the meaning of the terms *i-point* and *point-base*, see note on "Multiple Points. I.," below.]

* [*Multiple Points. I.*—It is well known that the number of conditions supplied to a curve by an $i$-fold point is $\frac{1}{2}i\,(i+1)$, and that these conditions, when combined with others, may quite possibly not be independent. It is very convenient in the geometrical theory of point-groups to replace, if possible, these

and in the last section of the paper, I have shown how the whole question is capable of generalization.

## II. Summary and Deductions.

We use the letter $\rho$ to denote the difference of two successive defects. Thus $\rho_n = q_n - q_{n-1}$. If $\rho_l$ is the first $\rho$ which does not vanish, we have $q_l + 1 = \rho_l$ and $q_p + 1 = \rho_l + \rho_{l+1} + \ldots + \rho_p$ $(p \geq l)$. The differences of the $\rho$'s, or second differences of the defects, are denoted by $\delta$, so that $\delta_l = \rho_l$, $\delta_p = \rho_p - \rho_{p-1}$.

(i.) Writing the general equation of a $C_n$, referred to two coordinate axes $Ox$, $Oy$, in the form

$$C_n \equiv u_0 + u_1 + \ldots + u_n = 0,$$

where $u_p$ is a homogeneous function of $x$, $y$ of order $p$, with $p+1$ coefficients, and substituting the coordinates of each of the $N$ points in $C_n = 0$, we have a set of $N$ linear equations for the coefficients of $C_n$. These equations have always a solution, whether a $C_n$ can be

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$\frac{1}{2}i(i+1)$ conditions by those of passage through $\frac{1}{2}i(i+1)$ points. (*Cf.* pp. 508, 509 of the *Proceedings*, Vol. xxvi.). The theorem which renders such an interchange of conditions practicable is as follows:—"Given a curve $C_n$, with any number and kind of multiple points, it is always possible to find a curve $C'_{n'}$ ($n'$ being either equal to or greater than $n$) whose coefficients differ only to an infinitely small extent from those of $C_n$, and such that corresponding to each and every $i$-fold point $A$ of $C_n$ the curve $C'_{n'}$ passes through $\frac{1}{2}i(i+1)$ points chosen arbitrarily and generally about and infinitely near to $A$." All that is necessary then to effect the change required is to place at each $i$-fold point $(i>1)$ a general set of $\frac{1}{2}i(i+1)$ points on an infinitely small scale, and to consider in the place of any curve $C_n$, with the given multiple points, a proximate curve $C'_{n'}$, which passes through all the sets of $\frac{1}{2}i(i+1)$ points, and of which $C_n$ is the limit.

Such a set or *cluster* of $\frac{1}{2}i(i+1)$ points may itself be called a point of order $i$; we shall therefore call it an *$i$-point*, reserving the term *$i$-fold point* for a multiple point of order $i$ on a curve. Thus $i$-point and $i$-fold point are practical equivalents; but the one refers to an element of a point-group, and the other to an element of a curve. It should be noticed that an ordinary point is a point of order 1, that is, a 1-point. An $i$-point is equivalent to $\frac{1}{2}i(i+1)$ simple points. The degree $N$ of a point-group, that is, the total number of simple points to which it is equivalent, is given by $N = \frac{1}{2}\Sigma i(i+1)$, the summation extending to all the points of the group, including those for which $i = 1$.

We also give the name *point-base* (meaning "base of points" = *gruppo base*, Castelnuovo) to a point-group made up of points of assigned orders. In Section vi. we generalize the meaning of this term, and distinguish the point-base here defined by the added epithet *simple*. We shall then, hereafter, only use the term *point-group* in the restricted sense of a group of points which are all of order 1. This is the sense in which it is used throughout the text. The term *point-base* must be distinguished from *base-point*; the latter, however, we shall have no need to use.

The chief importance of this method of dealing with multiple points is that any two curves drawn through an $i$-point must each have there an $i$-fold point, and intersect there again in an $(i-1)$-point. Consequently, in the reduction of a point-base, each step leads to a reduction throughout in the orders of the points, while the new points introduced are all of order 1.—*October 13th.*]

drawn through $N$ or not. If the $N$ points lie on a $C_n$, the general solution of the $N$ equations simply determines a certain number of the coefficients of $C_n$ in terms of the rest, which are left arbitrary. The number so determined is $N-r_n$; and the number left arbitrary is $q_n+1$, since a $C_n$ through $N$ has still a degree of freedom $q_n$. If $N$ does not lie on a $C_n$ the solution requires all the coefficients to vanish, and the number determined is $N-r_n=\frac{1}{2}(n+1)(n+2)$, and the number left arbitrary is $q_n+1=0$. *Hence we say that the n-ic defect of a point-group N which does not lie on a $C_n$ is always* $-1$.

Returning to the case in which a $C_n$ can be drawn through $N$, the number of arbitrary coefficients in $u_0+u_1+\ldots+u_p$ $(p\leqslant n)$ is $q_p+1$, by the same reasoning as before; and the number in $u_0+u_1+\ldots+u_{p-1}$ is $q_{p-1}+1$. *Hence the number of arbitrary coefficients in $u_p$ is* $q_p-q_{p-1}$, *i.e.*, $\rho_p$. The number $\rho_p$ has therefore a precise analytical interpretation; and, consequently, the properties mentioned below can easily be interpreted analytically. It follows that one limitation to the value of $\rho_p$ is given by $\rho_p\leqslant p+1$, since $u_p$ contains only $p+1$ coefficients in all; but it will be seen later that this limitation disappears when we regard the $\rho$'s from a slightly different point of view.

(ii.) *In order that a point-group N may be a possible one it is necessary and sufficient that the values of the $\rho$'s, after ceasing to be zeros, should consist of continually increasing positive integers, subject to the limitations* $\rho_p\leqslant p+1$ *and* $N+q_p+1\geqslant\frac{1}{2}(p+1)(p+2)$.

Both limitations disappear when we regard the number $N$ and the orders of the curves as being given by the values of the $\rho$'s. A still simpler way of enunciating the theorem is:—

*A point-group is possible if the second differences of the defects, after once ceasing to be zeros, are positive integers, not including zero; otherwise a point-group is impossible.*

It follows from this theorem that if only a $C_l$ can be drawn through $N$ for which the excess of $N$ is not zero, then must $q_l+r_l\leqslant l$. For, in such a case, we have

$$q_{l-1}=-1,\quad r_{l+1}=0;$$

therefore $$\rho_l=q_l-q_{l-1}=q_l+1;$$

and $$r_l-r_{l+1}=l+2-(q_{l+1}-q_l),\quad \text{by (2), p. 675;}$$

therefore $$\rho_{l+1}=l+2-r_l,$$

and $$\rho_{l+1}-\rho_l-1=l-q_l-r_l\geqslant 0.$$

*If* $q_l+r_l>l$, *and* $q_l \geq 0$, $N$ *must have excess for more than a* $C_l$ *through* $N$.*

(iii.) *If* $\delta_p \equiv \rho_p - \rho_{p-1} = 1$, *and* $p > \rho_{p-1} > 0$, *then any* $C_p$ *through* $N$ *must contain a fixed constituent curve of order* $p - \rho_{p-1}$.

Not only the order $p-\rho_{p-1}$ of the fixed curve, but the number of points $N'$ on it, is known, and also the full characterization of $N'$ and that of the remainder $N''$ of the $N$ points.

Thus a point-group $N$, fully characterized, can be separated into as many constituent point-groups as there are sets of one or more successive $\delta$'s equal to 1, each set being preceded and succeeded by one or more $\delta$'s greater than 1, together with a remainder.

The points of each constituent (including the remainder when it is not a general point-group) must have a certain number of interconnexions among themselves; but, so far as the characterization of $N$ affects the result, there will not be any connexions between any two of the constituents. Each constituent has its own characterization, and, when its construction has been found, can be placed in any position, without reference to the positions of the other constituents. The sum thus obtained forms the most general point-group $N$ with the assigned characterization. This property of the independence of the constituents, which is not easily apparent by intuition, is here emphasized, since it evidently results in a considerable simplification.†

(iv.) Two theorems suffice for the quickest reduction of a point-group. The first is the theorem mentioned in (iii.), which, in terms of the second differences of the defects, may be expressed as follows:—

($\alpha$) *If* $\delta_a = 1$, *then*

$$(\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b) = (\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

The other theorem is

($\beta$) $(1^a, \delta_b+1, \delta_{b+1}+1, \ldots, \delta_c+1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta, a+\Sigma\delta)$,

*where* $I(l, m)$ *denotes the complete intersection of a* $C_l$ *and* $C_m$.

The notation is explained in the next section; but the following remarks will perhaps be intelligible. The characterization of $N$ is fully represented by the numbers

$$\ldots, 0, 0, \delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_{b-1}, \delta_b, 1, 1, 1, \ldots, \textit{ad inf.}$$

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* In the same way it can be proved that if $q_l \geq 0$, $r_{n-2} > 0$, $l < n-2$, then $q_{l-1}+1$ cannot be zero unless $q_{l+1}+1 > 2(q_l+1)$, and $r_{n-1}$ cannot be zero unless $r_{n-3} \geq 2r_{n-2}$.

† This property does not appear to hold in general for a point-base.

Any $\delta$, or any set of successive equal $\delta$'s, preceded and succeeded by $\delta$'s of higher value is a minimum; and any minimum which reaches its lowest possible value 1 may be called a breaking-point. Theorem ($\alpha$) exhibits the result after breaking. If there is no breaking-point, theorem ($\beta$) shows that from $N$ we can derive a point-group $N'$ such that the $\delta$'s of $N'$ are simply the $\delta$'s of $N$ each diminished by 1, in reversed order. It is thus evident that all the minima reach breaking-point, some time or other, before the reduction of $N$ has been completed. If the $\delta$'s never decrease until the greatest value is reached, and after that never increase, there will be no minimum, and no breaking-point during the whole reduction.

The second differences of the excesses of $N$ are simply $\delta_1-1$, $\delta_2-1$, ..., $\delta_a-1$, ..., $\delta_b-1$, and exhibit the same properties as the second differences of the defects; but the breaking-point is at the value 0 instead of 1.

(v.) *If a $C_l$ is the lowest curve through a point-group $N$, and a $C_{n-2}$ the highest curve for which the excess of $N$ does not vanish, then the number of the independent interconnexions of the $N$ points, due to the characterization, is*

$$\rho_l(\rho_{l+2}-\rho_{l+1}-1)+\rho_{l+1}(\rho_{l+3}-\rho_{l+2}-1)+\ldots+\rho_{n-2}(\rho_n-\rho_{n-1}-1).*$$

If $N$ breaks up into constituents, *i.e.*, if, for one or more values of $p$ between $l$ and $n$, $\rho_p-\rho_{p-1}-1=0$, then there are zero terms in the above series, which divide the whole into shorter series. These give

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* This result can be compared with the formula $(q+1)r$ given by Brill and Nöther as the number of conditional equations for the existence of a point-group $G^r_R$ on a $C_n$. (*Cf.* Benoist's translation of Clebsch, *Leçons sur la Géométrie*, Vol. III., pp. 53 ff.) In the notation adopted above this formula would be written $(q_{n-3}+1)r_{n-3}$. The validity of the formula $(q+1)r$ really rests, however, on the hypothesis that the point-group $G^r_R$ has excess only for a $C_{n-3}$ adjoined to $C_n$, which requires, as we have seen in (ii.) above, $q+r\leqslant n-3$. If we suppose the excess of the point-group $N$ to be similarly restricted, the number of its interconnexions reduces to

$$\begin{aligned}\rho_{n-3}(\rho_{n-1}-\rho_{n-2}-1) &= (q_{n-3}+1)(n-1-\overline{q_{n-2}-q_{n-3}})\\ &= (q_{n-3}+1)(r_{n-3}-r_{n-2}) = (q_{n-3}+1)r_{n-3},\end{aligned}$$

so that, on this supposition, the two results agree. If $q+r>n-3$, the formula $(q+1)r$ is not in general valid.

It is easy to show that on a non-hyperelliptic curve in which $n>p>1$ the condition $q+r\leqslant n-3$ is necessarily satisfied. For, by the Riemann-Roch equations, $q+r=p-1-(R-2r)$, and, by Bertini's addition to Clifford's theorem, $R-2r>0$. Hence $q+r\leqslant p-2$; and $n\geqslant p+1$; therefore $n-q-r\geqslant 3$, *i.e.*, $q+r\geqslant n-3$. (*Cf.* Miss F. Hardcastle, p. 133 of this volume of the *Proceedings*, and references there given.)

the numbers of the interconnexions of the points of the several constituents, the first series corresponding to what has been called the remainder. As already remarked, the different constituents are unconnected with each other.

There is no reason why a certain number of the points should not coincide in pairs. A point-pair determines a direction, and is expressible in terms of three parameters, viz., the direction and two coordinates. If there are $D$ point-pairs, the $2N$ coordinates of the $N$ points thus reduce to $2N-D$ parameters, and the formula of the theorem then gives the number of independent conditional equations satisfied by these $2N-D$ parameters. A $C_n$ through $N$ will in general have fixed tangents at the point-pairs; but one additional condition applied to $C_n$ at any point-pair will cause it to have there a double point, leaving the directions of its tangents free. It is not so clear as to what interpretation should be put upon the formula if some of the $N$ points coincide in threes or more.

## III. Preliminary Theorems and Notation.

(i.) *The n-ic excess $r_n$ of any point-group $N$ diminishes as $n$ increases, until it becomes zero.*

For a $C_n$ can be drawn through any $N-r_n-1$ of the $N$ points without passing through (all) the remainder, and a straight line $C_1$ can be drawn through any one of the $N$ points without passing through any more. Hence a $C_{n+1}$, viz., $C_n C_1$, can be drawn through any $N-r_n$ of the $N$ points without passing through (all) the remainder; not so a $C_n$ (Note, p. 674). Hence $r_{n+1} < r_n$.

(ii.) (a) *If a $C_p$ through $N$ is necessarily degenerate, then one constituent of $C_p$ must be fixed.*

Choose any $q_p$ general points in the plane. Then there is one and only one $C_p$ through the $N+q_p$ points. Suppose that this $C_p$ breaks up into $C_{p'} C_{p''}$, and let $N'$ of the $N$ points and $q'$ of the $q_p$ points lie on $C_{p'}$, and the remaining $N''$ of the $N$ points and $q''$ of the $q_p$ points lie on $C_{p''}$. Then, since the $q'+q''$ points are general, $q'$ must be the $p'$-ic defect of $N'$, and $q''$ the $p''$-ic defect of $N''$. Hence, if neither $q'$ nor $q''$ is zero, a $C_{p'}$ can be drawn through $N'$ and any $q'$ of the $q'+q''$ points, and a $C_{p''}$ through $N''$ and the remaining $q''$ of the $q'+q''$ points; and this $C_{p'}$ and $C_{p''}$ would make up a second $C_p$

through the $N+q_p$ points, which is impossible.* Hence $q'$ or $q''$ is zero; let $q'$ be zero; then $q_p$ is the $p''$-ic defect of $N''$. Hence, in whatever new position we choose $q_p$ general points, the $C_p$ through them and the $N$ points will consist of the $C_{p''}$ through the $N''+q_p$ points, and a fixed $C_{p'}$ through the $N'$ points (whose $p'$-ic defect is zero).

Also any curve lower than a $C_p$ through $N$ must have the same fixed constituent $C_{p'}$.

($\beta$) *If a $C_{n-2}$ is the highest curve for which the excess of $N$ does not vanish, then a proper $C_n$ can be drawn through $N$.*

For, if $C_n$ is necessarily degenerate, it must have a fixed constituent $C_{n'}$. Also a $C_{n-1}$ can be drawn through all the $N$ points except one (chosen on $C_{n'}$) without passing through the last, since $r_{n-1}=0$. This $C_{n-1}$ cannot have $C_{n'}$ for a constituent, for, if it had, it would pass through the last point. Also a $C_1$ can be drawn through the last point in any arbitrary direction. Then $C_{n-1}\,C_1$ is a $C_n$ through $N$, not having $C_{n'}$ for a constituent. Hence a proper $C_n$ can be drawn through $N$.

It may be that a $C_{n-1}$ through $N$ is necessarily degenerate, or that there is no $C_{n-1}$ through $N$; but a $C_{n-1}$ can be drawn through any $N-1$ of the $N$ points.

(iii.) We express the orders of curves in terms of $l$, $m$, $n$, $p$, the first three being generally fixed, and the last, $p$, having any value. A $C_l$ is the lowest curve through $N$, and a $C_{n-2}$ the highest curve for which $N$ has excess; and we suppose, in general, that $N$ lies on a $C_{n-2}$. A $C_m$ is the lowest curve through $N$ which has not any fixed constituent, and is not fixed as a whole. We always have $l \leqslant m \leqslant n$.

If the suffixes of $q$, $r$, $\rho$, $\delta$ are expressed in terms of $l$, $m$, $n$, $p$, they are to be understood as having explicit reference to the orders of curves; thus $q_p$ is the $p$-ic defect of $N$, and $\rho_p = q_p - q_{p-1}$, $\delta_p = \rho_p - \rho_{p-1}$.

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* [*Multiple Points. II.*—This reasoning fails in a special case when we are dealing with a point-base $N$. This happens when the constituents of the second $C_p$ are a simple rearrangement of those of the first, so that the $C_p$ itself is not changed. One constituent of $C_p$, say $C_{p'}$, may be assumed to be a proper curve. If then $q' \geqslant 2$, we could interchange 1 of the $q'$ points with 1 of the $q''$ points, thus obtaining a second set of $q'$ points, and a second $C_{p'}$, which is certainly not a constituent of the original $C_p$; in this case therefore the theorem will hold, as also when $q'=0$. But if $q'=1$, it might happen that any second $C_{p'}$, was necessarily a constituent of the original $C_p$. In this case every proper constituent of the general $C_p$ through $N$ must have defect 0 or 1, and the $q_p$ constituents with defect 1 must belong to a fixed pencil of curves. The conclusion is that, *when a $C_p$ through a point-base $N$ is necessarily degenerate, either* (i.) *one constituent of $C_p$ is fixed, and may break up into several parts, while the remaining constituent is a proper curve with defect $q_p$; or* (ii.) *$C_p$ breaks up into $q_p$ constituents belonging to a fixed pencil of curves, and the remaining constituents of $C_p$, if any, are fixed absolutely.—October 13th.*]

But if the suffixes are 1, 2, 3, ..., $a$, $a+1$, ..., $b$, ..., they refer only implicitly to the orders of curves; $q_a$ is not the $a$-ic defect of $N$, but we still have $\rho_a = q_a - q_{a-1}$, $\delta_a = \rho_a - \rho_{a-1}$.

The symbol $(\rho_1, \rho_2, \ldots, \rho_a)$ represents a point-group whose successive $\rho$'s are $\rho_1, \rho_2, \ldots, \rho_a$, every $\rho$ before $\rho_1$ being zero, and $\rho_{b+1}$ being equal to $\rho_b+1$ when $b \geq a$. We suppose also that $\rho_1 > 0$, and $\rho_a > \rho_{a-1}+1$; for, if not, we can simply leave out the $\rho$'s at the beginning and end until these inequalities are established.

The following properties of the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ will be easily seen by supposing it to be the same as $(\rho_l, \rho_{l+1}, \ldots, \rho_n)$, so that $\rho_1 = \rho_l$, $\rho_2 = \rho_{l+1}$, ..., $\rho_a = \rho_n = n+1$ (since $r_n = r_{n-1} = 0$), and $a = n-l+1$.

($\alpha$) $\rho_a$ is the difference of the defects of $N$ for a $(\rho_a-1)$-ic and $(\rho_a-2)$-ic.

($\beta$) The highest curve for which $N$ has excess is a $(\rho_a-3)$-ic.

($\gamma$) The lowest curve through $N$ is a $(\rho_a-a)$-ic, *remembered as the last $\rho$ diminished by the number of the $\rho$'s.*

($\delta$) The number of the $\rho$'s exceeds by 2 the number of the different orders of curves through $N$ for which $N$ has excess.

($\epsilon$) The number $N$ is given by

$$N + q_{n-1} + 1 = \tfrac{1}{2}n(n+1),$$

*i.e.*,
$$N + \rho_l + \ldots + \rho_{n-1} = \tfrac{1}{2}\rho_n(\rho_n - 1),$$

*i.e.*,
$$N = \tfrac{1}{2}\rho_a(\rho_a-1) - (\rho_1 + \rho_2 + \ldots + \rho_{a-1}).$$

($\zeta$) Thus the values of the $\rho$'s in the point-group $N = (\rho_1, \rho_2, \ldots, \rho_a)$ express the *whole* effect of $N$ in the determination of algebraic curves of all orders. The only restrictions on the values of the $\rho$'s are $0 < \rho_1 < \rho_2 \ldots < \rho_a$, and $\rho_{b+1} = \rho_b + 1$ if $b \geq a$.

The point-groups $(\rho_1, \rho_2)$ and $(\rho_1)$, as indicated in ($\delta$), are general, having zero excess for curves of any order which can be drawn through them. This is easily verified. The number $N$ in $(\rho_1, \rho_2)$ is $\frac{1}{2}\rho_2(\rho_2-1) - \rho_1$, and in $(\rho_1) = (\rho_1, \rho_1+1)$ is $\frac{1}{2}\rho_1(\rho_1-1)$.

It may also be verified that the point-group $(1, 2, \ldots, a, \rho_b, \rho_{b+1})$ consists of $\frac{1}{2}\rho_{b+1}(\rho_{b+1}-1) - (1+2+\ldots+a+\rho_b)$ general points on a $(\rho_{b+1}-a-2)$-ic [*cf.* ($\gamma$) and ($\epsilon$) above]. Similarly $(1, 2, \ldots, a, \rho_b)$, or $(1, 2, \ldots, a, \rho_b, \rho_b+1)$, consists of $\frac{1}{2}\rho_b(\rho_b-1) - \frac{1}{2}a(a+1)$ general points on a $(\rho_b-a-1)$-ic. The point-group $(1, 2, \ldots, a)$ is a zero one.

Expressed in terms of the second differences of the defects $(\rho_1, \rho_2, \ldots, \rho_a, \ldots, \rho_b)$ becomes $(\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b)$; and it is easy to change from one expression to the other ($\delta_1 = \rho_1$, $\delta_a = \rho_a - \rho_{a-1}$). All

the $\delta$'s before $\delta_1$ are zeros, and all the $\delta$'s after $\delta_b$ are units,* while $\delta_1 > 0$, $\delta_b > 1$. The only restriction on the values of the $\delta$'s is that they are all positive integers, not including zero.

The lowest curve through $(\delta_1, \delta_2, \ldots, \delta_b)$ is of order $l = \rho_b - b$, or

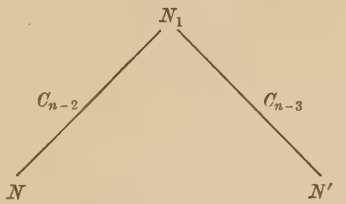
$$l = (\delta_1 - 1) + (\delta_2 - 1) + \ldots + (\delta_b - 1) = \Sigma(\delta - 1).$$

The point-group $(1, 2, \ldots, a, \rho_b, \rho_{b+1}, \ldots, \rho_c)$, when expressed in $\delta$'s, is written in the form $(1^a, \delta_b, \delta_{b+1}, \ldots, \delta_c)$; $\delta_b = \rho_b - a$.

## IV. Reduction of Point-Groups.

Theorem I.—*A point-group* $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1)$ *can always be found coresidual to a given point-group* $N = (\rho_1, \rho_2, \ldots, \rho_a)$ *on a* $(\rho_a - 1)$*-ic.*

For $(\rho_1, \rho_2, \ldots, \rho_a)$ substitute $(\rho_l, \rho_{l+1}, \ldots, \rho_n)$; then $\rho_n = n + 1$. The highest curve for which $N$ has excess is a $C_{n-2}$; therefore $N$ lies on a proper $C_n$ [($\beta$), p. 681]. Take this $C_n$ as base-curve. We suppose that $a \geq 3$, otherwise the point-group $N$ is a general one (p. 682); hence $n - 2 \geq l$. Let a $C_{n-2}$ be drawn through $N$. This will cut $C_n$ again in a finite point-group $N_1$, since $C_n$ is a proper curve. Also, since the excess of $N$ for a $C_{n-2}$ is not zero, $N_1$ must, by a known theorem, lie on a $C_{n-3}$. Let a $C_{n-3}$ through $N_1$ cut $C_n$ again in a finite point-group $N'$.†



*Base-curve* $C_n$.

Then $N$, $N'$ are coresidual on $C_n$, and have identically the same sets of residuals. Also we know that the multiplicity of a series of point-groups on $C_n$ cut out by curves of order $p$ through $N$ is $q_p$ when $p < n$, and $q_p - \frac{1}{2}(p-n+1)(p-n+2)$ when $p \geq n$. The same series is cut out by curves of order $p-1$ through $N'$, and the multiplicity is $q'_{p-1}$ or $q'_{p-1} - \frac{1}{2}(p-n)(p-n+1)$ according as $p < n$ or $p \geq n$. Equating these values,

$$q'_{p-1} = q_p \text{ when } p < n, \quad \text{and} \quad q'_{p-1} = q_p - (p-n+1) \text{ when } p \geq n.$$

* The full expression for $(\delta_1\, \delta_2, \ldots, \delta_b)$ is $(0, \delta_1, \delta_2, \ldots, \delta_b, 1^\infty)$, where $1^\infty$ stands for 1 repeated *ad inf.*

† *Cf. Proceedings*, Vol. xxvi., p. 525. The $N$ points might necessarily comprise all the $N'$ points. This would be the case if, and only if, $N$ consisted of $n$ points on a straight line, and $N-n$ remaining points having no excess for a $C_{n-3}$. The $N'$ points would then be identical with the $N-n$ points, but the reasoning would not be affected.

The first of these equations holds not only when $p \geqslant l$, but also when $p < l$, for then $q'_{p-1} = q_p = -1$. Both equations hold when $p = n-1$.

Taking differences, we have

$$\rho'_{p-1} = \rho_p \text{ when } p < n, \quad \text{and} \quad \rho'_{p-1} = \rho_p - 1 = p \text{ when } p \geqslant n.$$

The successive values of $\rho'$ are therefore

$$\ldots,\ 0,\ \rho_l,\ \rho_{l+1},\ \ldots,\ \rho_{n-1},\ n,\ n+1,\ n+2,\ \ldots,$$

or

$$\ldots,\ 0,\ \rho_1,\ \rho_2,\ \ldots,\ \rho_{a-1},\ \rho_a - 1,\ \rho_a,\ \rho_a + 1,\ \rho_a + 2,\ \ldots;$$

hence

$$N' = (\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 1),$$

which proves the theorem.

If $\rho_a - 1 = \rho_{a-1} + 1$, then $N' = (\rho_1, \rho_2, \ldots, \rho_{a-1})$; and, if $\rho_a - 1 > \rho_{a-1} + 1$, we can repeat the process on $N'$, thus obtaining the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a - 2)$. After $\rho_a - \rho_{a-1} - 1$ steps in the reduction we arrive at the point-group $(\rho_1, \rho_2, \ldots, \rho_{a-1})$. In this we leave out all $\rho$'s at the end, if there are any, which exceed the preceding by 1, and continue the reduction. We ultimately arrive at the general point-group $(\rho_1, \rho_2)$, which is a zero point-group if $\rho_1 = 1$ and $\rho_2 = 2$.

*This proves that the point-group* $N = (\rho_1, \rho_2, \ldots, \rho_a)$ *is a possible one, if* $0 < \rho_1 < \rho_2 < \ldots < \rho_a$.*

It also follows that the point-group $N$ is impossible if $\rho_{b+1} \leqslant \rho_b > 0$. For, if the point-group is possible, the process of reduction is valid. But if $\rho_{b+1} \leqslant \rho_b > 0$, then at some stage we arrive at a point-group for which $\rho_b$ is the difference of the defects for curves of order $\rho_b - 2$ and $\rho_b - 1$, and $\rho_{b+1}$ the difference of the defects for curves of order

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* [*Multiple Points. III.*—In the reduction of a point-group which is *general of its kind* (that is, of the kind which has the assigned characterization), the distinguishing feature is that all the derived point-groups are general of *their* kind. It is this that leads to a determinate (not unique) construction for such point-groups. For point-bases the process of reduction is still valid, and so also is the reverse process of construction; but the derived point-bases may not be general of their kind (the kind which possesses a known characterization and consists of points of known order). In such a case the construction is not determined by the reduction alone. Thus if $N$, in the figure, is a point-base, then for any $i$-point belonging to $N$ we have an $(i-1)$-point belonging to $N_1$. The orders of the points of $N_1$ are all known, and also the characterization (Theorem III.). But from $N_1$ we have to construct $N$ by passing a $C_{n-2}$ and $C_n$ through it, having $i$-fold points at the $(i-1)$-points of $N_1$. Such curves could certainly be drawn through the general point-base of the kind of $N_1$ if the $(n-2)$-ic defect of $N_1$ is equal to or greater than $\Sigma i$, the sum of the orders of all the points of $N$ for which $i > 1$. There is also the condition, in order that the construction may be valid, that the $C_n$ should be a proper curve, or at least should not have any fixed constituent. We return to this again in Note V., p. 688. The question of the generality or speciality of these derived point-bases is not correctly stated in Vol. XXVI., p. 541, § 30.—*October 13th.*]

$\rho_b - 1$ and $\rho_b$. The corresponding differences of the excesses are then 0 and $\rho_b + 1 - \rho_{b+1} > 0$, by (2), p. 675. Hence the excesses of the derived point-group for curves of order $\rho_b - 2$ and $\rho_b - 1$ are equal, but not zero, which is impossible, by (i.), p. 680.

## General Theorems in Reduction.

Theorem II.—*If* $\rho_a - \rho_{a-1} = 1$, *then the point-group*

$$N = (\rho_1, \ldots, \rho_a, \ldots, \rho_b)$$

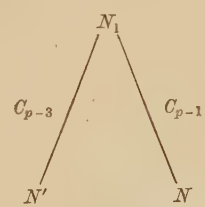
*necessarily breaks up into the two independent point-groups*

$$(\rho_1, \rho_2, \ldots, \rho_{a-1}) \quad \textit{and} \quad (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).*$$

If $\rho_1 = 1$, $\rho_2 = 2$, ..., $\rho_{a-1} = a - 1$, then $(\rho_1, \rho_2, \ldots, \rho_{a-1})$ is a zero point-group, and the theorem is nugatory. We shall therefore assume the contrary.

Let $(\rho_1, \ldots, \rho_a, \ldots, \rho_b)$ be the same as $(\rho_l, \ldots, \rho_p, \ldots, \rho_n)$, where $\rho_1 = \rho_l$, $\rho_a = \rho_p$, $\rho_p - \rho_{p-1} = 1$, $\rho_b = \rho_n = n + 1$. We have $p \geq l + 1$, since $a \geq 2$.

Then a $C_p$ through $N$ must have a fixed factor; for, if not, a proper $C_p$ can be drawn through $N$ [(a), p. 680], which is different from $C_l$, since $p > l$. Taking this $C_p$ as base-curve, we can find on it a point-group $N' = N - 2p$ coresidual to $N$, as shown in the figure;† and, by the same reasoning as in the last theorem, we have



*Base-curve* $C_p$.

$$q'_{p-4} = q_{p-2}, \qquad q'_{p-3} = q_{p-1}, \qquad q'_{p-2} = q_p - 1;$$

therefore

$$\rho'_{p-3} = \rho_{p-1}, \qquad \rho'_{p-2} = \rho_p - 1;$$

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* [*Multiple Points. IV.*—It is almost certain that this theorem, except as regards the independence of the constituents, holds in general for a point-base; but it is to be observed:—(i.) that the theorem does not determine in what manner the points of various orders are distributed among the two constituent point-bases, but only the totality of simple points in each; (ii.) that the assigning of orders to the points of $N$ may cause $N$ to break up further than would be required by the characterization alone; and (iii.) that there are limits to the possible values of the orders, combined with the numbers and the characterization, of the points, of which very little is at present known.

It would in many cases be feasible to effect such a distribution of the points of $N$ among $N'$ and $N''$ that no $i$-point should belong in part to one and in part to the other; but, if this were not possible, the theorem would not thereby be invalidated. —*October 13th.*]

† A $C_{p-1}$ can be drawn through $N_1$, for $p - 1 \geq l$; and $N_1$ lies on a $C_{p-3}$, since $N$ has excess for a $C_{p-1}$. If $N$ does not lie on a $C_{p-2}$, then $N'$ does not lie on a $C_{p-4}$, and $q'_{p-4} = q_{p-2} = -1$.

therefore $$\rho'_{p-3} = \rho'_{p-2} > 0,$$

which is impossible (p. 684).

Then let $C_p = C_{p'} C_{p''}$, where $C_{p'}$ is fixed and $C_{p''}$ has no fixed constituent; and let $N'$ be the number of the $N$ points which lie on $C_{p'}$, and $N''$ the remainder which lie on $C_{p''}$.

All curves lower than a $C_p$ through $N$ must have the fixed factor $C_{p'}$. Consequently, $q_p, q_{p-1}, q_{p-2}, \ldots$, which are the defects of $N'+N''$ for curves of order $p, p-1, p-2, \ldots$, must be the defects of $N''$ for curves of order $p'', p''-1, p''-2, \ldots$. Hence the successive differences of the defects of $N''$ are

$$\rho_1, \rho_2, \ldots, \rho_{a-1}, \rho_a.$$

But $\rho_a - \rho_{a-1} = 1$; and a $C_{p''}$, to which $\rho_a$ corresponds, has no fixed constituent, by hypothesis; therefore all the $\rho$'s of $N''$ from $\rho_{a-1}$ increase by units. Hence $$N'' = (\rho_1, \rho_2, \ldots, \rho_{a-1}).$$

This determines the full characterization of $N''$, and the number $N''$.

Also, since $\rho_{a-1}$ corresponds to a $C_{p''-1}$, we have

$$p''-1 = \rho_{a-1}-1 = \rho_{p-1}-1, \quad \text{and} \quad p' = p-p'' = p-\rho_{p-1}.$$

The fixed constituent of $C_p$ is therefore of order $p-\rho_{p-1}$.

Again the excess of $N''$ for a $(\rho_{a-1}-2)$-ic, that is, a $C_{p''-2}$, is zero [($\beta$), p. 682]. Therefore the excess of $N'+N''$ for a $C_{p-2}$ is entirely contributed by $N'$. The same must be true for all curves higher than a $C_{p-2}$, whether degenerate, as in the case of a $C_{p-2}, C_{p-1}, C_p$, or not. But the defects of $N'+N''$ for curves of order $p-2, p-1, \ldots, n$ are

$$q_{p-2}, q_{p-1}, q_p, \ldots, q_n;$$

therefore the defects of $N'$ for curves of the same orders are

$$q_{p-2}+N'', q_{p-1}+N'', q_p+N'', \ldots, q_n+N'',$$

and the differences of these are

$$\rho_{p-1}, \rho_p, \ldots, \rho_n, \quad \text{or} \quad \rho_{a-1}, \rho_a, \ldots, \rho_b.$$

And the differences of successive defects of $N'$ before $\rho_{a-1}$ are

$$1, 2, 3, \ldots, \rho_{a-1}-2, \rho_{a-1}-1,$$

since any curve lower than a $C_p$ through $N'$ has the fixed constituent $C_{p'}$. Hence $$N' = (1, 2, 3, \ldots, \rho_{a-1}, \rho_a, \rho_{a+1}, \ldots, \rho_b).$$

This determines the full characterization of $N'$. And it can be seen that any two point-groups having the characterizations found for $N'$ and $N''$, when placed in any positions relatively to one another, make

up a composite point-group with the same characterization as that of

$$N = (\rho_1, \rho_2, \ldots, \rho_a, \ldots, \rho_b).$$

Expressed in terms of the second differences of the defects the theorem is, since $\rho_a = \delta_1 + \delta_2 + \ldots + \delta_a$:—

*If* $\delta_a = 1$, *then the point-group* $N = (\delta_1, \delta_2, \ldots, \delta_a, \ldots, \delta_b)$ *breaks up into*

$$(\delta_1, \delta_2, \ldots, \delta_{a-1}) + (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \delta_{a+2}, \ldots, \delta_b).$$

Here we may suppose that $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ are all greater than 1. The point-group $(\delta_1, \delta_2, \ldots, \delta_{a-1})$ may, of course, break up further.

The lowest curve through $N = (\delta_1, \ldots, \delta_a, \ldots, \delta_b)$ is a $C_l$, where $l = \Sigma(\delta - 1)$; and this is a proper curve if $\delta_1, \delta_2, \ldots, \delta_a$ are all equal to 1 and $\delta_{a+1}, \delta_{a+2}, \ldots, \delta_b$ all greater than 1. It follows from what is proved in the next theorem that the lowest curve through $N$ without any fixed constituent is a $C_m$, where $m = a + \Sigma(\delta - 1)$ if $\delta_a$ is equal to 1 and $\delta_{a+1}, \ldots, \delta_b$ are all greater than 1, and $m = \Sigma(\delta - 1) = l$ if $\delta_1, \ldots, \delta_a, \ldots, \delta_b$ are all greater than 1. If $\delta_a = 1$, the fixed curve common to $C_l, C_{l+1}, \ldots, C_{m-1}$ is the lowest curve through

$$N' = (1^{\delta_1+\delta_2+\ldots+\delta_a}, \delta_{a+1}, \ldots, \delta_b),$$

and is therefore of order

$$(\delta_{a+1} - 1) + (\delta_{a+2} - 1) + \ldots + (\delta_b - 1).$$

THEOREM III.—*If* $\delta_b, \delta_{b+1}, \ldots, \delta_c$ *are all positive integers, not including zero, then*

$$(1^a, \delta_b + 1, \delta_{b+1} + 1, \ldots, \delta_c + 1) + (\delta_c, \delta_{c-1}, \ldots, \delta_b) = I(\Sigma\delta, a + \Sigma\delta),$$

$I(l, m)$ *denoting the complete intersection of a* $C_l$ *and* $C_m$.

This can be deduced from the following theorem (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 526):—

*If a* $C_l$ *and* $C_m$ *can be drawn through a point-group* $N$, *cutting again in a finite point-group* $N'$ $(N + N' = lm)$, *then*

$$q'_{l+m-p-3} = r_p - 1 + [\tfrac{1}{2}(l-p-1)(l-p-2)] + [\tfrac{1}{2}(m-p-1)(m-p-2)],$$

square brackets indicating, as before, that the product enclosed is to be retained only if its individual factors are positive. This theorem is true for all values of $p$ from 0 to $l+m-3$, taking $q_p + 1 = 0$ if $N$ does not lie on a $C_p$.

In applying the theorem we take

$$N = (1^a, \delta_b + 1, \delta_{b+1} + 1, \ldots, \delta_c + 1) = (\delta_l, \delta_{l+1}, \ldots, \delta_m, \ldots, \delta_n),$$

and assume for the moment that a proper $C_l$ and $C_m$ can be drawn through $N$, where $l = \Sigma\delta$, $m = a + \Sigma\delta$; the assumption being justified at the end. We then have

$$\delta_{l-1} = 0, \quad \delta_l = \delta_{l+1} = \ldots = \delta_{m-1} = 1,$$

$$\delta_m = \delta_b + 1, \ \ldots, \quad \delta_n = \delta_c + 1, \quad \delta_{n+1} = \delta_{n+2} = \ldots = 1.$$

Taking differences in the above theorem, we have

$$\rho'_{l+m-p-2} - \rho'_{l+m-p-3} = r_{p-1} - r_p + [l-p-1] + [m-p-1],$$

or

$$\rho'_{l+m-p-2} = p + 1 - \rho_p + [l-p-1] + [m-p-1].$$

Let $p$ diminish from $l+m-3$ to 1, so that $l+m-p-2$ increases from 1 to $l+m-3$. Then

| | | |
|---|---|---|
| from $p = l+m-3$ to $n$, | | $\rho'_{l+m-p-2} = p+1-\rho_p$; |
| „ $p = n$ | „ $m-1$, | $\rho'_{l+m-p-2} = p+1-\rho_p$; |
| „ $p = m-1$ | „ $l-1$, | $\rho'_{l+m-p-2} = m-\rho_p$; |
| „ $p = l-1$ | „ 1, | $\rho'_{l+m-p-2} = l+m-p-1-\rho_p$. |

Again taking differences, subtracting each equation from the next succeeding, we have

| | | |
|---|---|---|
| from $p = l+m-3$ to $n+1$, | | $\delta'_{l+m-p-1} = \delta_p - 1 = 0, 0, 0, \ldots, 0$; |
| „ $p = n$ | „ $m$, | $\delta'_{l+m-p-1} = \delta_p - 1 = \delta_c, \delta_{c-1}, \ldots, \delta_b$; |
| „ $p = m-1$ | „ $l$, | $\delta'_{l+m-p-1} = \delta_p = 1, 1, \ldots, 1$; |
| „ $p = l-1$ | „ 2, | $\delta'_{l+m-p-1} = \delta_p + 1 = 1, 1, \ldots, 1$. |

Hence the successive values of $\delta'$, for ascending orders of curves, are

$$\ldots, 0, 0, \delta_c, \delta_{c-1}, \ldots, \delta_b, 1, 1, 1, \ldots;$$

therefore

$$N' = (\delta_c, \delta_{c-1}, \ldots, \delta_b).*$$

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* [*Multiple Points. V.*—We have given only that form of the theorem which is adapted for the quickest reduction of a given point-group. The general theorem is

$$(\ldots \delta_a, \ldots, \delta_b, \ldots, \delta_c) + (\delta_c - 1, \ldots, \delta_b, \ldots, \delta_a + 1, \ldots) = I(f + \Sigma\overline{\delta - 1}, g + \Sigma\overline{\delta - 1}). \quad \ldots\ldots\ldots(A)$$

Here, in order to be perfectly general, we do not suppose the suffixes $\ldots a \ldots b \ldots c$ to be consecutive positive integers. The $\delta$'s of $N' = (\delta_c - 1, \ldots, \delta_b, \ldots, \delta_a + 1, \ldots)$, taken in reversed order, are formed from the $\delta$'s of $N = (\ldots \delta_a, \ldots, \delta_b, \ldots, \delta_c)$, in direct order, those up to $\delta_a$ being increased by 1, thence up to $\delta_b$ being unchanged, and thence up to the last $\delta_c$ being diminished by 1; $f$ is the number of the $\delta$'s of $N$ which are increased, *i.e.*, the number of the $\delta$'s in $\ldots \delta_a$, and $g$ is the number of the $\delta$'s of $N$ which are not diminished, *i.e.*, the number of the $\delta$'s in $\ldots \delta_a, \ldots, \delta_b$. The $\delta$'s may have any values which make both $N$ and $N'$ possible; *i.e.*, in the case of point-groups, any number of the $\delta$'s at the beginning of both $N$ and $N'$ may be

Hence also we see that it is possible to draw proper curves of order $\Sigma\delta$ and $a+\Sigma\delta$ through $N=(1^a, \delta_b+1, \ldots, \delta_c+1)$; for two proper curves of such order can certainly be drawn through $N'=(\delta_c, \delta_{c-1}, \ldots, \delta_b)$,

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zeros, but after once ceasing to be zeros they must be positive integers, excluding zero.

The specially important case in which $f=0$ gives

$$(\ldots \delta_b, \ldots, \delta_c)+(\delta_c-1, \ldots, \delta_b, \ldots)=I(\Sigma\overline{\delta-1},\ g+\Sigma\overline{\delta-1}), \quad \ldots\ldots\ldots\ldots(B)$$

where $g$ is the number of the $\delta$'s in $N$ which remain unchanged in $N'$.

Taking $g$ equal to the whole number of the $\delta$'s in $N$, we have

$$(\delta_1, \ldots, \delta_c)+(\delta_c, \ldots, \delta_1)=I(\Sigma\overline{\delta-1},\ \Sigma\delta). \quad \ldots\ldots\ldots\ldots(C)$$

If the two curves of order $\Sigma(\delta-1)$, $\Sigma\delta$ touch at all the points where they meet, and if $N$ is the point-group formed by all the points of contact (each counted once only), then $N'$ coincides with $N$; hence the series $\delta_1, \ldots, \delta_c$ is unaltered when reversed. The general case, as may be seen from (A), is almost as simple. In order that a point-group $N$ may be such that two curves can touch at all the $N$ points, without further intersection, then either $N=(\delta_1, \delta_2, \ldots, \delta_2, \delta_1)$, as above, or

$$(\delta_1, \ldots, \delta_{a-1}, \delta_a, \ldots, \delta_a, \delta_{a-1}+1, \ldots, \delta_1+1), \quad \text{or} \quad (\delta_1, \ldots, \delta_a, \delta_a+1, \ldots, \delta_1+1),$$

or one of the two last reversed. It is very remarkable that we can apparently assume any characterization for $N$, provided it comes under one of these forms and gives the correct value of the number $N$, without increasing the total number of independent interconnexions of the $N$ points, that is, without increasing the specialization of $N$.

{For, let $C_n$, $C_{n'}$ $(n \leqslant n')$ be two curves which cut altogether in two point-groups $N$, $N'$ having any the same characterization, and let $k$ be the number of independent interconnexions of either point-group due to this characterization. The number of points which can be chosen at will on a given $C_n$ which form part of a point-group on $C_n$ with the same characterization as $N$ is $N+r_n-k$ (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 529). If therefore a $C_{n'}$ can be drawn through $N$ which touches any $C_n$ through $N$ at $N+r_n-k$ of the $N$ points, it will touch it at the remainder. But a $C_{n'}$ can be drawn through $N$ touching $C_n$ at $N-\frac{1}{2}(n-1)(n-2)$ of the $N$ points. Hence the number of conditions that a $C_{n'}$ can be drawn touching $C_n$ at all the $N$ points is

$$(N+r_n-k)-(N-\tfrac{1}{2}\overline{n-1}\,.\,\overline{n-2})=r_n-k+\tfrac{1}{2}(n-1)(n-2)=N-k+q_n+1-3n.$$

Hence, since the degree of freedom of $C_n$ accounts for $q_n$ of the conditions, it follows that $N-k-3n+1$ is the number of conditions to be satisfied by the $N$ points; and to this we can now add the $k$ conditions due to the characterization. Thus the total number is $N-3n+1$, which remains unaltered, whatever the assumed characterization may be.}

The properties expressed by (A), (B), (C) hold equally for point-groups and point-bases; and (B), (C) have applications especially to the latter. What is required to complete the theory in regard to point-bases is, first, to determine the limits of possibility of the $\delta$'s as depending on the assigned orders of the points, and, second, to show how to deal with specialized derived point-bases when they cannot be excluded, as in the case of any nine 2-points which lie on a proper sextic.

Whether we are given the characterization of a point-group, or of a point-base, we know the order $\Sigma(\delta-1)$ of the lowest curve which passes through it. Also in the case of a point-group we know the order of the lowest curve without fixed constituents which passes through it; but we do not know it at present with any certainty for a given point-base. Hence for a point-base we have to use (B) in the place of the theorem in the text; and we can only apply (B) by way of trial, for we do not know the lowest value of $g$ which will make the derived point-base $N'$ possible.

We give now an example of a point-base which does not contain any points of

since $N'$ has no excess for a curve of order $\Sigma\delta - 2$ [*cf.* ($\beta$), p. 682, and ($\beta$), p. 681]; and these determine $N$, and pass through it.

We add an example of the reduction of a point-group. Take the

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order 1. Very few such examples of a constructional kind are known, the best known being that of the nine 2-points on a proper sextic. Constructional point-bases which include 1-points can be obtained in any number by the methods we have described; not so those which do not include 1-points. (*Cf.* Cayley, *Proc. Lond. Math. Soc.*, Vol. III., p. 197; *Collected Works*, Vol. VII., p. 254.) Consider the point-base $N$ formed by $lm$ 2-points, situated at the $lm$ points in which a $C_l$ and $C_m$ intersect, these $lm$ points being all finitely separated. We know the form of the equation of the general algebraic curve through the point-base, viz.,

$$C_l^2 S + C_l C_m S' + C_m^2 S'' = 0;$$

and we can thence deduce the characterization. The result is

(i.) $N = (3, 3, \ldots \text{repeated } l \text{ times}) = (3^l)$, if $l = m$;

(ii.) $N = (1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l})$, if $l < m$, $2l > m$;

(iii.) $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, if $2l \leqslant m$.

Suppose now that all that is given with respect to $N$ is its characterization, viz. that in (i.), (ii.), or (iii.), and the fact that $N$ is made up of $lm$ separate 2-points. (The degree $N = 3lm$.) We shall consider first the application of (B) to case (ii.). The lowest curve through $N$ is of order $\Sigma(\delta - 1) = 2l$; call it $C_{2l}$. (If $N$ is constructed as originally supposed, $C_{2l} = C_l^2$.) If $N$ were a point-group, the lowest curve without fixed constituents through it would be of order $m - l + 2l = l + m$. But, as regards the point-base, this is too low a limit; for, since the two curves must intersect in $4lm$ points at the least, the curve without fixed constituents must be at least of order $2m$. Assume then, by way of trial, that a $C_{2m}$ without fixed constituents can be drawn through $N$, *i.e.*, that the value of $g$ in (B) is $2(m - l)$. Then the first $2(m - l)$ of the $\delta$'s in $N$ are to remain unchanged in $N'$, and the rest are to be diminished each by 1. Thus (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m}, 2^{m-l}) = I(2l, 2m).$$

Hence
$$N' = (1^{m-l}, 2^{2l-m}, 2^{m-l}) = (1^{m-l}, 2^l) = I(l, m).$$

Now it is possible to draw through $N' = I(l, m)$ two curves $C_{2l}$, $C_{2m}$ which have double points at all the points of $N'$, and which have no common constituent. The point-base $N$ can therefore be reduced in a single step to an unspecialized point-group $N' = I(l, m)$; and the construction thus found for $N$ is the one originally supposed.

This $N' = I(l, m)$ is the smallest derived of $N$. The next smallest $N''$ is obtained by drawing a $C_{2l}$ and $C_{2m+1}$ through $N$. For this, (B) gives

$$(1^{m-l}, 2^{m-l}, 3^{2l-m}, 2^{m-l}) + (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l}) = I(2l, 2m+1).$$

It can be proved that this $N'' = (1^{m-l}, 2^{2l-m-1}, 3, 2^{m-l})$ must be specialized. For $N''$ is a point-group containing $lm + 2l$ points, and, if general of its kind, lies on a proper $C_{l+1}$ and a proper $C_{m+1}$. But through $N''$ a curve $C_{2l}$ can be drawn, having double points at $lm$ of the $N''$ points. This $C_{2l}$ and the proper $C_{l+1}$ therefore cut in

$$2lm + 2l = 2l(m+1) > 2l(l+1) \text{ points, since } l < m.$$

Hence $C_{2l}$ must have the proper $C_{l+1}$ for a constituent. Thus

$$C_{2l} \equiv C_{l+1} C_{l-1}.$$

But again, since $C_{2l}$ has double points at each of the $lm$ points, $C_{l-1}$ must pass through the $lm$ points; and $C_{l-1}$ has only $l^2 - 1$ points in all in common with the proper $C_{l+1}$. Thus $l^2 - 1 \geqslant lm$, and $l^2 > lm$, which is not true. Thus $N''$ must be

point-group, expressed in terms of the second differences of the defects, $N = (4, 4, 6, 1, 1, 5, 1, 3, 7, 2)$. Working upwards to the defects, we find that this is a group of 404 points whose defects fo curves of order 23 to 33 are $-1, 3, 11, 25, 40, 56, 77, 99, 124, 156, 190$.

$$N = (4, 4, 6) + (1^{16}, 5) + (1^{22}, 3, 7, 2) \quad \text{(Theorem II.)}.$$

(α) (i.) $(4, 4, 6) + (5, 3, 3) = I\,(11, 11)$ (Theorem III.);

(ii.) $(5, 3, 3) + (2, 2, 4) = I\,(8, 8)$ „

(iii.) $(2, 2, 4) + (3, 1, 1) = I\,(5, 5)$ „

(iv.) $(3, 1, 1) = (3, 4)$ in first differences

$= 3$ general points (p. 682).

This gives us the construction of the point-group $(4, 4, 6)$. The number of its points is $11^2 - 8^2 + 5^2 - 3 = 79$.

(β) $(1^{16}, 5) = (1, 2, \ldots 16, 21)$ in first differences

$= 74$ general points on a $C_4$ (p. 682).

(γ) (i.) $(1^{22}, 3, 7, 2) + (1, 6, 2) = I\,(9, 31)$ (Theorem III.);

(ii.) $(1, 6, 2) = (1, 7, 9)$ in first differences

$= 28$ general points on a $C_6$ (p. 682).

Thus $(1^{22}, 3, 7, 2)$ is constructed by drawing a $C_9$ and $C_{31}$ through 28 general points on a $C_6$ to cut again in $9 \times 31 - 28 = 251$ points.

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specialized; in fact, $C_{l+1}$ cannot be a proper curve. If the construction found above for $N$ is the only solution, $N''$ consists of the $lm$ points in which $C_l$, $C_m$ cut, and $l$ point-pairs on a straight line, viz., at the points where any straight line cuts $C_l$. This straight line is an $l$-fold tangent to $C_{2m+1}$.

This reasoning suggests the inference that the *smallest* derived point-base $N'$ of a given point-base $N$ is the one which *is the most likely to be general of its kind.* Hence the importance of discovering the order of the lowest curve without any fixed constituent which passes through a given point-base.

Taking case (i.), $N = (3^l)$, the lowest curve through $N$ is a $C_{2l}$, as before. If we assume, by way of trial, that $C_{2l}$ has no fixed constituent, (B) gives (taking $g = 0$)

$$(3^l) + (2^l) = I\,(2l, 2l).$$

Here $N' = (2^l) = I(l, l)$ satisfies the premised conditions, and is general of its kind. The $C_{2l}$ through $N$ is not a proper curve, but has the requisite property that it does not possess any fixed constituent.

Taking case (iii.), $N = (1^{m-l}, 2^l, 1^{m-2l}, 2^l)$, we see that $N$ breaks up, by Theorem II., into

$$(1^{m-l}, 2^l) + (1^{2m-l}, 2^l) = I\,(l, m) + I\,(l, 2m).$$

The two constituents of $N$, $I\,(l, m)$ and $I\,(l, 2m)$, are not independent of one another, nor is the second general of its kind. If the first is general of its kind, the second must consist of point-pairs having the same situation as the single points of the first. This gives a correct analysis of $N$.—*October 13th.*]

## V. Number of Interconnexions of a Point-Group.

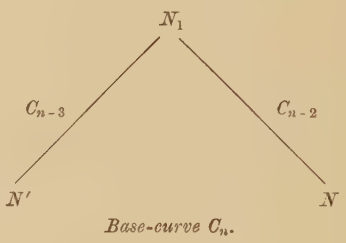
*The number of the independent interconnexions of the points of the point-group* $(\rho_1, \rho_2, \ldots, \rho_a)$, *due to the characterization, is*

$$\rho_1(\rho_3-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+\ldots+\rho_{a-2}(\rho_a-\rho_{a-1}-1).$$

To find the number of independent interconnexions of the points of

$$N=(\rho_1, \rho_2, \ldots, \rho_a)=(\rho_l, \rho_{l+1}, \ldots, \rho_n)$$

we go back to Theorem I. Suppose that $k$ is the required number of interconnexions of $N$, and $k'$ that of $N'$. We find the value of $k-k'$ by obtaining and equating two different expressions for the least number of parameters in terms of which the base-curve $C_n$, and the positions of all the points of $N$ and $N'$ upon $C_n$, can be expressed.



*Base-curve* $C_n$.

Taking any two coordinate axes, the least number of parameters in terms of which the positions of the $N$ points in the plane can be expressed is $2N-k$, since $k$ is the number of independent interconnexions of the $N$ points. These $2N-k$ parameters determine the $N$ points; $q_n$ more parameters, and not less, will determine the $C_n$, since $q_n$ is the defect of $N$ for a $C_n$; and $r'_{n-3}$ more parameters, and not less, will determine $N'$. This last result follows from the fact that $r'_{n-3}$ is the multiplicity of $N'$ on $C_n$.* Thus one of the required expressions is $2N-k+q_n+r'_{n-3}$; and the other is $2N'-k'+q'_n+r_{n-3}$, obtained by starting with $N'$. Equating these, and noticing that $N+q_n=N'+q'_n$, we have

$$\begin{aligned} k-k' &= (N-r_{n-3})-(N'-r'_{n-3}) \\ &= q'_{n-3}-q_{n-3}, \text{ from (1), p. 675,} \\ &= q_{n-2}-q_{n-3}=\rho_{n-2}=\rho_{a-2}. \end{aligned}$$

Hence, in changing from the point-group $(\rho_1, \ldots, \rho_a)$ to $(\rho_1, \ldots, \rho_{a-1}, \rho_a-1)$, $k$ is diminished by $\rho_{a-2}$. After $\rho_a-\rho_{a-1}-1$ such steps $(\rho_1, \ldots, \rho_a)$ is reduced to $(\rho_1, \ldots, \rho_{a-1})$, and $k$ is diminished by $\rho_{a-2}(\rho_a-\rho_{a-1}-1)$. But, when $(\rho_1, \ldots, \rho_a)$ has been reduced to the

* A concise statement of the Riemann-Roch theorem is that *the multiplicity of any point-group on a curve* $C_n$ *is equal to the* $(n-3)$*-ic excess of the point-group.* (*Proceedings*, Vol. xxvi., p. 523.)

general point-group $(\rho_1, \rho_2)$, $k$ is diminished to zero. Hence

$$k = \rho_1(\rho_3-\rho_2-1)+\rho_2(\rho_4-\rho_3-1)+ \dots +\rho_{a-2}(\rho_a-\rho_{a-1}-1).*$$

Also, since $k-k' = \rho_{a-2}$, and $N-N' = \rho_a-1$,

therefore $(N-k)-(N'-k') = \rho_a-\rho_{a-2}-1 \geqslant 1$;

therefore $N-k > N'-k' > \dots \geqslant 0.$

It can be easily proved that

$$N-k = \tfrac{1}{2}(\delta_1-1)\delta_1+\tfrac{1}{2}(\delta_2-1)(\delta_2+2\delta_1)+ \dots +\tfrac{1}{2}(\delta_a-1)(\delta_a+2\delta_{a-1}).$$

## VI. Rational Transformation of Point-Bases.†

In rational transformation the whole system of curves of order $n$ which satisfy given conditions transforms into the whole system of curves of another order $n'$ satisfying another set of conditions. If the original conditions are simply those of passage through a given point-base, the transformed conditions are also, so long as $n$ remains fixed, *simply* those of passage through another point-base. But, as the orders $n$ of the original curves increase by units, the orders $n'$ of the transformed curves increase in arithmetical progression, while also the orders of their multiple points may some increase in arithmetical progression, and others remain constant. As $n$ varies, the orders of the points of the transformed point-base also vary.

Rational transformation thus leads to a generalized view of the questions treated above. Instead of investigating the properties in respect to excess and defect of a *simple* point-base, whose points are all of fixed orders, we have to consider these same properties for a

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* [*Multiple Points. VI.*—In the application of this result to point-bases each $i$-point is to be regarded as a single point. But the reasoning by which the result is obtained fails in the majority of cases, since the proof depends on the use of a slow process of reduction, which would generally cause the derived point-bases to be specialized. The proof can, however, be extended to any reduction of a point-group. Thus it appears that one condition (and probably not the only one) for the correctness of the result, when applied to a given point-base $N$, is that it should be possible to reduce $N$ by means of a series of unspecialized point-bases; and, for this purpose, as we have seen (Note V., p. 691), the most rapid reduction seems the most likely of any to prove effective.

It seems probable that the result holds for a point-base $N$ so long as it does not exceed twice the number of the points of order 1 contained in $N$. Further it appears that the correct result, if different from, is less than that found above. Thus in case (i.) of the example in Note V. the value of $k$, given by the formula $\Sigma\rho_{a-2}(\rho_a-\rho_{a-1}-1)$, is just three times the correct value; and in cases (ii.) and (iii.) the formula gives a value which is *more* than three times too great.—*October 13th.*]

† See footnotes on "Multiple Points" for the meaning of *point-base*, and the applicability to point-bases of the results proved for point-groups.

*generalized* point-base, including some points of fixed, and others of variable, order. The orders of the curves drawn through the point-base increase with constant difference $\nu$, while the orders of the points of the base increase correspondingly, but each with its own constant difference $\iota$, which varies for different points, and may, in particular, be zero.

In this generalized view the *virtual* number of the conditions supplied for a $C_n$ is $N_n = \frac{1}{2}\Sigma i\,(i+1)$, while the *actual* number is $N_n - r_n$, $r_n$ being the $n$-ic excess. The $n$-ic defect, $q_n$, is still the degree of freedom of a $C_n$ which satisfies the $N_n$ conditions. Instead of formula (1), p. 675, we have

$$N_n - r_n + q_n + 1 = \tfrac{1}{2}\,(n+1)\,(n+2).$$

It is evident that all the defects are invariants in rational transformation, since the number of general points through which a $C_n$, satisfying the $N_n$ conditions, can be drawn is equal to the number of general points through which the transformed $C_{n'}$, satisfying the transformed $N'_{n'}$ conditions, can be drawn, and *vice versâ*. The invariance of the excesses is not so evident; but this can be easily shown by proving it to hold for any quadric transformation. Thus $q_n$, $r_n$, $\rho_n$, $\delta_n$, $N_n - \frac{1}{2}\,(n+1)\,(n+2)$ are all invariants, while $N_n$ and $n$ are not. Here $\rho_n = q_n - q_{n-\nu}$, $\delta_n = \rho_n - \rho_{n-\nu}$; and we may further put $\sigma_n = r_{n-\nu} - r_n$, $\epsilon_n = \sigma_{n-\nu} - \sigma_n$.

The invariant $N_n - \frac{1}{2}\,(n+1)\,(n+2)$ involves three others, viz., $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$, and $n\nu - \Sigma i\iota$. In a simple point-base $\nu = 1$, and all the $\iota$'s vanish; hence $\nu^2 - \Sigma\iota^2 = 1$, and $3\nu - \Sigma\iota = 3$; and a generalized point-base derived by rational transformation from a simple one must satisfy these equations. Conversely, if the equations hold for a generalized point-base, it can be rationally transformed into a simple one; for the equations $\nu^2 - \Sigma\iota^2 = 1$, $3\nu - \Sigma\iota = 3$ show that a net of curves $C_\nu$ can be described with multiple points of order $\iota, \ldots$ at the points of the point-base. This net rationally transforms the point-base into a simple one; for $C_\nu$ transforms to $C_{\nu'}$, consequently $\nu' = 1$, and all the $\iota'$'s vanish.

I do not know whether generalized point-bases with similar properties of excess and defect, but having other values than 1 and 3 for $\nu^2 - \Sigma\iota^2$ and $3\nu - \Sigma\iota$, are possible, or not. Assuming them to be possible, the curves $C_\nu$, if they exist, still transform into the curves $C_{\nu'}$, but cannot themselves be used for rational transformation. The numbers 1, 3 are perhaps the lowest possible values of $\nu^2 - \Sigma\iota^2$, $3\nu - \Sigma\iota$ respectively; and, this being so, the curves $C_\nu$ certainly exist. If $\nu^2 - \Sigma\iota^2$ were

negative, it would appear that there must be a *superior* limit to the order $n$ of a curve which could satisfy the $N_n$ conditions.

If we write $\pi$ for the deficiency of $C_\nu$, we have $\delta_n - \epsilon_n = \nu^2 - \Sigma\iota^2 =$ *constant* number of ordinary points in which two curves $C_\nu$ intersect, and $\rho_n + \sigma_n + \pi - 1 = n\nu - \Sigma i\iota =$ number of ordinary points in which $C_n$ and $C_\nu$ intersect. Reasoning from analogy we should expect 0 to be the least possible value of $\epsilon_n$, corresponding to the breaking-point (p. 679), and $\nu^2 - \Sigma\iota^2$ to be the least possible value of $\delta_n$.

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## *The Conformal Representation of a Pentagon on a Half Plane.*

*By* Miss M. E. Barwell. Read June 9th, 1898. Received, in revised form, September 15th, 1898.

1. The conformal representation of a rectilinear polygon on a half plane was first attempted by Schwarz and Christoffel, who arrived independently at the same result. They have shown that the area of the $w$-plane included by a polygon, whose sides do not cross, can be conformally represented by the northern half of the $z$-plane, the boundary of the polygon corresponding to the axis of real quantities on the $z$-plane.

The necessary transformation is

$$w = M\int (z-a)^{\alpha-1}(z-b)^{\beta-1}\dots(z-b)^{\lambda-1}\,dz + M',$$

where $a, b, \dots, l$ are the points on the real axis of $z$ corresponding to the angular points of the polygon taken in order, and all lying in the finite part of the $z$-plane.

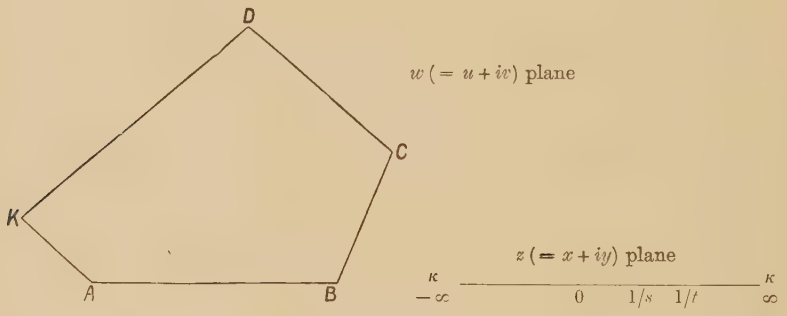
$\alpha\pi, \beta\pi, \dots, \lambda\pi$ are the internal angles of the polygon at the respective points. The constant $M'$ is determined by fixing the origin in the $w$-plane. Any three of the real quantities $a, \dots, l$ may be chosen arbitrarily, and the remainder must be determined in terms of these three, and the constants of the polygon $\alpha . \beta \dots \lambda$. The case of the quadrilateral is given in Forsyth's *Theory of Functions*, p. 546. There is one unknown quantity besides $M$ to be determined, and the solution involves Gauss' hypergeometric functions.

With the pentagon, there are two unknown quantities, in addition to $M$, and the necessary equations contain functions analogous to those of Gauss, but satisfying differential equations of the third order (Schäfli, *Crelle*, Vol. LXXVIII.). They may be called "hypergeometric functions of the third order," and they form a special case of the functions investigated by Pochhammer, in his paper on "Hypergeometrische Functionen $n^{\text{ter}}$ Ordnung" (*Crelle*, Vol. LXXI.).

Let $\alpha\pi$, $\beta\pi$, $\gamma\pi$, $\delta\pi$, $\kappa\pi$ be the internal angles of the pentagon at $A$, $B$, $C$, $D$, $K$ respectively. Then

$$\alpha+\beta+\gamma+\delta+\kappa = 3.$$

Also let 0, 1, and $\infty$ be the homologues in the $z$-plane of the points $A$, $B$, $K$.




The required transformation is

$$\frac{dw}{dz} = Mz^{\alpha-1}(1-z)^{\beta-1}(1-sz)^{\gamma-1}(1-tz)^{\delta-1},$$

where $\frac{1}{s}$ and $\frac{1}{t}$ correspond to the points $C$ and $D$, and where

$$1>s>t>0.$$

Suppose $w=0$ at the point $A$, and that $AB$ is taken as the real axis of $w$.

Then $$w = M\int_0^z z^{\alpha-1}(1-z)^{\beta-1}(1-sz)^{\gamma-1}(1-tz)^{\delta-1}\,dz,$$

and $M$ must be real.

Integrating along the real axis of $z$, we obtain the following equations :—

$$AB = M\int_0^1 X\,dx, \tag{1}$$

$$BC.e^{i\pi(1-\beta)} = M\int_1^{1/s} X\,dx, \tag{2}$$

$$CD.e^{i\pi(2-\beta-\gamma)} = M\int_{1/s}^{1/t} X\,dx, \tag{3}$$

$$DK.e^{i\pi(3-\beta-\gamma-\delta)} = M\int_{1/t}^{\infty} X\,dx, \tag{4}$$

$$KA.e^{i\pi(4-\beta-\gamma-\delta-\epsilon)} = M\int_{-\infty}^{0} X\,dx, \tag{5}$$

where $$X = x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}.$$

There are thus five equations to determine $M$, $s$, and $t$; and they must all be consistent by virtue of the relations between the sides and angles of the pentagon. Any three may be selected for the determination of the three unknown quantities.

It is first necessary to reduce equations (1) to (5) to a real form.

Now $$\int_1^{1/s} X\,dx = \int_1^{1/s} x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

$$= e^{\pm i\pi(\beta-1)}\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx.$$

The lower sign must be taken, for the argument of $1-x$ is $-\pi$, when

$$\frac{1}{s} > x > 1.$$

Hence $$\int_1^{1/s} X\,dx = e^{i\pi(1-\beta)}\int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx.$$

Similarly,

$$\int_{1/s}^{1/t} X\,dx = e^{i\pi(2-\beta-\gamma)}\int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

$$\int_{1/t}^{\infty} X\,dx = e^{i\pi(3-\beta-\gamma-\delta)}\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

and $$\int_{-\infty}^{0} X\,dx = e^{\pm i\pi(\alpha-1)}\int_{-\infty}^{0}(-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx$$

$$= e^{i\pi(\alpha-1)}\int_{-\infty}^{0}(-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

(since the argument of $x$ is $+\pi$, for $0>x>-\infty$)

$$= e^{i\pi(2-\beta-\gamma-\delta-\kappa)}\int_{-\infty}^{0}(-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx,$$

since $$\alpha+\beta+\gamma+\delta+\kappa = 3.$$

Hence the five equations become

$$AB = M\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(i.)}$$

$$BC = M\int_{1}^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(ii.)}$$

$$CD = M\int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(1-tx)^{\delta-1}\,dx, \qquad \text{(iii.)}$$

$$DK = M\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx, \qquad \text{(iv.)}$$

$$KA = M\int_{-\infty}^{0}(-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx. \qquad \text{(v.)}$$

From these equations we can obtain two which will determine $s$ and $t$, and for this purpose we are at liberty to select any two pairs of equations. For instance, from (ii.) and (iv.),

$$\left.\begin{aligned} &DK\int_{1}^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx \\ &= BC\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx, \end{aligned}\right\}$$

and, from (i.) and (v.),

$$\left.\begin{aligned} &AB\int_{-\infty}^{0}(-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}dx \\ &= KA\int_{0}^{1} x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx. \end{aligned}\right\} \qquad \text{(A)}$$

2. It is to be observed that the definite integrals in equations (1) to (5) on p. 697 are hypergeometric functions as defined by Pochhammer

in his paper already quoted (*Crelle*, Vol. LXXI.). He has established as hypergeometric functions of the $n^{\text{th}}$ order the integrals typified by

$$\int_g^h (\xi-a_1)^{b_1-1}(\xi-a_2)^{b_2-1}\dots(\xi-a_n)^{b_n-1}(\xi-s)^{\lambda-1}\,d\xi,$$

where $g$ and $h$ may be any two of the quantities $a_1, a_2, \dots, a_n, s, \infty$.

Now, in the typical function of the third order, let

$$\xi = \frac{1}{x}, \quad a_1 = 0, \quad a_2 = t, \quad a_3 = 1,$$

$$\lambda = \gamma, \qquad b_1 = \kappa, \quad b_2 = \delta, \quad b_3 = \beta;$$

and therefore $3-\lambda-b_1-b_2-b_3 = 3-\gamma-\kappa-\delta-\beta = \alpha.$

We then obtain the definite integrals given on p. 697.

3. *Expansions of the Integrals on p.* 698.

i. $$\phi_1 = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx \qquad (1>s>t>0)$$

is finite, uniform, and continuous, if $\alpha>0$, $\beta>0$, and if $s$ and $t$ have no real positive value between 1 and $\infty$. Hence the integral can, in the interior of a circle of radius mod $(1-t)$ and centre $s=t$, be developed in a converging series of ascending powers of $s-t$.

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

$$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta-1}\{1-tx-(s-t)\,x\}^{\gamma-1}\,dx$$

$$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-2}\left\{1-(\gamma-1)\,\frac{s-t}{1-tx}x+\dots\right\}dx$$

$$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-2}\,dx$$
$$-(\gamma-1)(s-t)\int_0^1 x^{\alpha}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-3}\,dx$$
$$+\dots$$
$$+(-1)^k\frac{(\gamma-1)(\gamma-2)\dots(\gamma-k)}{k!}(s-t)^k$$
$$\times\int_0^1 x^{\alpha+k-1}(1-x)^{\beta-1}(1-tx)^{\delta+\gamma-k-2}\,dx$$
$$+\dots$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} F(2-\delta-\gamma,\ \alpha,\ \alpha+\beta,\ t)$$

$$-(\gamma-1)\frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} F(3-\delta-\gamma,\ \alpha+1,\ \alpha+\beta+1,\ t)(s-t)$$

$$+\ldots$$

$$+(-1)^k \frac{\Gamma(\alpha+k)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+k)} \frac{(\gamma-1)(\gamma-2)\ldots(\gamma-k)}{k!}$$

$$\times F(k+2-\delta-\gamma,\ \alpha+k,\ \alpha+\beta+k,\ t)(s-t)^k$$

$$+\ldots,$$

where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\ldots.$$

Since $$2-\delta-\gamma = \alpha+\beta+\kappa-1,$$

the series may be denoted by $F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$, where the first suffix means that the expansion is that of a hypergeometric function of the *third* order, the second that $\alpha$ is the element which increases by unity from term to term.

ii. $$\phi_2 = \int_1^{1/s} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

after division by $(1-s)^{\beta+\gamma-1}$, is finite, uniform, and continuous, if $\beta>0$, $\gamma>0$, and if $t$ has no real positive value between $s$ and 1. The integral can therefore be expanded in the neighbourhood of $t=0$ in a converging series of powers of $t$, and also, in the vicinity of $s=1$, in a converging series of powers of $1-s$.

Let $$x = \frac{1}{1-(1-s)\,\xi};$$

then $$dx = \frac{(1-s)\,d\xi}{\{1-(1-s)\,\xi\}^2},$$

$$1-sx = \frac{(1-s)(1-\xi)}{1-(1-s)\,\xi}, \quad x-1 = \frac{(1-s)\,\xi}{1-(1-s)\,\xi},$$

$$1-tx = 1-\frac{t}{1-(1-s)\,\xi}.$$

Then $$\phi_2 = (1-s)^{\beta+\gamma-1}\int_0^1 \xi^{\beta-1}(1-\xi)^{\gamma-1}(1-\overline{1-s}\,\xi)^{1-\alpha-\beta-\gamma} \times\left\{1-\frac{t}{\overline{1-(1-s)\,\xi}}\right\}^{\delta-1} dx$$

$$= (1-s)^{\beta+\gamma-1}\int_0^1 {}^{\beta-1}(1-\xi)^{\gamma-1}(1-\overline{1-s}\,\xi)^{1-\alpha-\beta-\gamma} \times\left\{1-(\delta-1)\frac{t}{\overline{1-1-s}\,\xi}+\ldots\right\} d\xi$$

$$= (1-s)^{\beta+\gamma-1}F_{3,\,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t),$$

in accordance with our notation.

iii. $$\varphi_3 = \int_{1/s}^{1/t} x^{\alpha-1}(x-1)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx,$$

after division, by $(s-t)^{\gamma+\delta-1}$, is uniform, finite, and continuous, if $\gamma>0$, $\delta>0$, and if $s$ and $t$ have only values lying between 0 and 1. Thus $\varphi_3$ can, in the vicinity of $t=s$, be developed in a converging series of powers of $s-t$, the coefficients of which will involve $s$.

Let $$\frac{sx-1}{x\,(s-t)} = \xi;$$

then $$x = \frac{1}{s-(s-t)\,\xi},\quad dx\,\frac{s-t}{\{s-(s-t)\,\xi\}^2}\,d\xi,$$

$$x-1 = \frac{1-s+(s-t)\,\xi}{s-(s-t)\,\xi},\quad sx-1 = \frac{(s-t)\,\xi}{s-(s-t)\,\xi},$$

$$1-tx = \frac{(s-t)(1-\xi)}{s-(s-t)\,\xi}.$$

$$\varphi_3 = (s-t)^{\gamma+\delta-1}\int_0^1 \xi^{\gamma-1}(1-\xi)^{\delta-1}\{s-(s-t)\,\xi\}^{2-\alpha-\beta-\gamma-\delta} \times\{1-s+(s-t)\,\xi\}^{\beta-1}\,d\xi$$

$$= (s-t)^{\gamma+\delta-1}s^{\kappa-1}(1-s)^{\beta-1}\int_0^1 \xi^{\gamma-1}(1-\xi)^{\delta-1}\left(1-\frac{s-t}{s}\,\xi\right)^{\kappa-1} \times\left(1-\frac{s-t}{1-s}\,\xi\right)^{\beta-1} d\xi$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\delta)}{\Gamma(\gamma+\delta)} s^{\kappa-1}(1-s)^{\beta-1}(s-t)^{\gamma+\delta-1}$$

$$\times\Big[1-\frac{\gamma}{\gamma+\delta}\Big(\frac{\kappa-1}{s}+\frac{\beta-1}{1-s}\Big)(s-t)+\frac{\gamma(\gamma+1)}{(\gamma+\delta)(\gamma+\delta+1)}$$

$$\Big\{\frac{(\kappa-1)(\kappa-2)}{2!\,s^2}+\frac{(\kappa-1)(\beta-1)}{s(s-1)}+\frac{(\beta-1)(\beta-2)}{2!\,(1-s)^2}\Big\}(s-t)^2+\ldots\Big].$$

iv. $$\phi_4 = \int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}(tx-1)^{\delta-1}\,dx.$$

When $t=0$ the integral becomes

$$\mathrm{Lt}_{t=0}\int_{1/t}^{\infty} x^{\alpha-1}(x-1)^{\beta-1}(sx-1)^{\gamma-1}\,dx,$$

and is therefore infinite and of the order $\alpha+\beta+\gamma-2$, *i.e.*, $1-\kappa-\delta$. Hence, after division, by $t^{\kappa+\delta-1}$, $\phi_4$ will be finite, uniform, and continuous, if $\delta>0$, $3-\alpha-\beta-\gamma-\delta>0$, *i.e.*, if $\delta>0$ and $\kappa>0$, and if, moreover, $s$ has no real positive value between 0 and $t$. Hence $\frac{\phi_4}{t^{\kappa+\delta-1}}$ can be developed in a converging series of powers of $t$, and also of $1-s$.

Let $$x = \frac{1}{t\xi};$$

then $$dx = -\frac{1}{t\xi^2}\,d\xi,$$

$$x-1 = \frac{1-t\xi}{t\xi}, \quad sx-1 = \frac{s-t\xi}{t\xi}, \quad tx-1 = \frac{1-\xi}{\xi},$$

$$\phi_4 = t^{2-\alpha-\beta-\gamma}\int_0^1 \xi^{2-\alpha-\beta-\gamma-\delta}(1-\xi)^{\delta-1}(1-t\xi)^{\beta-1}(1-t\xi-\overline{1-s})^{\gamma-1}\,d\xi$$

$$= t^{\kappa+\delta-1}\int_0^1 \xi^{\kappa-1}(1-\xi)^{\delta-1}(1-t\xi)^{\beta+\gamma-2}\Big\{1-(\gamma-1)\,\frac{1-s}{1-t\xi}+\ldots\Big\}\,d\xi$$

$$= t^{\kappa+\delta-1}F_{3,\,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s).$$

This result might have been obtained from the expansion of $\phi_2$ on p. 701, by interchanging $\beta$ and $\kappa$, $\gamma$ and $\delta$, $t$ and $1-s$.

v. $$\phi_5 = \int_{-\infty}^{0} (-x)^{\alpha-1}(1-x)^{\beta-1}(1-sx)^{\gamma-1}(1-tx)^{\delta-1}\,dx$$

is uniform, finite, and continuous, if $\alpha>0$, and $3-\alpha-\beta-\gamma-\delta>0$ (*i.e.*, if $\alpha>0$ and $\kappa>0$), and if, moreover, $s$ and $t$ have no real negative

value. Hence, in the neighbourhood of $s = 1$, $\phi_5$ can be developed in a converging series of powers of $1-s$, and again, when considered as a function of $t$, it can be expanded in a similar series of powers of $s-t$.

Let $$x = \frac{\xi}{\xi-1} = 1 + \frac{1}{\xi-1}, \quad dx = \frac{-d\xi}{(1-\xi)^2},$$

$$1-x = \frac{1}{1-\xi}, \quad 1-sx = \frac{1-(1-s)\xi}{1-\xi}, \quad 1-tx = \frac{1-(1-t)\xi}{1-\xi},$$

$$\phi_5 = \int_0^1 \xi^{\alpha-1}(1-\xi)^{2-\alpha-\beta-\gamma-\delta}(1-\overline{1-s}\,\xi)^{\gamma-1}(1-\overline{1-t}\,\xi)^{\delta-1}\,d\xi$$

$$= \int_0^1 \xi^{\alpha-1}(1-\xi)^{\kappa-1}(1-\overline{1-s}\,\xi)^{\gamma+\delta-2}\left\{1-(\delta-1)\frac{s-t}{1-\overline{1-s}\,\xi}\xi+\dots\right\}d\xi$$

$$= F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).$$

The two equations on p. 698 for the determination of $s$ and $t$ can now be written in the form

$$\left.\begin{aligned}
&(1)\quad (1-s)^{\beta+\gamma-1}F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t\\
&\qquad = \frac{BC}{DK}t^{\kappa+\delta-1}F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s),\\
&(2)\quad F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ x-\gamma,\ t,\ s-t)\\
&\qquad = \frac{AB}{KA}F_{3,\alpha}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).
\end{aligned}\right\} \quad (B)$$

### 4. *The Numerical Determination of s and t.*

The questions naturally arise: Are the equations (B) at the end of the last paragraph suitable for the determination of the numerical values of $s$ and $t$, and why should one pair of the equations (i.) to (v.) on p. 698 be selected rather than another?

With regard to the latter question, let us consider the expansions of the functions $\phi$, that is, of the five definite integrals.

$\phi_1$ can be expanded in a double series of powers of $t$ and $s-t$;

$\phi_2$, after division by $(1-s)^{\beta+\gamma-1}$, in powers of $1-s$ and $t$;

$\phi_3$, after division by $(s-t)^{\beta+\delta-1}$, in powers of $s-t$, the coefficients of which contain $s$;

$\phi_4$, after division by $t^{\delta+\kappa-1}$, in powers of $1-s$ and $t$;

$\phi_5$ in a double series of powers of $s-t$ and $1-s$.

It is in general advantageous to have an equation connecting $\phi_2$ and $\phi_4$, since they can both be expanded in powers of $1-s$ and $t$, and for this reason equations (ii.) and (iv.) on p. 698 would be selected. The choice of the other pair would probably depend on the numerical data of the pentagon in question.

Thus an equation connecting $t$ and $1-s$ is

$$(1-s)^{\beta+\gamma-1}.F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t)$$

$$=\frac{BC}{D\kappa}.t^{\kappa+\delta-1}F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ t,\ 1-s).$$

Suppose $\beta+\gamma-1$ and $\kappa+\delta-1$ can each be expressed as the ratio of two integers.

Let $$\beta+\gamma-1=\frac{p'}{p},\quad \kappa+\delta-1=\frac{q'}{q},$$

where $p$, $p'$, $q$, $q'$ are integers, which will be in general positive, and let

$$\sigma^p=1-s,\quad \tau^q=t;$$

$\sigma$ and $\tau$ may be supposed small.

We have

$$\sigma^{p'}.F_{3,2-\kappa-\delta}(2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ \sigma^p,\ \tau^q)$$

$$=\frac{BC}{DK}.\tau^{q'}.F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \kappa,\ \kappa+\delta,\ 1-\gamma,\ \tau^q,\ \sigma^p).$$

This may be written in the form

$$f(\sigma,\tau)\equiv(p'.0)\,\sigma^{p'}+(0.q')\,\tau^{q'}+\ldots=0,$$

where $(m, n)$ denotes the coefficient of $\sigma^m\tau^n$, the series being absolutely convergent for small values of $\sigma$ and $\tau$.

First, suppose that $p'$ and $q'$ are both unity, *i.e.*, that $\beta+\gamma-1$ and $\kappa+\delta-1$ are aliquot parts of unity. Then, if $f(\sigma,\tau)=0$ be considered as a curve, the origin is an ordinary point, and $\sigma$ can be expanded in a series of the form

$$c_0\tau+c_1\tau^2+\ldots,$$

convergent for sufficiently small values of $\sigma$ and $\tau$. Similarly $\tau$ can be expanded in a like series of powers of $\sigma$.

Next, let one of the quantities $p'$, $q'$ be unity, $p'$ say. Then $\sigma$ can be expanded in a converging series of integral powers of $\tau$, the exponent of the first term being $q'$. But $\tau$ cannot be expanded in a similar power-series of $\sigma$. Thus, in both these cases, we can obtain an expansion for $1-s$ in a series of fractional or integral powers of $1-t$.

Another equation connecting $s$ and $t$ is equation (2) of (B) viz.,

$$F_{3,2}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$$

$$=\frac{AB}{KA}\cdot F_{3,2}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\kappa,\ 1-\delta,\ 1-s,\ s-t).$$

Substituting in this equation the expansion for $1-s$ in terms of $t$, we have an equation for $t$ in the form of an infinite series.

Now suppose that neither $p'$ nor $q'$ is unity. Then the origin is a singular point of the curve $f(\sigma,\tau)=0$, and $\sigma$ and $\tau$ have respectively $p'$-ple and $q'$-ple values. There are, however, $p'$ different convergent expansions for $\sigma$ of the form

$$\sigma=\Sigma C_r\tau^r,$$

where the exponents $r$ form a series of increasing positive fractional numbers (for the theorem proved in Chrystal's *Algebra*, Vol. II., ch. xxx., (*d*), § 22, for an algebraic function, can be extended to a sufficiently converging infinite series).

Hence for a singular point, by a process such as the use of Newton's parallelogram, corresponding to that employed for an algebraic equation, we can obtain expansions of $\sigma$ in fractional powers of $\tau$, and consequently expansions of $1-s$ in powers of $t$. These can be substituted in equation (2) of (B), or in any other of equations (B) which may be convenient.

In the special case of a pentagon which is symmetrical about a line joining $A$ to the middle point of $CD$, we have

$$\beta=\kappa,\quad \gamma=\delta,\quad AB=AK,\quad BC=DK.$$

The equation (1) of (B) gives $1-s=t$,

and B (2) will be identically satisfied. Combining equations (i.) and (ii.) on p. 698, we have

$$F_{3,2}(\alpha+\beta+\kappa-1,\ \alpha,\ \alpha+\beta,\ 1-\gamma,\ t,\ s-t)$$

$$=\frac{AB}{BC}(1-s)^{\beta+\gamma-1}\cdot F_{3,2-\kappa-\delta}\ 2-\kappa-\delta,\ \beta,\ \beta+\gamma,\ 1-\delta,\ 1-s,\ t);$$

*i.e.*, $F_{3,2}(\alpha+2\beta-1,\ \alpha,\ \alpha+\beta,\ 1-\gamma,\ t,\ 1-2t)$

$$= \frac{AB}{BC}\, t^{\beta+\gamma+1} F_{3,2-\beta-\gamma}(2-\beta-\gamma,\ \beta,\ \beta+\gamma,\ 1-\gamma,\ t,\ t)$$

or $$\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \Big\{ F(2-2\gamma,\ \alpha,\ \alpha+\beta,\ t)$$

$$+ (1-\gamma)\,\frac{\alpha}{\alpha+\beta}\, F(3-2\gamma,\ \alpha+1,\ \alpha+\beta+1,\ t)(1-2t)$$

$$+ \frac{(1-\gamma)(2-\gamma)}{2!}\ \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\times F(4-2\gamma,\ \alpha+2,\ \alpha+\beta+2,\ t)(1-2t)^2 + \dots \Big\}$$

$$= \frac{AB}{BC}\, t^{\beta+\gamma-1}\, \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$\times \Big\{ F(2-\beta-\gamma,\ \beta,\ \beta+\gamma,\ t) + (1-\gamma)\, F(3-\beta-\gamma,\ \beta,\ \beta+\gamma,\ t)\, t$$

$$+ \frac{(1-\gamma)(2-\gamma)}{2!}\, F(4-\beta-\gamma,\ \beta,\ \beta+\gamma,\ t)\, t^2 + \dots \Big\},$$

an equation for $t$ in the form of a power-series.

If $$\beta+\gamma = 1,$$

that is, if $AB$ and $CD$ are parallel, the powers of $t$ are all integral.

If the pentagon is regular, the equation becomes

$$F(\tfrac{4}{5},\ \tfrac{3}{5},\ \tfrac{6}{5},\ t) + \tfrac{2}{5}\cdot\tfrac{3}{6}\cdot F(\tfrac{9}{5},\ \tfrac{8}{5},\ \tfrac{11}{5},\ t)(1-2t) + \dots$$

$$= t^{\frac{1}{5}} \Big\{ F(\tfrac{4}{5},\ \tfrac{3}{5},\ \tfrac{6}{5},\ t) + \tfrac{2}{5}\cdot F(\tfrac{9}{5},\ \tfrac{3}{5},\ \tfrac{6}{5},\ t)\, t + \dots \Big\}.$$

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*On certain Regular Polygons in Modular Network.* *By* L. J. Rogers. Received June 7th, 1898; in revised form, August 29th, 1898. Read June 9th, 1898.

1. In connexion with the theory of modular functions, it is proved that a plane may be entirely covered with a network of circles whose interstices are curvilinear triangles equiangular to one another whose angles are all sub-multiples of two right angles. These networks are of three kinds, according as the sum of the angles in each triangle is greater, equal, or less than two right angles. In the

first case, the triangles are limited in number and diverge. They can, moreover, be stereographically projected on to the surface of a sphere, which is thereby divided into equal spherical triangles (see Klein, *Lectures on the Icosahedron*, or Klein-Fricke, *Theorie der elliptischen Modulfunctionen*). In the second case, the network consists of plane triangles, and the circles degenerate into straight lines. In the third kind, the circles all intersect a certain circle orthogonally, and the portions of them internal to this circle form a network which converges to it, while the external portions form a network inverse to the former and re-diverging to infinity (see Klein-Fricke, Vol. I., p. 108). The network of the third kind is the subject of the present paper.

It is easy to see that a property analogous to the stereographic property of the first kind also exists for the third kind. For, if

$$x^2+y^2=1$$

be the orthogonal circle, and

$$a(x^2+y^2+1)+2gx+2fy=0, \quad a'(x^2+y^2+1)+2g'x+2f'y=0$$

be any two circles of the network, then they intersect at an angle

$$\cos^{-1}\frac{gg'+ff'-aa'}{\sqrt{g^2+f^2-a^2}\,\sqrt{g'^2+f'^2-a'^2}},$$

so that, if imaginary lines be drawn whose direction-cosines with respect to three rectangular axes in space are proportional to $(g, f, ai)$ and $(g', f', a'i)$, then planes through the origin perpendicular to them will meet a sphere $x^2+y^2+z^2=1$ in imaginary great circles which intersect in the same real angle as the above circles. The network is thus projected on to an imaginary part of the sphere, and divides it into imaginary spherical triangles. The method, however, has obvious drawbacks, as we cannot re-project any plane section of the sphere back into a real circle unless the normal to the plane has direction cosines of the form $(\lambda, \mu, \nu i)$, where $\lambda$, $\mu$, $\nu$ are real.

However, the triangles of the network possess a trigonometry with formulæ analogous to those in spherical trigonometry, and which we may easily establish by means of homographic complex transformation, as is indicated by Poincaré in *Act. Math.*, Vol. I., p. 6, &c.

Three portions of network are given in the diagrams accompanying this paper, viz., Fig. 1, in which the angles of every triangle are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$; Fig. 2, in which they are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{9}$; and Fig. 3, in which they are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$. In general, a network with angles

$\frac{\pi}{\nu_1}$, $\frac{\pi}{\nu_2}$, $\frac{\pi}{\nu_3}$ will be alluded to as $(\nu_1, \nu_2, \nu_3)$, and in the purely geometric treatment of the subject, irrespective of its connexion with modular functions, it is immaterial in what order we write the symbols. We shall, however, always assume that the centre of the orthogonal circle $(x^2+y^2=1)$ is a node of the net.

Nodes of the net will be called $\nu_1$-points, &c., according as the circles meeting there intersect at angles $\pi/\nu_1$, &c. The origin we shall take as a $\nu_3$-point, and the initial line of coordinates will be one of the diameters of the circle $x^2+y^2=1$ belonging to the network.

If we trace along any circle of the net, it is easy to see the law for the order in which the kinds of nodes follow one another, viz., that a node of even order (*i.e.*, where $\nu$ is even) lies between two like nodes, but a node of odd order lies between two unlike.

Thus, if $\nu_1$, $\nu_2$, $\nu_3$ are all odd, they follow one another in rotation.

If, however, one is even (say $\nu_1$), and $\nu_2$, $\nu_3$ are odd, the order is $\dot{\nu}_1\nu_2\nu_3\nu_1\nu_3\dot{\nu}_2$, where dots have the same significance as in circulating decimals. Moreover, this order, or this reversed, is the only possible one.

If two are even (say $\nu_1$, $\nu_2$), while $\nu_3$ is odd, some circles only contain $\dot{\nu}_1\dot{\nu}_2$, while others contain $\dot{\nu}_3\nu_1\nu_3\dot{\nu}_2$.

If the $\nu$'s are all even, the orders are $\dot{\nu}_1\dot{\nu}_2$, $\dot{\nu}_2\dot{\nu}_3$, $\dot{\nu}_3\dot{\nu}_1$.

2. It will be convenient to write down for reference the formulæ connecting the sides and angles of any one of these curvilinear triangles which compose the modular network, as indicated by Poincaré in the *Act. Math.*, Vol. I., p. 6. The formulæ in the two following sections will therefore be stated without proof.

The principle involved in the modular function theory is that any circles of the network can be conformally transformed into any other on which the nodes lie in the same order as on the original circle, by the homographic relation $\zeta=(za+b)/(zc+d)$, where $z$ is the complex of any point on the first circle and $\zeta$ its copy on the second, and where the orthogonal circle $x^2+y^2=1$ is transformed into itself.

The most general form for such a substitution is

$$\zeta = e^{\psi i}\frac{z-re^{\phi i}}{zr-e^{\phi i}}, \qquad (1)$$

where $r$ is real.

If a circle cut the unit orthogonal circle in points whose complexes

are $x$ and $y$, the complex of any point on it is of the form

$$\sqrt{xy}\,\frac{\kappa\sqrt{x}+i\sqrt{y}}{\kappa\sqrt{y}+i\sqrt{x}}, \tag{2}$$

where $\kappa$ is some real quantity.

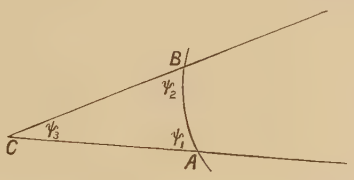
It will be convenient to allude to $\kappa$ as the *parameter* of the point with respect to the circle.

If $\kappa$, $\lambda$ are the parameters of two points on any circle of the system, then the ratio $\lambda/\kappa$ is unaltered for any transformation (1).

The natural logarithm of this ratio will be called the *parametral distance* between the points $\lambda$ and $\kappa$ on the circle. It is called by Poincaré the "$L$" of the arc.

3. The trigonometrical formulæ connecting the angles $\psi_1$, $\psi_2$, $\psi_3$ of a curvilinear triangle with the parametral lengths $u_1$, $u_2$, $u_3$ of the opposite sides can be obtained by supposing the triangle to be made up of two diameters of the orthogonal circle and a circle of the network which lies nearest to the origin.

If $CA$ is the axis of real quantities, the $x$ and $y$ in § 2, (2), are 1




and $-1$; the parameter of $C$, the origin, is unity, and the parameter of $A$ is given by

$$CA = \frac{\kappa-1}{\kappa+1} = \frac{e^{u_2}-1}{e^{u_2}+1}$$

By elementary geometry, we have

$$\frac{\sinh u_1}{\sin\psi_1} = \frac{\sinh u_2}{\sin\psi_2} = \frac{\sinh u_3}{\sin\psi_3}. \tag{1}$$

$$\cosh u_1 = \frac{\cos\psi_1 + \cos\psi_2\cos\psi_3}{\sin\psi_2\sin\psi_3}, \tag{2}$$

$$\begin{aligned}\sinh u_1 \sin\psi_2 \sin\psi_3 &= \sqrt{2\cos\psi_1\cos\psi_2\cos\psi_3 + \cos^2\psi_1 + \cos^2\psi_2 + \cos^2\psi_3 - 1} \\ &= n \text{ (say)},\end{aligned} \tag{3}$$

$$\cosh(u_1+u_2+u_3) = \frac{\cos\psi_1+\cos\psi_2+\cos\psi_3-1}{4\sin\frac{1}{2}\psi_1\sin\frac{1}{2}\psi_2\sin\frac{1}{2}\psi_3}. \tag{4}$$

4. The object of the present paper is to investigate under what conditions it is possible that a network should contain a parametrally regular polygon whose angles are all equal to one of the angles $\frac{\pi}{\nu_1}$, $\frac{\pi}{\nu_2}$, $\frac{\pi}{\nu_3}$, and whose sides traced in the same order are conformally equivalent, and are composed of the shortest circular arcs connecting conformal nodes (the angular points of the polygon).

The method employed will be to suppose that two adjacent diametrical lines of the network shall be adjacent sides of the polygon, which may afterwards be transformed so as to have its centre at the origin and thus appear in a symmetrical shape.

Such a polygon, having an angle $\frac{\pi}{\nu}$, I propose to call a *primary* polygon; while a polygon having angles equal to $\frac{m\pi}{\nu}$ will be called *m-ary* polygons. The existence of primary polygons is proved in Klein's *Modulfunctionen*, Vol. I., p. 369, where

$$\psi_1, \psi_2, \psi_3 = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7};$$

also by R. Fricke, *Math. Ann.*, Vol. XLI., p. 458, for the same values of the angles.

Secondary polygons, moreover, exist, as is shown by R. Fricke in *Act. Math.*, Vol. XVII., p. 373, for the network (2, 4, 5), but the aim of the present paper is to prove the existence, and to find the limit to the number, of primary polygons only.

To develop the theory, we need reference to the general treatment of the substitution § 2, (1).

It is shown in Klein's *Modulfunctionen*, Vol. I., p. 163, and in Forsyth's *Theory of Functions*, p. 512, &c., that the substitution in question may be reduced to the form

$$\frac{\zeta - z_1}{\zeta - z_2} = k \frac{z - z_1}{z - z_2},$$

where $z_1$, $z_2$, the fixed points of the substitution, are

$$\frac{1}{r} e^{\frac{1}{2}(\theta+\phi)i} \left\{ \cos \tfrac{1}{2}(\theta-\phi) \pm \sqrt{\cos^2 \tfrac{1}{2}(\theta-\phi) - r^2} \right\}, \qquad (1)$$

while

$$k = \frac{1}{1-r^2} \left\{ \sin \tfrac{1}{2}(\theta-\phi) + i \sqrt{\cos^2 \tfrac{1}{2}(\theta-\phi) - r^2} \right\}^2. \qquad (2)$$

If

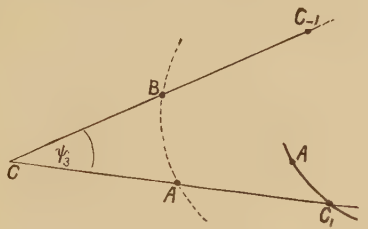
$$r^2 < \cos^2 \tfrac{1}{2}(\theta-\phi),$$

this value of $k$ is a complex of form $e^{\psi i}$, and the substitution is *elliptic*, while the fixed points are mutually inverse with respect to $x^2+y^2=1$.

If $$r^2 > \cos^2 \tfrac{1}{2}(\theta-\varphi),$$

$k$ is real, while the moduli of the fixed points are unity; that is, the substitution is hyperbolic, and the fixed points lie on the orthogonal circle.

Returning to the subject of the polygons, we see that, if $CA \dots C_1$ be the initial line and $CB \dots C_{-1}$ the next diameter in the direction of positive rotation, then the order of nodes from $C_{-1}$ to $C$ must be the same as from $C$ to $C_1$.




A substitution can then be found to bring $C_{-1}$ to $C$ and $C$ to $C_1$. Now $CC_1 = CC_{-1}$, for the parametral lengths are equal, and such lengths measured along straight lines from the origin imply equality of actual distance also.

If, then, $CC_1 = \lambda$, the substitution required is

$$\zeta = \frac{z-\lambda e^{\psi_3 i}}{z\lambda - e^{\psi_3 i}} \quad \text{(say } S\text{)}, \tag{3}$$

since it brings $C\,(z=0)$ to $C_1\,(z=\lambda)$ and $C_{-1}\,(z=\lambda e^{\psi_3 i})$ to $C$.

The same substitution $S$ will convert $CC_1$ into the nearest circle $C_1A_1$, which passes through $C_1$, making an angle of $\psi_3$ with $CC_1$ towards $C$, and a continued application of $S$ produces a series of circles forming a parametrally regular polygon having angles all equal to $\psi_3$, in fact, a polygon of the type we are seeking.

The only condition for the existence of a closed polygon is that $S$ should be elliptic.

This condition is, by (2), that

$$\lambda^2 < \cos^2 \tfrac{1}{2}\psi_3,$$

in which case, $$k = e^{2\psi i} \quad \text{(say)}$$

$$= \frac{1}{1-\lambda^2}\,(\sin \tfrac{1}{2}\psi_3 + i\sqrt{\cos^2 \tfrac{1}{2}\psi_3 - \lambda^2})^2,$$

*i.e.*, $$\cos\psi = \frac{\sin \tfrac{1}{2}\psi_3}{\sqrt{1-\lambda^2}}. \tag{4}$$

Conversely, if this requirement is fulfilled, $S$ will be elliptic, and must belong to the group of substitutions which leave the network unaltered. Hence $\psi$ must be a multiple of either $\psi_1$, $\psi_2$, or $\psi_3$.

Let us first suppose the numbers $\nu_1, \nu_2 = \frac{\pi}{\psi_1}, \frac{\pi}{\psi_2}$ are both odd.

Then the period $CC_1$ is of three segments, and its parametral length is $u_1+u_2+u_3$, so that

$$\lambda = \tanh \tfrac{1}{2}(u_1+u_2+u_3).$$

The relation (4) then becomes

$$\begin{aligned}\cos\psi &= \sin\tfrac{1}{2}\psi_3 \cosh\tfrac{1}{2}(u_1+u_2+u_3)\\ &= \frac{\cos\psi_1+\cos\psi_2+\cos\psi_3-1}{4\sin\tfrac{1}{2}\psi_1\sin\tfrac{1}{2}\psi_2}, \quad \text{by § 3.} \qquad (5)\end{aligned}$$

The condition for a closed polygon is, therefore,

$$\cos\psi_1+\cos\psi_2+\cos\psi_3-1 < 4\sin\tfrac{1}{2}\psi_1\sin\tfrac{1}{2}\psi_2,$$

*i.e.*,
$$\cos^2\tfrac{1}{2}\psi_3 < (\sin\tfrac{1}{2}\psi_1+\sin\tfrac{1}{2}\psi_2)^2$$

or
$$\cos\tfrac{1}{2}\psi_3 < \sin\tfrac{1}{2}\psi_1+\sin\tfrac{1}{2}\psi_2, \qquad (6)$$

since all the angles are acute.

The condition gives a very limited number of cases :—

Case I.—If $\psi_1 = \psi_2 = \frac{\pi}{3}$,

(6) is satisfied, and $\cos\psi = \cos\psi_3$,

so that for any network $(3, 3, \nu)$ we have a polygon of the type sought. Moreover, since

$$\psi = \psi_3 = \frac{\pi}{\nu},$$

the period of the substitution is $\nu$, so that the polygon has $\nu$ sides.

If $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{5}$,

then $\cos\tfrac{1}{2}\psi_3 < \frac{\sqrt{5}+1}{4} = \cos\frac{\pi}{5}$,

so that $\psi_3 > \frac{2\pi}{5}$.

The only possible value for $\psi_3$ is therefore $\frac{\pi}{2}$; but, since

$$\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{5} > \pi$$

(the icosahedral case), we exclude this type as not belonging to the network of the third species.

CASE II.—If $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{7}, \quad \psi_3 = \frac{\pi}{2},$

then $$\cos\psi = \frac{2\cos\frac{\pi}{7} - 1}{4\sin\frac{\pi}{14}} = \frac{\cos\frac{3\pi}{14}}{2\sin\frac{\pi}{7}} = \frac{\sin\frac{2\pi}{7}}{2\sin\frac{\pi}{7}} = \cos\frac{\pi}{7},$$

so that we here have a rectangular heptagon.

However, if $\psi_1 = \frac{\pi}{3}, \quad \psi_2 = \frac{\pi}{9},$

$$\cos\tfrac{1}{2}\psi_3 < \tfrac{1}{2} + \sin 10^\circ = \cdot 6736 < \cdot 7071 = \cos\frac{\pi}{4},$$

as is seen by tables, so that $\psi_3 > \frac{\pi}{2}$,

which is impossible.

*A fortiori*, if $\psi_1 = \frac{\pi}{3},$

$\psi_2$ cannot be less than $\frac{\pi}{9}$, so that we can exclude all values of $\psi_1$ and $\psi_2$, except $\frac{\pi}{5}$ and $\frac{\pi}{7}$.

Again, when $\psi_1 = \psi_2 = \frac{\pi}{5},$

$$\cos\tfrac{1}{2}\psi_3 < 2\sin 18^\circ = \cdot 6180 < \cos\frac{\pi}{4},$$

so that we may reject this case also, and all other cases in which $\nu_1$ and $\nu_2$ are greater than 5.

We have, therefore, investigated all cases in which $\nu_1$ and $\nu_2$ are odd, and have shown that $(3, 3, \nu)$ and $(3, 7, 2)$ are the only networks for odd $\nu_1$ and $\nu_2$ in which primary polygons occur. Also, when $\nu_3$ is even, $\nu_1$ and $\nu_2$ must be odd, since otherwise we cannot have the same set of nodes on the two lines $CA$ and $CB$. For along $CA$, if $\nu$ is even, they follow the order $\nu_3\,\nu_1\,\nu_3\,\nu_1\ldots$, and along $CB$, if $\nu_2$ is odd, the order $\nu_3\,\nu_2\,\nu_1\,\nu_2\,\nu_3\ldots$.

We will now proceed to the case in which $\nu_1$ is even.

Here there are six segments of the line $CC_1$ before arriving at $C_1$ (§ 1), viz., two of each kind, so that the parametral distance $CC_1$ is $2(u_1+u_2+u_3)$ and $\lambda = \tan(u_1+u_2+u_3)$.

This gives $$\sqrt{1-\lambda_2} = \operatorname{sech} 2u = \frac{1}{2\cosh^2 u - 1},$$

so that $\cos\psi = \sin\frac{1}{2}\psi_3 (2\cosh^2 u - 1)$,

which, by § 3, (4),

$$= \frac{c_1c_2c_3 + c_2c_3 + c_1c_2 + c_1^2 + c_2^2 + c_3^2 - c_1 - c_2 - c_3}{2\sin\frac{1}{2}\psi_3 (1-c_1)(1-c_2)}, \qquad (7)$$

where $c_1$, $c_2$ $c_3$ denote the cosines of $\psi_1$, $\psi_2$, $\psi_3$, respectively.

Now it is easy to show, by differentiating $\cos\psi$ partially with respect to $\psi_1$, $\psi_2$, $\psi_3$, that $\cos\psi$ increases when any of these angles decrease, so that we easily obtain a limit as to the smallness of these angles. For instance, when

$$\psi_1 = \psi_3 = \frac{\pi}{3} \quad \text{and} \quad \psi_2 = \frac{\pi}{4},$$

then $$\cos\psi = \frac{3+2\sqrt{2}}{2},$$

which is impossible, and consequently no case is possible unless at least one angle is greater than one of these. Thus the even denominator must be 2. Let, then,

$$\psi_1 = \frac{\pi}{2}.$$

Now, testing for $$\psi_2 = \psi_3 = \frac{\pi}{5},$$

we find $$\cos\psi = \frac{3\sqrt{5}+5}{4},$$

so that one of the odd denominators must be 3.

Case III.—Let, then, $\psi_2 = \frac{\pi}{3}$ (supposing $\psi_2 > \psi_3$).

We now have

$$\cos\psi = \frac{2\cos^2\psi_3 - \cos\psi_3 - \frac{1}{2}}{2\sin\frac{1}{2}\psi_3} = \frac{\cos\frac{5}{2}\psi_3}{2\sin\psi_3}.$$

If $\psi_3 = \frac{\pi}{7}$, then $\cos\psi = \frac{1}{2}$,

which, since the period of the substitution is 3, gives the triangle mentioned in Klein's *Modulfunctionen*, Vol. I., p. 369.

Case IV.—If $\psi_3 = \frac{\pi}{9}$, then

$$\cos\psi = \cos\frac{\pi}{9},$$

which gives a nine-sided polygon.

However, if $\psi_3 = \frac{\pi}{10}$, then

$$\cos\psi = \frac{\sqrt{5}+1}{2\sqrt{2}} > 1,$$

so that $\psi_3$ cannot be less than $\frac{\pi}{10}$.

Again, we may suppose that

$$\psi_2 < \psi_3 \quad \text{and} \quad \psi_3 = \frac{\pi}{3},$$

so that

$$\cos\psi = \frac{\cos\frac{5}{2}\pi_2}{4\sin\frac{1}{2}\psi_2 \sin\psi_2}.$$

If $\psi_2 = \frac{\pi}{7}$, then

$$\cos\psi = \frac{1}{4\sin\frac{\pi}{14}};$$

but, since

$$\sin^{-1}\tfrac{1}{4} = 14^\circ 29',$$

approximately, it follows that

$$\sin\frac{\pi}{14} < \sin^{-1}\tfrac{1}{4} \quad \text{and} \quad \cos\psi > 1.$$

Hence $\psi_2$ cannot be equal to $\frac{\pi}{7}$ or any smaller angle.

We see, then, that the only cases for networks in which only one of the numbers $\nu_1, \nu_2, \nu_3$ is even, and the corresponding angle is not the angle of the polygon, are (2, 3, 7) and (2, 3, 9).

We now come to the last case in which $\nu_1$, $\nu_2$ are both even.

Here the recurrence of the $\nu_3$-points on the line $CA$ (§ 3) takes place after four segments of parametral length $u_2, u_2, u_1, u_1$, so that

$$\lambda = \tanh(u_2+u_1);$$

whence

$$\cos\psi = \sin\tfrac{1}{2}\psi_3 \cosh(u_1+u_2).$$

Now

$$\cosh(u_1+u_2) = \cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2$$

$$= \frac{(c_2c_3+c_1)(c_1c_3+c_2)+n^2}{\sin\psi_1 \sin\psi_2 \sin^2\psi_3}$$

(by the formulæ in § 3)

$$= \frac{c_1c_2c_3+c_1c_2+c_1^2+c_2^2+c_3-1}{2\sin^2\frac{1}{2}\psi_3 \sin\psi_1 \sin\psi_2}.$$

Therefore

$$\cos\psi = \frac{c_1c_2c_3+c_1c_2+c_1^2+c_2^2+c_3-1}{2\sin\frac{1}{2}\psi_3 \sin\psi_1 \sin\psi_2}.$$

Again, in this case, by differentiating, we may show that $\cos\psi$ always increases when any of the angles $\psi_1$, $\psi_2$, $\psi_3$ diminish, so that,

on testing for

$$\psi_1 = \psi_2 = \frac{\pi}{4}, \quad \psi_3 = \frac{\pi}{3},$$

which gives $\cos\psi = \frac{5}{2}$,

we see that no cases are possible unless one of the angles, say

$$\psi_1 = \frac{\pi}{2} \quad \text{and} \quad \psi_2 = \frac{\pi}{4}.$$

Thus $$\cos\psi = \frac{c_3^2 + c_3 - 1}{2\sin\frac{1}{2}\psi_3 \sin\psi_2} = \frac{2\cos\psi_3 - 1}{2\sqrt{2}\sin\frac{1}{2}\psi_3} = \frac{\cos\frac{3}{2}\psi_3}{\sqrt{2}\sin\psi_3}.$$

Case V.—We omit the case $\psi_3 = \frac{\pi}{3}$,

(the octahedral case), since then

$$\psi_1 + \psi_2 + \psi_3 > \pi,$$

and put $$\psi_3 = \frac{\pi}{5},$$

which gives $$\cos\psi = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}.$$

We have then, for the network (2, 4, 5), a polygon of four sides.

Further than this we cannot go, for, even if $\psi_3 = \frac{\pi}{6}$, $\cos\psi = 1$;

and, hence, if $\psi_3 < \frac{\pi}{6}$, $\cos\psi > 1$.

Recapitulating, we see that the only possible primary polygons occur in the following networks:—

(3, 3, $\nu$), containing a polygon of $\nu$ sides for all integral values of $\nu$;
(3, 7, 2), „ „ heptagon;
(2, 3, 7), „ „ triangle;
(2, 3, 9), „ „ polygon of nine sides;
(2, 4, 5), „ „ quadrilateral.

If $\nu$ denote the number of sides in each case, it is evident that, by transforming the network conformally so as to bring a $\nu$-point to the origin, the $\nu$-gon becomes actually symmetrical instead of only parametrally so. However, in the cases (3, 3, $\nu$), (3. 7, 2), and (2, 3, 9) the origin was, in the first instance, a point of the required type, so that the network will contain the actually symmetrical polygon without further transformation.

Subjoined are figures illustrating the triangle (Fig. 1), the nonagon (Fig. 2), and the quadrilateral (Fig. 3) in their symmetrical position. The figure of the heptagon is given in *Math. Ann.*, Vol. xli., p. 458.

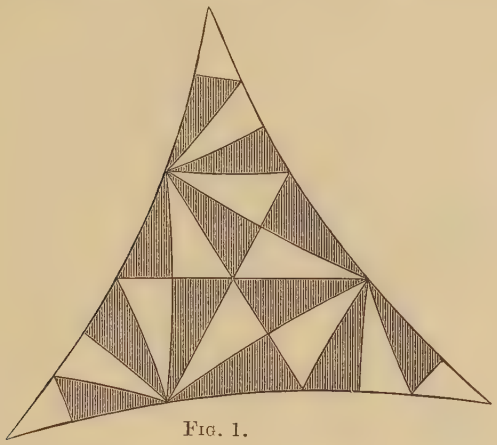


Fig. 1.

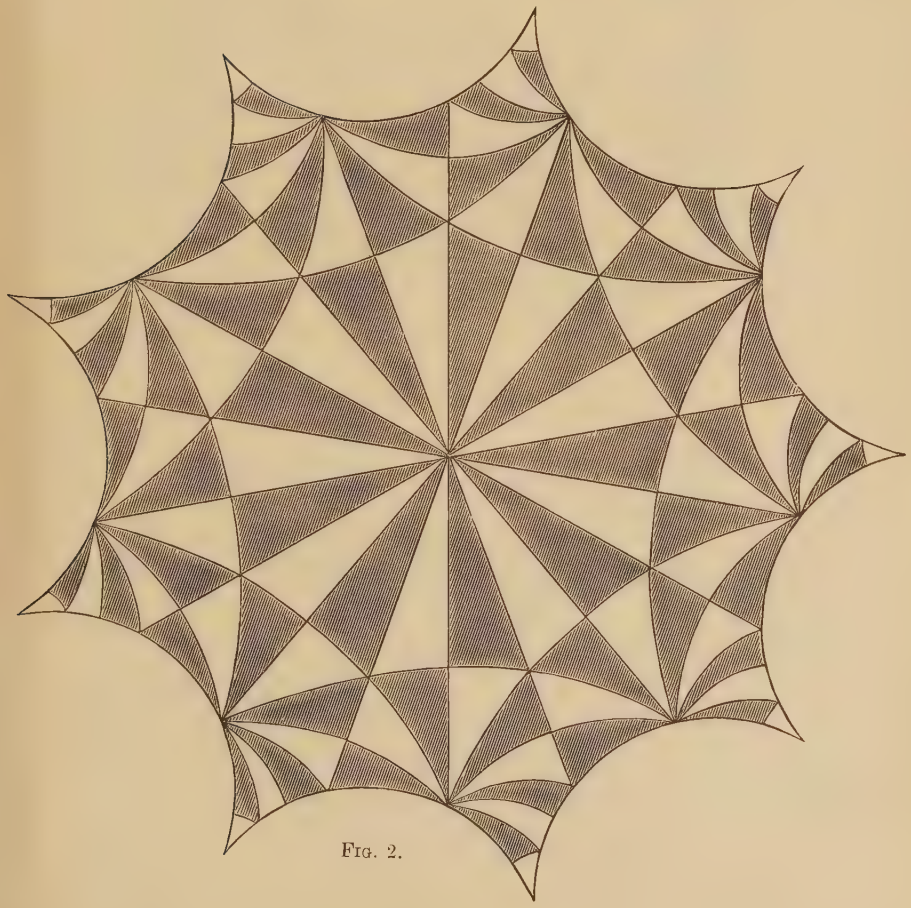


Fig. 2.

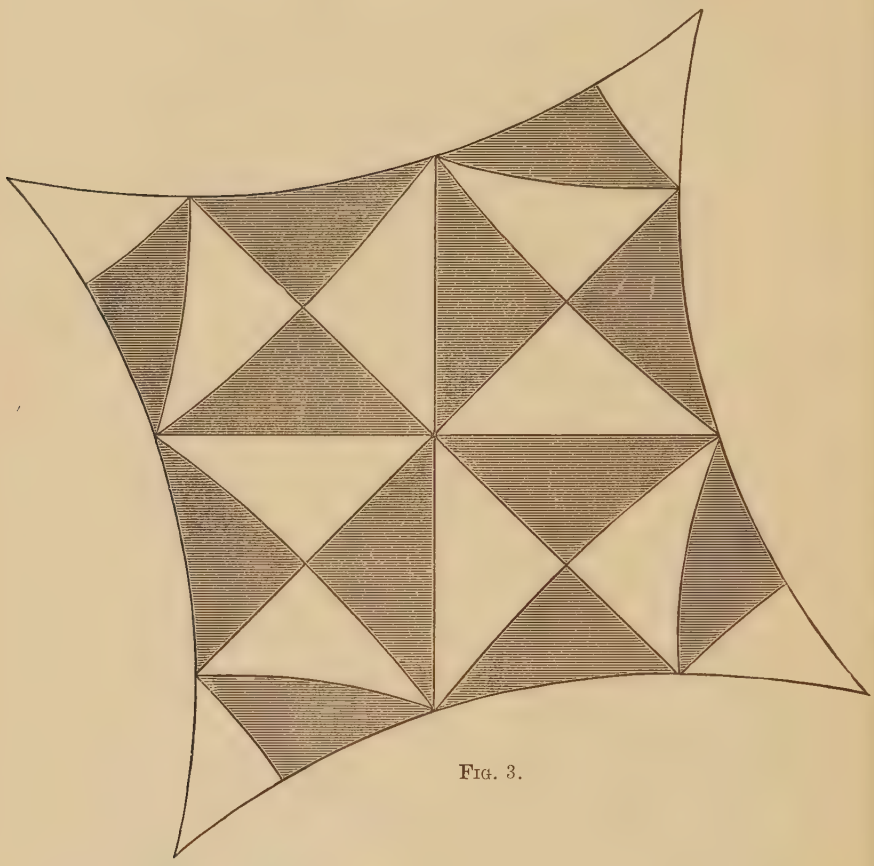


FIG. 3.

5. As an assistance in describing a network, it may be noticed that the centre of every circle, which contains the same order of nodes as the diameters, will be an external node conformal with the origin. Thus the centre $\frac{2xy}{x+y}$, and therefore the inverse $\frac{x+y}{2}$ of the centre of the circle $(x, y)$, will be a $\psi_3$-point. To prove this, we can first show that, if an elliptic substitution

$$\frac{\zeta-z_1}{\zeta-z_2} = e^{2\psi i}\frac{z-z_1}{z-z_2}, \tag{1}$$

where $z_1$, $z_2$ lie on $(x, y)$, transform $(x, y)$ into $(X, Y)$, then the square of the substitution will transform $\frac{x+y}{2}$ into $\frac{X+Y}{2}$.

Let $\kappa$, K be the parameters of $z_1$ on $(x, y)$ and $(X, Y)$, respectively, so that

$$z_1 = \sqrt{xy}\,\frac{\kappa\sqrt{x}+i\sqrt{y}}{\kappa\sqrt{y}+i\sqrt{x}} = \sqrt{XY}\,\frac{\mathrm{K}\sqrt{X}+i\sqrt{Y}}{\mathrm{K}\sqrt{Y}+i\sqrt{X}},$$

and

$$z_2 = \sqrt{xy}\,\frac{\kappa\sqrt{x}-i\sqrt{y}}{\kappa\sqrt{y}-i\sqrt{x}} = \sqrt{XY}\,\frac{\mathrm{K}\sqrt{X}-i\sqrt{Y}}{\mathrm{K}\sqrt{Y}-i\sqrt{X}}.$$

Now (1) changes $x$ into $X$, and, substituting these values for $z$ and $\zeta$, we obtain

$$\frac{\mathrm{K}\sqrt{Y}-i\sqrt{X}}{\mathrm{K}\sqrt{Y}+i\sqrt{X}} = e^{2\phi i}\,\frac{\kappa\sqrt{y}-i\sqrt{x}}{\kappa\sqrt{y}+i\sqrt{x}}. \qquad (2)$$

Again, if

$$z = \frac{x+y}{2},$$

we easily get

$$\frac{z-z_1}{z-z_2} = \left(\frac{\kappa\sqrt{y}-i\sqrt{x}}{\kappa\sqrt{y}+i\sqrt{x}}\right)^2;$$

so that, by (2), we see that

$$\frac{\frac{1}{2}(X+Y)-z_1}{\frac{1}{2}(X+Y)-z_2} = e^{4\phi i}\,\frac{\frac{1}{2}(x+y)-z_1}{\frac{1}{2}(x+y)-z_2},$$

which shows that the square of (1) brings $\frac{x+y}{2}$ to $\frac{X+Y}{2}$.

Now, for a diameter, $\frac{x+y}{2}$ is the origin, and, as we arrive at any circle conformal with a diameter by a series of rotations about nodes, the inverses of the centres are successively derived from the origin by the series of the squares of the same rotations, and are therefore all conformal with the origin.

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The following presents were made to the Library during the Recess :—

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# APPENDIX.

The Society's loss, through the deaths of its members, has been again a large one. For the following notice of Signor Francesco Brioschi we are indebted to documents put at our disposal by Prof. Luigi Cremona.

La sera del 13 mese corrente moriva in Milano, sua città natale, il nostro collega Francesco Brioschi, dopo breve malattia, aggravata forse dalla fiducia inspiratagli dalla costante immunità. La sua morte, del tutto impreveduta, ci ha colpiti del più doloroso stupore: sebbene egli fosse alla vigilia del compimento del suo 73$^{mo}$ anno, a vederlo così vivace ed agile, forte e grigio come acciaio, nessuno avrebbe dubitato che la sua operosità non fosse per durare ancora molti anni: tanta era in lui la robustezza della fibra e della mente, che non avevano mai conosciuto stanchezza. Era nato il 22 dicembre 1824, aveva fatto gli studi universitari a Pavia e vi si era laureato ingegnere con onore, nel 1845. Ad innamorarlo vieppiù dell' alta matematica contribuì Gabrio Piola, il dotto commentatore delle opere di Lagrange.*

* From the *Resoconto Sommario* (Senato del Regno), Lunedi, 20 Dicembre 1897, the President, for the time, being Prof. Cremona. The rest of the speech treats rather of Brioschi as a public man than as a mathematician.

The following estimate of Brioschi, as a mathematician, is taken from that furnished by Signor Beltrami to *La Perseveranza* of Thursday, December 23rd, 1897.*

Straordinaria versatilità d'ingegno, singolare potenza di assimilazione rapida e profonda, instancabile tenacia in ogni ricerca come in ogni intrapresa, tali furono le doti onde rifulse la mente, davvero eccelsa, di Francesco Brioschi.

Nelle scienze matematiche poi, in cui egli raggiunse più presto la gloria, e gloria solida e duratura, recò un' altra preziosissima dote, quella che or si direbbe una impareggiabile virtuosità, cioè una agilità elegante di forma e di pensiero, la quale, se sgomentava gli impazienti ed i meno provetti, formava l' ammirazione dei dotti e degli studiosi di lena.

Allevato alla scuola di Bordoni nel culto, forse un po' troppo esclusivo, dei metodi lagrangiani, entrò poco appresso con Piola nell' ambiente meno rigido delle ricerche fisico-matematiche di Fourier, di Poisson e di Cauchy. Ma, per quanto grande e feconda fosse allora la produzione matematica francese, che era la sola cui attingessero i pochi studiosi d' Italia, egli intuì ben presto la necessità d' allargare la cerchia delle fonti, estendendola alla produzione delle altre nazioni colte d' Europa, massimamente della Germania e dell' Inghilterra. Fra noi egli fu indubbiamente il primo a mettersi risolutamente per questa via, e ad indirizzarvi quanti allievi e studiosi potè attirare con sè in quest' opera, che può ben dirsi di risanamento, giacchè soltanto per essa cessò quel tal quale ristagno che da lungo tempo pesava sulla scienza italiana, e incominciò quel sempre più attivo e fecondo ricambio intellettuale colla scienza e cogli scienziati di fuori, che fu certamente favorito e promosso dalla fortunata ricostituzione dell' unità nazionale, ma che sarebbe ingiustizia non revocare a lui, per ciò che spetta alle sue prime origini, ben più modeste, ma ben più laboriose.

È incredibile la quantità di lavori che il Brioschi seppe comporre e produrre in luce, nei più svariati indirizzi, non appena si fu rapidamente orientato nel vastissimo campo delle ricerche che occupavano, alla metà di questo secolo, i più valorosi matematici d' Europa. La recente scomparsa del grande Jacobi, col quale il Brioschi aveva

* The same writer furnishes a "Necrologia" to the *Annali di Matematica*, Tomo XXVI., Fasc. 4, pp. 343-347. *Cf.* also "Francesco Brioschi, Cenno Necrologico per Alfredo Capelli," in the *Giornale di Matematiche*, Vol. XXXVI., Jan.–Feb., 1898, p. 51.

tanta affinità di temperamento scientifico, richiamava allora la attenzione sul grande problema delle equazioni dinamiche, e fu questo uno dei primi soggetti a cui egli si rivolse, trattovi anche dalla natura dell' insegnamento che impartiva all' Università di Pavia.

Numerosi ed apprezzatissimi sono i suoi lavori, sia sul problema d' integrazione, sia sulle affini teorie delle equazioni a derivate parziali e delle equazioni isoperimetriche. Nè cessò mai, anche a maggior distanza di tempo, di ritornare su quei primi studii con altre geniali pubblicazioni, come, a cagion d' esempio, con quelle relative all' ellissoide fluido di Dirichlet ed al problema dei tre corpi.

Altro argomento di numerose ed interessantissime pubblicazioni fu la teoria analitica delle superficie, rimessa allora in onore da una celebre Memoria di Gauss, che era passata per lungo tempo inosservata, ma di cui i geometri riconoscevano finalmente la fondamentale importanza. Il Brioschi prese parte grandissima allo svolgimento (divenuto poi sempre più largo e più complesso) di questa teoria, e vi arrecò più d' un contributo essenziale, tra altro col concetto di coordinate curvilinee tangenziali, da lui primamente adombrato in una nota sulla superficie delle onde.

Non volle rimanere estraneo agli studi di pura geometria, il cui decisivo risveglio risale a un dipresso alla medesima epoca, benchè l' indole peculiare del suo ingegno lo chiamasse di preferenza alle ricerche di pura analisi; e fu felicissimo nella trattazione di quelle questioni in cui l' una e l' altra disciplina gareggiano nel raggiungere una stessa meta, del che basterà citare l' esempio fornito dai poligoni di Poncelet.

Ma l' indirizzo in cui il Brioschi si slanciò con vera passione e con istraordinario successo fu quello delle ricerche sulle equazioni algebriche e sui nuovi algoritmi, che si riassumono nell' uso sistematico dei determinanti, degli invarianti, dei covarianti e delle forme algebriche e simboliche. Basterebbe già il libro dei Determinanti, che risale ai primissimi anni della sua carriera scientifica e che fu tradotto in pressochè tutte le lingue colte, per constatare le eminenti sue doti d' assimilazione e d' invenzione, como il profondo e sicuro possesso d' ogni più disparato dominio dello scibile matematico. Ma sarebbe impossibile analizzare anche sommariamente, senza entrare in particolari troppo disformi dall' indole d' un giornale, l' infinita copia di nuove vedute, di nuove proposizioni, di nuovi procedimenti che si trovano disseminati nelle numerosissime Memorie di lui sulle indicate teorie, fra le quali basterà menzionare quella Monografia sulle forme binarie che doveva riassumere gran parte dei suoi studii e che,

sebbene rimasta incompieta, contiene pur tuttavia un ricco tesoro di materiali preziosi. E, per citare almeno uno dei moltissimi argomenti speciali in cui maggiormente brillarono l' acume e la genialità del Brioschi, giovi ricordare le sue elegantissime ricerche sulle serie analoghe a quella di Sturm.

Per ciò poi che spetta alla dottrina delle equazioni algebriche, basti il dire che, nella memorabile scoperta della risoluzione dell' equazione di 5° grado, il nome di Brioschi è indissolubilmente legato a quelli di Hermite e di Kronecker, con questo di più, ch' egli non ha poi mai cessato di illustrare con nuove ricerche questo campo cosi irto di difficoltà, preparando il terreno e partecipando attivamente ad altri non meno cospicui progressi.

Un altro larghissimo campo di studii ai quali, non meno che ai precedenti, il Brioschi si trovò spontaneamente attratto dalle sue peculiari attitudini e preferenze scientifiche, e che del resto si collegava necessariamente ed intimamente coll' ultimo dei dianzi accennati, fu quello delle funzioni trascendenti, inaugurato da Legendre e recato d' un tratto a smisurate altezze dai lavori di Abel, di Jacobi e da quelli, allora recentissimi, di Weierstrass. Qui forse, più che altrove, il Brioschi era destinato a raccogliere una messe oltremodo feconda e rigogliose, la materia prestandosi mirabilmente a quel suo genio, ormai maturo, di analista supremamente classico, e già egli era entrato gloriosamente nell' arringo, ispirandosi ai lavori di Weierstrass, quando, sopravvenuti gli eventi del 1859, si trovò d' un tratto chiamato a spendere in altro modo le forze esuberanti del suo ingegno. Cosi si chiuse il periodo eroico della sua operosità scientifica, periodo che durò non più d' un decennio, ma che bastò a circondare per sempre il suo nome d' una aureola di gloria purissima così presso di noi, come presso tutte le culte nazioni, di cui in così breve tempo egli aveva saputo assimilare ed eguagliare la poderosa produzione scientifica.

Se tuttavia, in tutto il tempo successivo, egli non potè mai più consacrare alla scienza pura l' intera somma delle sue smisurate energie intellettuali, neppur cessò mai di tener sempre ed amorosamente fiso in essa lo sguardo, tornando ad ogni tratto, e più d' una volta abbastanza intensamente, al culto di essa, così da aggiungere molto al moltissimo già prodotto, e nulla trascurando di ciò che poteva, direttamente ed indirettamente, favorire la diffusione ed il progresso degli alti studii nel nostro paese. In quest' ultimo senso merita principalmente d' esser ricordata l' opera indefessa da lui data nel mantenere in vita dapprima, e nel recare poscia a rigogliosa

fioritura quella pubblicazione periodica che s' intitola *Annali di matematica pura ed applicata*, e che da non breve tempo rappresenta degnamente l' Italia fra le congeneri e più apprezzate pubblicazioni d' Europa e d' America. Già fin dal 1858, quando questo periodico sorse in Roma, in continuazione d' un altro più modesto che lo precedette, egli aveva contribuito moltissimo a dargli alimento e notorietà; ma nel 1867, quando la vita ne era divenuta alquanto languida e stentata, egli ne trasportò la sede da Roma a Milano, e ne assunse la direzione, dapprima insieme col collega Cremona, poi, dopo la partenza di questo, da solo. Nei trent' anni trascorsi dopo questo rinnovamento dell' antico periodico romano, ne sono apparsi in luce ben 26 volumi in-4°, ai quali collaborarono tutti i migliori matematici italiani e non pochi fra gli stranieri d' ogni nazione, attivando così anche fra noi quel ricambio d' ospitalità scientifica che già s' era iniziato altrove ed al quale il Brioschi aveva già tanto e così ampiamente contribuito coll' esempio e col consiglio.

È inutile dire a lungo dell' opera data, in un senso più universale, a pro della scienza italiana durante la lunga presidenza dell' Accademia dei Lincei. Niuno ignora come primo pensiero del Brioschi sia sempre stato quello di far convergere i maggiori mezzi possibili all' ampliamento delle pubblicazioni accademiche, così da potervi accogliere, come può ora ben dirsi che avvenge, ogni degna manifestazione degli studii nazionali.

Senonchè l' analisi delle varie, per non dire infinite forme sotto cui si estrinsecò l' ardore inestinguibile del Brioschi per tutto ciò che s' attiene agli studii ed agli studiosi del nostro paese, ardore che andò facendosi sempre più largo e comprensivo di ogni sana manifestazione del pensiero scientifico, mentre da un lato condurrebbe troppo lontano e sconfinerebbe dal campo prefisso, riuscirebbe forse dall' altro ad offuscare un cotal poco, specialmente agli occhi di chi non ebbe la ventura di conoscere da vicino l' uomo, l' immagine integra e distinta che di lui dobbiamo formarci, e che è bene rimanga scolpita indelebilmente nella storia degli intelletti d' Italia. La quale immagine è quella d' un forte campione della schietta stirpe latina, la cui mente, sovranamente equilibrata, fu sempre aperta ad ogni più alta aspirazione ideale come ad ogni più intimo bisogno di vertiginosa attività esterna, e che nell' opera sua, di qualunque natura si fosse, recò invariabilmente la benevolenza, la sicurezza, la serenità e, non ultima attrattiva, quell' amabile scioltezza che, nell' esercizio delle discipline e delle cose severe, lascia come spirare un sottile profumo di squisita artisticità.

Signor Brioschi, who was an Honorary Fellow of the Cambridge Philosophical Society, was President of the Reale Accademia dei Lincei, Rome, and Director of the Istituto Tecnico Superiore of Milan. He was elected an Honorary Foreign Member of our Society May 9th, 1878. His only communications to the Society were "Sur une Propriété du Paramètre de la Transformée Canonique des Formes Cubiques Ternaires," Vol. XII., pp. 58–63 (read January 13th, 1881); "Sur la Transformation des Equations Algébriques," Vol. XX., pp. 127–131 (read February 14th, 1889).*

The Council are indebted to Mr. H. M. Taylor for the following notice of Dr. Frost:—

Percival Frost, the second son of Mr. Charles Frost, F.S.A., a solicitor, of Hull, was born in that town on September 1st, 1817. His schooldays were spent at Beverley and Oakham. In October, 1835, he commenced residence at St. John's College, Cambridge. At Oakham he had shown great aptitude for languages, and had competed on equal terms with James Atlay, afterwards Bishop of Hereford, who was ninth in the Classical Tripos of 1840.

At Cambridge, Frost devoted his time chiefly to mathematics, and, for three years, ran an almost neck-and-neck race with B. M. Cowie, the present Dean of Exeter. They were elected Fellows of St. John's College on the same day, March 18th, 1839. In 1841 Frost was ordained, and for some years acted as a curate, but he never held any preferment in the Church. In 1841 he vacated his Fellowship at St. John's on his marriage, and settled down in Cambridge to the work of a private tutor. In this work he was at once popular and successful. Among his pupils, the one who rose to the highest eminence in scientific reputation was the late W. K. Clifford; from him Frost was always ready to admit that he had learnt much. Frost held a Mathematical Lectureship at Jesus College for twelve years, and afterwards at King's College for thirty years. The latter College showed their appreciation of his services as a Lecturer by electing him to a Fellowship in 1882.

Frost examined in the Mathematical Tripos for the first time in January, 1846, and for the last time in the divided Tripos which took place in June, 1882, and January, 1883. Altogether Frost examined

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* *Cf.* also *Proc.*, Vol. XI., p. 157, and Vol. XII., p. 215.

ten times, and his record has only been beaten by Dr. Ferrers, who has examined eleven times.

In 1854 Frost published his first work, a treatise on part of Newton's *Principia*, and, in 1863, in conjunction with the late Joseph Wolstenholme, *A Treatise on Solid Geometry*. Each of these works has passed through several editions, and both are still popular text-books. In 1872 Frost published a third work, *A Treatise on Curve-Tracing*, a work of greater originality, of which Professor Chrystal says, in speaking of the reversion of series and of the expansion in power-series of an algebraic function: "The applications are very numerous, for example, to the finding of curvatures and curves of closest contact, and to curve-tracing generally. A number of beautiful examples will be found in that much-to-be-recommended text-book." He was the author of a considerable number of papers, a list of which will be found in the Royal Society's Catalogue.

On December 9th, 1869, Frost was elected a member of this Society. He never served on the Council or contributed a paper to the Society's *Proceedings;* but he acted as a referee on two or three occasions. In 1882 Frost took the degree of Sc.D. at Cambridge.

It would be an injustice to Frost's memory to omit all reference to his wide interests, his many and varied accomplishments, and to the characteristic simplicity of his generous nature. The playful spirit of the man, who was always ready to see the humorous side of every situation and of every subject, peeps out in the prefaces to his books and occasionally in his mathematical papers.

Frost inherited a splendid constitution. When he completed his eightieth year, there seemed to be a probability of his attaining extreme old age. His mother had reached the age of ninety-two, and her uncle, who held the living of Holy Trinity, Hull, for seventy years, died in his ninety-eighth year. A painful disorder, however, which attacked him, proved fatal after six weeks' illness, and Frost died at Cambridge on Trinity Sunday, June 5th, 1898.*

The following Notice of Dr. Hopkinson has been put at the service of the Council by Mr. R. T. Glazebrook:—

John Hopkinson was the eldest son of Mr. Alderman Hopkinson, of Manchester. He was born in 1849, and educated at Lindon Grove

* An *in memoriam* sketch by Dr. H. A. Morgan was published in the *Cambridge Review* of June 16th, 1898.

School and Queenswood College until the age of fifteen and a-half years. After two and a-half years spent at Owens College, he entered at Trinity College, Cambridge. He graduated as Senior Wrangler in 1871, and was first Smith's Prizeman of his year, The same year he was elected a Fellow of the College. Early in 1872 he became associated with Messrs. Chance Bros. & Co. as engineer at their lighthouse works near Birmingham. After a residence of about six years in that city, he removed to London, and laid the foundations of his large consulting practice as an electrical engineer. He had taken his D.Sc. degree at London University while still at Cambridge. In 1878 he was elected a Fellow of the Royal Society, and in 1890 he received the Royal Medal for his researches in electricity and magnetism. During his work at Messrs. Chance's, Dr. Hopkinson introduced many improvements into lighthouse machinery, and papers dealing with some of these were communicated to the Royal Society; among them may be named that on the group-flashing apparatus described in the *Proceedings* of the Society for 1874. In the year 1877, the first of his papers on the "Residual Charge of the Leyden Jar" was published, and this was followed by others on the "Measurement of Specific Inductive Capacity." The importance in the theory of electro-magnetism of Maxwell's relation connecting the refractive index with the specific inductive capacity of a substance was early apparent to him, and his observations on glass (*Phil. Trans.*, 1878) were made with a view to verifying this. Experiment showed that, for the materials used—various specimens of flint glass—the specific inductive capacity is not equal to the square of the refractive index. These results were contradicted by the work of Gordon (*Phil. Trans.*, 1879, Pt. 1.), and Hopkinson, in 1880 (*Phil. Trans.*, 1881, Pt. 2), repeated his observations, modifying, however, his method so as to use a rapidly alternating electromotive force—in his earlier experiments the electromotive force had been uniform in direction. Differences in the value of the specific inductive capacity were found in some cases; they were, however, small compared with the differences between his own results and those of Gordon, and Hopkinson's experiments showed conclusively that the large difference between them was due to the want of uniformity of the electric field in Gordon's apparatus, which causes errors greater than any one would suspect without actual trial. About the same time, the result of a series of experiments on the specific inductive capacity of liquids was made. These were extended in 1887 (*Proc. Roy. Soc.*, Vol. XLIII., 1887), and it appeared that, while for hydrocarbons and other liquids

the square of the refractive index is equal to the specific inductive capacity, for animal and vegetable oils it is not so. More recent research has fully confirmed the view that the discrepancy arises from the difference in the length of the waves for which the two quantities, inductive capacity and refractive index respectively, were necessarily measured, and Hopkinson's work has been greatly instrumental in establishing the above-mentioned fundamental relation of Maxwell's theory.

In handing the Royal Medal to Hopkinson in 1890, the President (Lord Kelvin) referred at length to his researches on the effect of temperature on the magnetic properties of iron and other metals. These researches were published in the *Phil. Trans.* for 1889. The experiments show that, for small magnetizing forces, the susceptibility of iron increases with rising temperature until a temperature varying from 690° to 870° C. in different specimens is reached. On approaching this critical temperature from below, the rise is very rapid; when the temperature has been passed, the iron ceases almost completely to be magnetic.

But Hopkinson's most important work is connected with the theory of dynamo-electric machinery. In one of the earliest of his papers on this subject (*Proc. Mech. Eng. Instit.*, 1879, p. 246; 1880, p. 266) he defines as the characteristic of a machine the curve which gives the relation between the electromotive force of the armature and the current round the magnets, and he shows how, from this curve, may be deduced "the result in all employments of the machine, whether as a motor, or to produce a current through a resistance, through an electric arc, or in charging accumulators; also the result of varying the winding of the machine, whether in the armature or the magnets."

In 1886 (*Phil. Trans.*, A, p. 331) appeared his paper on "Dynamo-electric Machinery," written in conjunction with his brother, Dr. E. Hopkinson. The paper, to quote from Prof. A. Gray when reviewing a collected edition of Hopkinson's papers on "Dynamo-electric Machinery" (*Nature*, July, 1893, p. 244), "has become justly famous as that in which his enormously useful idea of the magnetic circuit was first employed in a complete and consistent manner to the discussion of the results of experiments on different types of dynamo." One purpose, as the authors state, is to give "an approximately complete construction of the characteristic curve of a dynamo of given form from the ordinary laws of electro-magnetism and the known properties of iron, and to compare the result of such construction with the actual characteristic of the machine." This is done in a

masterly manner, and the results obtained are tested by experiment.

The latter part of the paper contains a method for measuring the efficiency of a dynamo. Two similar machines have their shafts coupled, and are connected electrically; one acts as a generator, the other as a motor. The motor helps to drive the generator which feeds it. If the two machines are similar, the only additional power necessary to drive the combination is the waste in the two dynamos and in the connexions. This is supplied from outside, and, being small compared with the power supplied from the generator, which can be measured electrically, can be determined by a dynamometer with sufficient accuracy. This idea of the Hopkinsons has borne fruit in various methods for testing dynamos and transformers since devised by others. The paper was completed in 1892 by one presented to the Royal Society by Dr. Hopkinson and E. Wilson. Another paper, published in 1884, dealt with the question of running alternating machines in parallel. In 1882 he invented and described the famous three-wire system of electrical distribution.

Hopkinson was the author of numerous other papers of importance, but sufficient has been written to show the fertility of his powers and the magnitude of his work. Moreover, he was known as a practical engineer no less than as a writer. He was connected with some of the most important of electrical undertakings; it may be sufficient to mention the City and South London Railway and the Manchester Electric Supply Station. At the time of his death he was engaged on the tramway systems of Leeds, Liverpool, and other towns. His work is distinguished by its clearness, its brevity, and its precision. Some of his papers are almost too condensed. At meetings of scientific societies, and in public, he did not usually speak at great length, but every word was to the point; he knew his own mind, and expressed his views concisely and clearly. The style in which he describes his experiments may be taken as a model by others; he states briefly the problem to be solved and then the method he intends to follow for its solution. Sufficient detail of each experiment is given to enable the reader to check and verify his results, and all is clearly arranged; there is no unnecessary verbiage, and no repetition. The loss to physical science by his tragic death is difficult for us who knew him to estimate; there are few whose influence was felt in so many fields of activity, few who are devoting themselves more completely to the advancement of their fellows. He was killed, with a son and two daughters, by a fall from the rocks of the Petite Dent de

Veisivi, at Arolla, on August 27 last; how the accident happened no one can say with certainty. Hopkinson himself, his son, and one of his daughters were mountaineers of exceptional skill and experience in rock climbing, and we may feel sure that no ordinary occurrence was the cause of so terrible a catastrophe. The scene in the beautiful cemetery at Territet, where they were buried the following week, will ever live in the minds of those who were present.

Dr. Hopkinson was elected a member of the Society, January 13th, 1873.

The Secretary is indebted to Mr. Joseph Edwards for the following brief notice of Mr. Walter Wren :—

One of the earliest members of our Society has passed away in the person of Mr. Walter Wren, whose election bears date October 16th, 1865.

Wren was the second son of Mr. Richard Wren, of Buntingford, and was born in 1831. He received his early education at Buntingford Grammar School and Elizabeth College, Guernsey, whence he proceeded to Christ's College, Cambridge, where he had for contemporaries a group of distinguished men, amongst whom were C. S. Calverley, Prof. Seeley, Sir Walter Besant, and Prof. Skeat. Wren's college career was blighted by his being attacked, whilst an undergraduate, by spinal disease, which prevented his taking an Honours degree. He, however, graduated in 1859, and proceeded to the M.A. degree in 1862.

His subsequent career has been a remarkable one, the more so in his success having been achieved in spite of a life of continuous suffering. His fame as a private tutor for candidates for the Government services is well known. For more than thirty years he has been engaged in preparing candidates for the Indian and Home Civil Service Examinations, as well as for Colonial Cadetships, the Army, the Woods and Forests Service, the Indian Police Service, &c. So great has been his success in this task that, at the present day, hundreds of the most distinguished members of the State Services owe their education and position to Mr. Wren's establishment.

Mr. Wren was an excellent judge of capacity. He always insisted upon thoroughness of work. He abhorred the acquirement of superficial knowledge in any subject, and would always urge with all the force of his powerful personality the importance of possessing a thorough mastery of all subjects taken up by his pupils.

As though the management of a great institution like his were not sufficient outlet for Wren's energy, we find him standing at the General Election of 1880, for Wallingford, where he was returned by a majority of forty-one. He was subsequently unseated on petition, but made several subsequent attempts to enter Parliament. He was a member of the first London County Council, and one of the founders of the National Liberal Club.

Wren died on August 5th, 1898, from paralysis, with which he had been seized only a week previously.*

Henry Perigal, who died on June 6th, 1898, in the ninety-eighth year of his age, came of a family noted for their longevity. He traced his descent from Sigurd, a Dane, who, after a successful raid into Normandy, in 908, settled there and assumed the name of Perigal. He was in the tenth generation from Gideon Perigal, and his wife Madeline Duval, of Dieppe, who as Huguenots escaped from prison and settled in London. Mr. Perigal's father, who reached the age of ninety-nine, was one of thirteen children, nine of whom reached a great age. Mr. Perigal, who was born April 1st, 1801, was the eldest of six, the youngest of whom, Mr. Frederick Perigal, is now in his eighty-seventh year. As a result of the great age to which his father lived, he was "Henry Perigal, junior," till he was an old man himself. In early life he was a clerk in the Privy Council Office for a short time, and subsequently he held for many years a position in the office of Messrs. Henry Tudor & Son, Stockbrokers, the senior partner of which firm was a personal friend of his. He was a member of the Astronomical, Microscopical, Meteorological, Photographic, and several other scientific Societies, and a very regular attendant at the meetings of all, usually staying to the end, and often being the last to leave; and it may be said that for thirty or forty years, during eight months of the year, he divided his time between daily office work at 29 Threadneedle Street and evening attendance at the meetings of scientific societies, the lectures at the Royal Institution, &c.

His connexion with the Meteorological Society was remarkable: the Society was founded in 1850; in May, 1853, he became Treasurer,

* [A letter from Sir Walter Besant in the *Times*, over the date August 7th, gives an interesting general sketch of Wren's work, founded upon "an almost life-long friendship."]

and retained this office up his death—for forty-five years—signing cheques and attending to the financial business of the Society almost to the last. When our Society was formed he would have liked to join it at once, but feared that his paradoxical views on the subject of astronomical motions would disqualify him; he was much pleased therefore when, some years later, De Morgan offered to propose him as a member and signed his nomination form.

As a young man Mr. Perigal was much interested in all kinds of lathe work, but especially in curve-tracing by means of Ibbetson's geometric chucks. To this subject he brought to bear great patience, perseverance, and delicacy of manipulation. Many of the curves so drawn by him between 1830 and 1840 are exquisite examples of fine geometrical tracings performed by means of the bow pen. He afterwards delineated mechanically a great number of bicircloidal curves, *i.e.*, curves described by two circular motions. These curves were generated by a point moving uniformly round a circle the centre of which also moved uniformly round another circle. The curve traced depended upon the ratio of the angular velocities in these two circles, this ratio being fixed during the description of the same curve. He also devised little apparatuses, which he called kinescopes, for exhibiting these curves to the eye as they were formed. The curve was traced by a bright point which, by a system of multiplying wheels, was made to move so rapidly as to exhibit a continuous bright line. He also occupied himself with curves generated by more than two circular motions, and as early as about 1835 had delineated the curves now generally known by the name of Lissajous, including the finite or retrogressive parabola, which he had obtained by points some years before he described it mechanically. He also displayed great ingenuity in proving geometrical theorems by means of dissection and transposition of parts. These dissections were mostly discovered between 1830 and 1840. Among the best are his proof of Euclid I. 47, and his decomposition of a square into three squares. Unfortunately nearly all his work was published merely for private distribution among his friends, and, as he had not the pen of a ready writer (as he himself used to say), his diagrams were generally issued without sufficient explanation, and sometimes without any at all. These fugitive pieces, which were being produced by him for sixty years or more, are very numerous. They consist principally of diagrams of bicircloidal curves (some on large sheets), of the "Lissajous" curves, and of dissections. Some of the dissections he made as puzzles, forming

them in card and wood, and gave to his friends. He also engraved them (as well as the finite parabola) upon some of his visiting cards. His proof of Euclid I. 47 was reproduced in Vol. II. of the *Messenger of Mathematics*.

Mr. Perigal's mechanical work in connexion with the lathe led him astray with respect to the motions of the moon and planets. As the Moon always turns the same face to the Earth, he considered that she should be regarded as subject to one simple circular motion, and not (as mathematicians said) to a motion of revolution round the Earth combined with one of rotation round her own axis. The point in dispute was not, as most people thought, merely a question of definitions, for Mr. Perigal assumed as an axiom that a central force necessarily acted upon a body describing a circular orbit round it exactly as if the one was attached to the other by a rigid bar. The "Moon paradox" was afterwards taken up by Mr. Jellinger Symons, and is frequently connected with his name. Mr. Perigal made a great many mechanical devices to illustrate his views, and wrote a number of little tracts and fly-sheets, in prose and verse, on "Revolution and Rotation," the "Fallacy of the Moon's Rotation," &c. These were generally circulated privately, but some appeared in the *Astronomical Register*. He was almost unique as a paradoxer in being a *persona grata* wherever he went, and neither at our Society, nor at the Astronomical, did he ever bring forward his own views aggressively or in a manner that was objectionable to those who differed from him. He is included in the "Budget of Paradoxes"; but De Morgan lets him off very gently, saying that "he has published valuable diagrams in profusion," and that he was "entirely indebted to him for the illustrations of the article on 'Trochoidal Curves,' published in the *Penny Cyclopædia*, without which the article would not have been written." There is no reason to doubt that he always believed that the "Perigalian system of Astronomy" would eventually supersede the Newtonian, but this did not interfere with his personal friendships with those who believed in a system which seemed to him obviously absurd. Mr. Proctor's book on the *Geometry of Cycloids* was chiefly illustrated by blocks cut in the lathe by Mr. Perigal; and Northcote in his work on *The Lathe* gives an illustration of the actual apparatus used, together with Perigal's rules for the use of Ibbetson's chucks. It may be mentioned that about 1868 Mr. Perigal publicly offered a prize of £10 for the discovery of a curve (one of his own curves produced by circular motions) satisfying certain conditions. The prize was won by Clerk

Maxwell, who obtained Mr. Perigal's curve, and also by Mr. W. D. Bushell. Many years ago he wrote a paper (in the *Journal of the Society of Arts*) on the way in which great monoliths, like those at Stonehenge, could be moved by such appliances as savages could use, and he also wrote upon the probable methods used in the construction of the Great Pyramid.

He was a warm friend, and a pleasant companion, gentle in manner, amiable, kind-hearted, and generous, interested in the topics of the day, and shrewd without being cynical. He rarely left London, and probably never went abroad in the last half of his life. At some time he had made a voyage to Madeira. Twenty years or more ago he let his beard grow, and since then his refined face, slight and fragile form, and long white beard and hair, made him a striking and picturesque figure. He ceased to attend scientific meetings two or three years ago, and for the last year or more had ceased to leave his lodgings; but there was no failure of mind or eye-sight. He was never married. By his own wish he was cremated at Woking.*

Mr. Perigal was elected a member of our Society, January 23rd, 1868, and was a frequent attendant at our meetings. There is a characteristic note by him, entitled "Link Trammels," in Vol. v., p. 144, in which he claims that Jopling had anticipated Peaucellier. The only other contribution he made was a verbal account of some "Geometrical Metamorphoses by Partition and Transformation" (January 8th, 1891). Many of these figures are to be seen in his pamphlet "Graphic Demonstrations of Geometric Problems," selected by H. Perigal, which he presented to the members of the Association for the Improvement of Geometrical Teaching (1891), of which Association he was elected a member in 1874.

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Prof. Burnside sends the following note:—

While the paper "On the Continuous Group that is defined by any Given Group of Finite Order," pp. 546–565, was passing through the press, the author had not the leisure to study carefully the memoir by Herr Frobenius "Über die Darstellung der endlichen Gruppen durch lineare Substitionen," referred to on p. 548. Having done so since, he wishes to acknowledge that the result contained in the concluding paragraph of his paper (p. 565) is anticipated in Herr Frobenius' memoir.

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* Dr. Glaisher, in a note which accompanied the above account of his old friend, writes "it was in 1855 that I first saw Perigal."

## ERRATA AND ADDENDA.

### VOL. XXVII.

Page 103, in result (111), *for* "$\left(\frac{X^2+XY+Y^2}{2}\right)^2$," *read* "$\left(\frac{X^2-2XY+Y^2}{2}\right)^2$."

,, 636, add to Lt.-Col. A. Cunningham's papers: "Note of Two High Primes, p. 328."

### VOL. XXVIII.

Page 379, line 9 from foot, *for* "of $(3^{105}+1) \div (3^{35}+1)$," *read* "of the algebraic prime factor of $(3^{105}+1)$."

### VOL. XXIX.

Page 405, result (53*d'*), *for* "$4y^4)+5$," *read* "$4y^4)^2+5$."

,, 410, table of factors of $(1+Y^{18})$, line of $Y=18$, col. of $L$, *insert* "33,388,093; B."

,, 428, col. of $M$, line of $n=39$, *for* "157 . 1041;" *read* "157 . 10141;"

,, 429, table of factors of $(1+Y^{15})$, col. of $M$, line of $Y=675$, *after* "31," *insert* "51,241 . 139,801; C."

,, 432, result (106), *for* "$(A^2+3B^2)$," *read* "$(A^2+3B^2)^2$."

# INDEX.

## INDEX OF PAPERS, &c.

[*The mark * indicates that the paper either has been printed* in extenso, *or an abstract of it given, in the "Proceedings."*]


BAKER, H. F., M.A., F.R.S.

*Note on a Property of Pfaffians, pp. 141–2.

BARWELL, Miss M. E.

*The Conformal Representation of a Pentagon on a Half Plane, pp. 695–706.

BURBURY, S. H., M.A., F.R.S.

*On the General Theory of Stationary Motion in an Infinite System of Molecules, pp. 225–248.

BURNSIDE, Prof. W., M.A., F.R.S.

*The Construction of the Straight Line joining Two given Points, pp. 125–132.

*On the Continuous Group that is defined by any given Group of Finite Order, pp. 207–224 ; (Second Paper), pp. 546–565.

*On Linear Homogeneous Continuous Groups whose Operations are Permutable, pp. 325–352.

CAMPBELL, J. E., M.A.

*On a Law of Combination of Operators (Second Paper), pp. 14–32 (see Note, p. 14).

*Transformations which leave the Lengths of Arcs on Surfaces unaltered, pp. 249–264.

CUNNINGHAM, Lt.-Col. A. J. C., R.E.

*On Aurifeuillians, pp. 381–438.—Appendix, *High Primes, 438.
Supplement to the above (incorporated in the said paper), p. 273.
The 77-Squares Puzzle, p. 518.

DAWSON, H. G., M.A.

*On the Numerical Value of $\int_0^h e^{x^2} dx$, pp. 519–522.

ELLIOTT, Prof. E. B., M.A., F.R.S.

*The Transformation of Linear Partial Differential Operators by extended Linear Continuous Groups, pp. 439–476.

FORSYTH, Prof. A. R., Sc.D., F.R.S.

*The Character of the General Integral of Partial Differential Equations, pp. 5–13.

*On those Transformations of the Coordinates which lead to new Solutions of Laplace's Equation, pp. 165–206.

*An Essay towards the Generating Functions of Ternariants, pp. 487–517.

Greenhill, Prof. A. G., M.A., F.R.S.
*The Catenary, and Associated Trajectory, on the Paraboloid, the Cone, &c. (read March 12th, 1896), pp. 585–670.

Hardcastle, Miss F.
*A Theorem concerning the Special Systems of Point-Groups on a Particular Type of Base-Curve, pp. 132–139.

Hargreaves, R., M.A.
*The Integral $\int P_n^2\,dx$, and Allied Forms in Legendre's Functions, between Arbitrary Limits, pp. 115–123.

Hopkinson, B., B.A.
*On Discontinuous Fluid Motions involving Sources and Vortices, pp. 142–164.

Lamb, Prof. H., M.A., F.R.S.
*On the Reflection and Transmission of Electric Waves by a Metallic Grating, pp. 523–544.

Love, A. E. H., M.A., F.R.S.
Note on some Fundamental Properties of Manifolds, p. 518.

Lovett, Prof. E. O.
*On the General Theory of Anharmonics, pp. 566–575.

Macaulay, F. S., D.Sc.
*Point-Groups in a Plane, and their effect in determining Algebraic Curves, pp. 673–695.

MacColl, H., B.A.
*On the Calculus of Equivalent Statements (Seventh Paper), pp. 98–109; (Eighth Paper), p. 545.

Macdonald, H. M., M.A.
*Note on Bessel Functions, pp. 110–115.
*Zeroes of the Bessel Functions, pp. 575–584.

Morley, Prof. F., Sc.D.
*On the Poncelet Polygons of a Limaçon, pp. 83–97.
*On a Regular Rectangular Configuration of Ten Lines, pp. 670–673.

Rogers, Prof. L. J., M.A.
*On certain Regular Polygons in Modular Network, pp. 706–719.

Sampson, Prof. R. A., M.A.
*A continuation of Gauss's "Dioptrische Untersuchungen," pp. 33–83.

Sheppard, W. F., M.A.
*On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Divisions of a Scale, pp. 353–380 (*cf.* Note, p. 274).

Taylor, H. M., M.A., F.R.S.
On the Intersections of Two Conics of a given type, p. 139.
*On the Intersections of Two Cubics, pp. 265–273.

Whitehead, A. N., M.A.
*The Geodesic Geometry of Surfaces in non-Euclidean Space, pp. 275–324.

Young, W. H., M.A.
*On Systems of One-Vectors in Space of $n$ Dimensions, pp. 478–487.

# GENERAL INDEX.


Acta Mathematica, 331, 707, 708, 710.
Album, 3.
Ampère, 5 &c.
Auditor, 2, 123.
Aurifeuille, 383.
Aurifeuillians, Notation of, 381 &c.
Authors cited :—
  Abel, 657.
  Ball (Math. Rec.), 518.
  Boltzmann, 235.
  Cayley, 182.
  Clebsch, 679.
  Darboux, 182, 252, 279, 287, 292.
  Euclid, 125.
  Euler (Alg.), 392, 394, 408.
  Forsyth (D. E.), 170, 195; (Th. of F.), 695, 710.
  Gauss, 33 &c., 292.
  Grassmann, 279.
  Guderman, 520.
  Halphen, 83, 586, 660.
  Jordan, 10.
  Keynes (F. Log.), 103.
  Lamé, 175, 182.
  Legendre (C. I.), 528; (F. E.), 585 &c., 648, 651, 659, 669.
  Lie, 14, 18, 254, 451.
  Lie-Engel, 329.
  Lie-Scheffers, 325, 327.
  Maxwell (E. and M.), 532.
  Monge, 176.
  Morley and Harkness, 87, 91, 95.
  Rayleigh, Ld., 523, 544.
  Reuschle, 406, 408, 427, 428.
  Reye, 671.
  Routh (A. S.), 623; (R. D.), 658.
  Salmon (S. G.), 292.
  Schroder, 107.
  Thomson, J. J., 523, 537.
  Vega, 520.
  Watson, 234.
  Whitehead, 276 &c.
Battaglini (Giorn. di), 478.
Beltrami, 275 &c.
Berliner (Monatsber.), 333; (Sitzungsber.), 207, 222, 330, 332, 351, 547, 548, 560.
Berry, 124.
Bertini, 133.
Bertram, 585.
Bickmore, 381 &c.
Bolyai, 276 &c.
Brill and Noether, 132, 679.
Brioschi, 139, 721.
Buchheim, 280.
Burbury, 225.
Burnside, W., 126, 310, 322, 546, 556, 735.
Cambridge Phil. Soc. Trans., 110, 520, 576.
——————— Proc., 142.
Campbell, J. E., 1, 4, 22.
Campbell, Lt.-Col. J. R., 1.
Candidates, Marks of, 363.
Cartan, 215, 216, 223.
Castelnuovo, 674.
Catenaries (Pseudo-Ellipt.), 273.
Catenary (on Paraboloid), 587; (Whirling Spherical), 594; (on Vertical Cone), 622; (Whirling, on Cone), 645.
Cayley, 265, 487, 512, 670, 674, 690.
Christoffel, 150, 695.
Clifford, 133, 566, 571, 573, 679.
Correspondence, Planes of, 68.
Council, Election of, 3.
Crelle, 110, 696, 699.
Cunningham, 384, 389, 396.
Definitions :—
  Absolute Differential Invariant, 450.
  Alternant, 211.
  Curves (Regular, Constructional), 674.
  Force, 478.
  Group, 326; (Abelian), 328.
  Hyposyzygetic Concomitants, 487.
  Of Geodesic Geometry, 275 &c.
  Point-base, 676.
  Toroidal Functions, 120.
De Francesco, 478.
Desboves, 437.
Dewar, T. I., 273, 606, 610, 638, 664.
Diagrams, 34, 70, 72, 77, 89, 94, 97, 101, 106–108, 127 &c., 153 &c., 237 &c., 275 &c., 525, 530, 533, 599, 607, 610, 611, 638, 665, 670, 671, 683 &c., 696, 709 &c., 717, 718.
Dorpat (Sitzungsber.), 548.
Educational Times, 109, 387.
Errata, 92, 736.
Elliott, 439, 452, 472, 474, 495, 496, 518.
Englishmen, Weight of 5552, 362.

Euler, 408.
Exchanges, 2, 3.
Factors, Tables of, 409–411, 420, 428 &c.
Forsyth, 166, 181, 488, 659.
Franklin, 496.
Frequency, Curves of, 371, 372.
Fricke, 710.
Frobenius, 207, 222, 330, 332, 351, 547, 548, 560, 735.
Frost, Dr. P., 545, 726.
Gallop, 123.
Glaisher, 520.
Gordan, 508.
Greenhill, 585 &c.
Grünert (Archiv), 614.
Hammond, 496.
Hardcastle, F., 679.
Hargreave, 114.
Hargreaves, 248, 518.
Hart, 265.
Hele Shaw, 542.
Helmholtz, 276.
Heppel, 3.
Hicks, 120.
Hobson, 579.
Hopkinson, J., 727.
Hudson, W. H. H., 125.
Hurwitz, 582.
Journal de l'Ecole Polytech., 5.
——— des Math. Pures et App., 392.
Journals subscribed for, 3.
Kasan (Memoirs), 528.
Kelvin, Ld., 165.
Kempe, A. B., 1.
Klein, 547, 707, 710, 714.
Klein-Fricke, 707.
Landry, 406.
Larmor, Dr., 33, 65, 68, 80, 523.
Lawrence, F. W., 408, 427.
Lebesgue, 392, 395.
Leipziger (Berichte), 207, 325.
Le Lasseur, 406, 427.
Library, Presents to, 4, 124, 140, 248, 274, 477. 518, 545, 719.
Lie, 450, 566.
Liouville (Journal), 165.
Lobatschewsky, 528.
Love, 142, 518.
Lucas, E., 383, 406, 408, 427.
Macaulay, F. S., 124, 673.
MacColl, H., 98, 108, 109.
MacMahon, 445, 453.
Matematica, Ann. di, 133.
Mathematical Journal, Camb., and Dubl., 265.
Mathematical Society, Bulletin of American, 675.
——— Proceedings of, 14, 22, 33, 65, 84, 98, 107, 109, 126, 225, 280, 330, 384, 396, 408, 439, 445, 451, 453, 472, 474, 495, 546, 556, 566, 579, 585 &c., 673, 679, 683, 687, 690, 692.
Mathematics, American Journal of, 451, 488, 496, 512, 579.
——— Messenger of, 181, 383, 659.
——— Quart. Jour. of, 265, 379.
Mathématiques, N. Ann. de, 437.
Mathematische Annalen, 110, 114, 132, 330, 508, 547, 582, 710, 716.
Mathews, 4, 140.
Meeting, Annual, 1.
Meetings, General (see Papers read).
Meldola, R., 139.
Members, Admission of, 1, 123, 248, 273, 518.
——— Election of, 123, 139, 518, 545.
——— Hon. Foreign, 545.
——— Number of, 2.
"Mind" (Proceedings), 108.
Molien, 547, 548.
Motions carried (Votes of thanks, &c.), 1, 124, 518.
Naval Architecture (Inst. of), 542.
Netto, 331.
Neumann, 585.
Newman, F., 520.
Obituary, 2, 139, 545, 721–735.
Osswald, 585.
Papers read, 2–4, 123, 139, 248, 273, (476), 518, 545.
Pearson, K., 378, 379.
Perigal, H., 545, 732.
Philosophical Magazine, 523, 536, 540.
——— Transactions, 114, 166, 378, 439, 445, 452.
Pincherle, 330.
Pochhammer, 696, 698.
Poincaré, 707, 708.
Primes (High), 438.
Proceedings, Exchange of, 2.
"Puzzle, 77-squares," 518.
Quorum, No, (476).
Rayleigh, Ld., 523, 536, 540.
Report, Auditor's, 124.
——— British Association, 362.
——— Secretary's, 2.
——— Treasurer's, 1, 124.
Richmond, 673.
Riemann-Roch Theorem (Concise statement of), 692.
Roberts, R. A., 84, 96.
Schwarz, 150, 695.
Schönlicht, 585.
Scott, C. A., 675.
Scrutineers, 3.
Sheppard, W. F., 362, 379.
Smith, H. J. S., 330.

Sonine, 110.
Sonntag, 585.
Statistical Soc., Jour. of R., 362.
Stöckert, 451.
Stokes, Sir G., 110, 576.
Study, 207, 325.
Sylvester, 3, 451, 496; (Memorial Fund), 139.
Tables, 509, 511, 521, 522.
Terry, 2, 123.
Trajectory (Particle on Smooth Paraboloid), 614; (Particle on Cone), 646.
Turton, Major, 518.
Van Vleck, 579.
Walker, J. J., 123.
Watson, Stephen, 109.
Weber, 109, 114, 115.
Weddle, 265.
Weierstrass, 329.
Western, 3.
Whittaker, 124.
Wren, W., 731.
Züge, 614.

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